Interpretations on the spectrum leads to understanding fundamental laws of nature.
The higher the mass, the more channels open.

A. Pilloni – The pole hunter

HadSpec PRD88, 094505

$\eta' (1300)$

$\phi$

$\omega$

$\rho$

$\eta$

$\pi$

$0^{-+}$

$1^{-+}$

$1^{++}$

$2^{++}$

$3^{+-}$

$3^{++}$

$4^{--}$

$4^{-+}$

$4^{++}$

$\pi_1 (1600)$

$a_1 (1260)$

$m_\pi = 392 \text{ MeV}$

$24^3 \times 128$

isoscalar

isovector

A. Pilloni – The pole hunter

3
**S-Matrix principles**

- **A(s,t)**
- **M-decay channel**
- **s-channel**
- **Crossing**

**Analyticity**

\[ t_l(s) = \lim_{\epsilon \to 0} t_l(s + i\epsilon) \]

**Im** \[ t_l = \rho_l |t_l|^2 \]

These are constraints the amplitudes have to satisfy, but do not fix the dynamics.

Impose the constraints, parameterize the ignorance, extract the physics.

+ Lorentz, discrete & global symmetries
S-Matrix principles

**Analyticity**

\[ t_l(s) = \lim_{\epsilon \to 0} t_l(s + i\epsilon) \]

**Im \( t_l = \rho_l |t_l|^2 \)**

**Unitarity**

Unitarity opens a cut on the real axis

Resonances (QCD states) are poles in the unphysical Riemann sheets

Unitarity controls the interference pattern between resonances and background

+ Lorentz, discrete & global symmetries
The hybrid $\pi_1$

HV
Hybrid Vehicle
Hybrid hunting

\[ \bar{\psi} \gamma^\mu F_{\mu\nu} \psi \text{ smells of glue} \]
\[ (J^{PC})_g = 1^{+-} \]
\[ \text{mass } \approx 1.0-1.5 \text{ GeV} \]

Lightest Hybrids

\[ S_{qq} = 1 \quad S_{q\bar{q}} = 0 \]

\[ J^{PC}: \quad 0^{-}, 1^{-+}, 2^{++} \quad 1^{--} \]

“exotic hybrid”

Small signal in data, requires refined PWA
Two hybrid states?

\[ \pi_1(1400) \quad I^G(J^{PC}) = 1^-(1^{-+}) \]

See also the mini-review under non-\( q \bar{q} \) candidates in PDG 2006. Journal of Physics G33 1 (2006).

<table>
<thead>
<tr>
<th>( \pi_1(1400) ) MASS</th>
<th>( 1354 \pm 25 \text{ MeV} ) (S = 1.8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_1(1400) ) WIDTH</td>
<td>( 330 \pm 35 \text{ MeV} )</td>
</tr>
</tbody>
</table>

**Decay Modes**

<table>
<thead>
<tr>
<th>Mode</th>
<th>( \frac{\Gamma}{\Gamma_{\text{tot}}} )</th>
<th>Scale Factor/Conf. Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>( q \bar{q}^0 )</td>
<td>seen</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>( q \bar{q}^- )</td>
<td>seen</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>( q \bar{q}^+ )</td>
<td></td>
</tr>
</tbody>
</table>

\[ \pi_1(1600) \quad I^G(J^{PC}) = 1^-(1^{-+}) \]

<table>
<thead>
<tr>
<th>( \pi_1(1600) ) MASS</th>
<th>( 1662^{+48}_{-38} \text{ MeV} ) (S = 1.4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \pi_1(1600) ) WIDTH</td>
<td>( 241 \pm 40 \text{ MeV} ) (S = 1.4)</td>
</tr>
</tbody>
</table>

**Decay Modes**

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<tr>
<td>( \Gamma_1 )</td>
<td>( \pi \pi \pi )</td>
<td>seen</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>( \rho^0 \pi^- )</td>
<td>seen</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>( f_2(1270)\pi^- )</td>
<td>not seen</td>
</tr>
<tr>
<td>( \Gamma_4 )</td>
<td>( h_1(1255)\pi )</td>
<td>seen</td>
</tr>
<tr>
<td>( \Gamma_5 )</td>
<td>( q'(958)\pi^- )</td>
<td>seen</td>
</tr>
<tr>
<td>( \Gamma_6 )</td>
<td>( f_1(1285)\pi )</td>
<td>seen</td>
</tr>
</tbody>
</table>
Two hybrid states?

Neither lattice nor models predict two isovector $1^{-+}$ states in that region.

A hybrid meson ($8 \otimes 8$) cannot decay into $\eta\pi$ in the chiral limit.

Tetraquark ($10 \oplus \overline{10}$)? Requires doubly charged.
A sharp drop appears at 2 GeV in P-wave intensity and phase.

No convincing physical motivation for it.

It affects the position of the $a'_2(1700)$.

We decided to fit up to 2 GeV only.
\[ \rho \pi \] channel and Deck amplitude

The production allows for a nonresonant component (Deck effect). The singularity is close to the physical region, peaking background.

We do not include this channel.
Amplitudes for $\eta^{(1)}\pi$

We build the partial wave amplitudes according to the $N/D$ method

\[
\text{Im } a(s) = \rho \ a(s) \ t^*(s)
\]

\[
\frac{d\sigma}{d\sqrt{s}} \propto \frac{\rho}{\sqrt{s}} \ |p^L q^{L-1} \ a(s)|^2
\]
Amplitudes for $\eta^{(i)}\pi$

We build the partial wave amplitudes according to the $N/D$ method

$\alpha(s)$ is an effective $2 \rightarrow 2$ process, where the Pomeron is treated as a vector quasi-particle with virtuality $t_{\text{eff}} = -0.1$ GeV$^2$
Amplitudes for $\eta^{(')}\pi$

We build the partial wave amplitudes according to the $N/D$ method

\[ t(s) = \frac{N(s)}{D(s)}, \quad a(s) = \frac{n(s)}{D(s)} \]

The $D(s)$ has only right hand cuts; it contains all the Final State Interactions constrained by unitarity $\rightarrow$ universal

\[ \text{Im} \ D(s) = -\rho \ N(s) \]
Amplitudes for $\eta^{(\pi)}\pi$

We build the partial wave amplitudes according to the $N/D$ method

\[ t(s) = \frac{N(s)}{D(s)}, \quad a(s) = \frac{n(s)}{D(s)} \]

The $n(s), N(s)$ have left hand cuts only, they depend on the exchanges → process-dependent, smooth
Recap: single channel $\eta\pi$

Test against the $D$-wave $\eta\pi$ data, where the $a_2$ and the $a_2'$ show up

A. Jackura, M. Mikhasenko, AP et al. (JPAC & COMPASS), PLB779, 464-472

Precise determination of pole position
Coupled channel: the model

Two channels, $i, k = \eta\pi, \eta'\pi$  
Two waves, $J = P, D$  
37 fit parameters

$$D_{ki}^J(s) = \left[ K^J (s)^{-1} \right]_{ki} - \frac{s}{\pi} \int_{s_k}^{\infty} ds' \frac{\rho N_{ki}^J(s')}{s'(s' - s - i\epsilon)}$$

$$K_{ki}^J(s) = \sum_R \frac{g_{kR}^{(R)} g_{iR}^{(R)}}{m_R^2 - s} + c_{ki}^J + d_{ki}^J s$$

$$\rho N_{ki}^J(s') = \delta_{ki} \frac{\lambda^{J+1/2}(s', m_{\eta'}^2, m_{\pi}^2)}{(s' + s_R)^{2J+1+\alpha}}$$

$$n_k^J(s) = \sum_{n=0}^{3} a_{n,k}^J T_n \left( \frac{s}{s + s_0} \right)$$
Coupled channel: the model

Two channels, \( i, k = \eta \pi, \eta' \pi \)  
Two waves, \( J = P, D \)  
37 fit parameters

\[
D_{ki}^J(s) = \left[K^J(s)^{-1}\right]_{ki} - \frac{s}{\pi} \int_{s_k}^{\infty} ds' \frac{\rho N_{ki}^J(s')}{s'(s' - s - i\epsilon)}
\]

\[
K_k^J(s) = \sum_R \frac{g_k^{(R)} g_i^{(R)}}{m_R^2 - s} + c_k^J + d_k^J s
\]

\[
\rho N_{ki}^J(s') = \delta_{ki} \frac{\lambda^{J+1/2} \left(s', m_{\eta'(\pi)}^2, m_{\pi}^2\right)}{(s' + s_R)^{2J+1+\alpha}}
\]

\[
n_k^J(s) = \sum_{n=0}^{3} a_n^{J,k} T_n \left(\frac{s}{s + s_0}\right)
\]

Left-hand scale (Blatt-Weisskopf radius) \( s_R = s_0 = 1 \text{ GeV}^2 \)  
\( \alpha = 2 \) as in the single channel, 3rd order polynomial for \( n_k^J(s) \)

1 K-matrix pole for the P-wave  
2 K-matrix poles for the D-wave
\[ \chi^2 / \text{dof} = 162/122 \sim 1.3, \text{ statistical error estimated via 50k bootstraps} \]

Bands show the 2\(\sigma\) error

A. Pilloni – The pole hunter
Polynomial in the numerator

The numerator should be smooth and have variation milder than the typical resonance width.

This happens indeed.
Correlations

Denominator parameters uncorrelated with the numerator ones ✅

Denominator parameters uncorrelated between $P$- and $D$-wave ✅

Production (numerator) parameters

K-matrix «pole» parameters

K-matrix «bkg» parameters

A. Pilloni – The pole hunter
For the best fit solution, we look at the closest Riemann sheet in the complex plane. We see a limited amount of poles in the relevant region. We also see some close-to-threshold poles, which are artifacts of the model for $N(s)$. How to distinguish the two?
We can identify the poles in the region \( m \in [1.2, 2] \text{ GeV}, \Gamma \in [0, 1] \text{ GeV} \)

Two stable isolated poles are identifiable in the \( D \)-wave

Only one is stable in the \( P \)-wave
The variance of the bootstrapped poles gives the statistical error

<table>
<thead>
<tr>
<th>Poles</th>
<th>Mass (MeV)</th>
<th>Width (MeV)</th>
</tr>
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<tbody>
<tr>
<td>$a_2(1320)$</td>
<td>1306.0 ± 0.8</td>
<td>114.4 ± 1.6</td>
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<td>$a_2'(1700)$</td>
<td>1722 ± 15</td>
<td>247 ± 17</td>
</tr>
<tr>
<td>$\pi_1$</td>
<td>1564 ± 24</td>
<td>492 ± 54</td>
</tr>
</tbody>
</table>

A. Pilloni – The pole hunter
Again into the complex plane

The strength of the pole propagates differently in the two channels

In $\eta\pi$ the strength move to lighter values
Systematic studies

• Change of functional form and parameters in the denominator

\[ \rho N_{ki}^J(s') = g \delta_{ki} \frac{\lambda^{J+1/2}(s', m_{\eta(\nu)}, m_{\pi})}{(s' + s_R)^{2J+1+\alpha}} \]

- Default: \( s_R = 1 \text{ GeV}^2 \). We try \( s_R = 0.8, 1.8 \text{ GeV}^2 \)
- Default: \( \alpha = 2 \). We try \( \alpha = 1 \)
- We also try a different function: \( \rho N_{ki}^J(s') = g \delta_{ki} \frac{Q_J(z_{s'})}{s'^{\alpha} \lambda^{1/2}(s', m_{\eta(\nu)}, m_{\pi})} \)
  with \( \alpha = 2, 1.5, 1 \)

• Change of parameters in the numerator

- Default: \( t_{\text{eff}} = -0.1 \text{ GeV}^2 \). We try \( t_{\text{eff}} = -0.5 \text{ GeV}^2 \)
- Default: 3rd order polynomial. We try 4th
For each class, the maximum deviation of mass and width is taken as a systematic error. Deviation smaller than the statistical error are neglected. Systematic of different classes are summed in quadrature.
Bootstrap for $s_R = 1.8 \text{ GeV}^2$

Our skepticism about a second pole in the relevant region is confirmed:
It is unstable and not trustable
Final results

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<td>492 ± 54 ± 102</td>
</tr>
</tbody>
</table>
The $a_1$
The $a_1(1260)$

Despite it has been known since forever, the resonance parameters of the $a_1(1260)$ are poorly determined. The production (and model) dependence is affecting their extraction.
The extraction of the resonance in the $\tau$ decay should be the cleanest, but the determination of the pole is still unstable.
3-body stuff

Having a $3\pi$ final state requires implementing 3-body unitarity

\[ \text{Diagram showing the calculation of 3-body unitarity.} \]

A. Jackura

M. Mai, B. Hu, M. Doring, AP, A. Szczepaniak EPJA53, 9, 177
A. Jackura, et al., 1809.10523

⇒ See Andrew’s talk on Monday
Factorizable model

\[ A(\sigma_j', s, \sigma_k) = f(\sigma_j') \hat{A}(\sigma_j', s, \sigma_k) f(\sigma_k) \]

One can amputate the 2-body amplitude from the initial and final state.
One can amputate the 2-body amplitude from the initial and final state.

If one neglects the effect of the disconnected diagrams to unitarity, is it possible to suppress the dependence of the reduced amplitude on the 2-body invariant masses?

$$\mathcal{A} (\sigma'_j, s, \sigma_k) = f(\sigma'_j) \hat{\mathcal{A}} (\sigma'_j, s, \sigma_k) f(\sigma_k)$$
The unitarity equation is now algebraic and easier to handle

\[ \text{Im} \hat{A}(s) = \hat{A}(s) \hat{A}(s)\dagger \int d\Phi_3 \left| \sum_j f(\sigma_j) \right|^2 \]

Integral over the Dalitz plot (aka quasi 2-body)
More about the model

We consider ALEPH data of $\tau^- \rightarrow \pi^- \pi^+ \pi^- \nu_\tau$

CLEO estimated the dominant decay mode to be $a_1(1260) \rightarrow \rho \pi$ in S-wave

This statement is model dependent, and would be desirable to perform a combined fit of the subchannels. However, no data are available → we consider $\rho^0 \pi^- \pi^- \pi^-$ S-wave only

The faible $\pi^- \pi^-$ interaction is neglected

\[
f(\sigma) = \mathcal{N} \frac{p(\sigma)R}{\sqrt{1 + (p(\sigma)R)^2}} \frac{1}{m_\rho^2 - \sigma - i m_\rho \Gamma_\rho(\sigma)}
\]

\[
\Gamma(\sigma) = \Gamma_\rho \times \frac{p^3(\sigma)}{\sqrt{\sigma} \sqrt{1 + (p(\sigma)R)^2}} \bigg/ \frac{p(m_\rho^2)}{m_\rho \sqrt{1 + (p(m_\rho^2)R)^2}}
\]

\[
\hat{A}(s) = \frac{c}{m^2 - s - ig^2 C(s)/2}
\]

Standard $P$-wave Breit-Wigner with Blatt-Weisskopf barrier factors

Standard $S$-wave Breit-Wigner with modified phase space
More about the model

We consider ALEPH data of $\tau^- \to \pi^- \pi^+ \pi^- \nu_\tau$

$$\hat{A}(s) = \frac{c}{m^2 - s - ig^2 C(s)/2}$$

$$C(s) = \left\{ \begin{array}{l}
\frac{1}{2} \int d\Phi_3 \left| \sum_{\lambda} f_\rho(\sigma_1)D^1_{0\lambda}(\Omega_1)D^1_{\lambda 0}(\Omega_{23}) - f_\rho(\sigma_3)D^1_{0\lambda}(\Omega_3)D^1_{\lambda 0}(\Omega_{12}) \right|^2 \equiv \rho_{\text{SYMM}}(s)

\int d\Phi_3 \left| \sum_{\lambda} f_\rho(\sigma_1)D^1_{0\lambda}(\Omega_1)D^1_{\lambda 0}(\Omega_{23}) \right|^2 \equiv \rho_{\text{QTB}}(s)

l_0 + \frac{s}{\pi} \int ds' \frac{\rho_{\text{SYMM}}(s')}{s'(s' - s - i\epsilon)} \equiv \rho_{\text{SYMM-DISP}}(s)

l_0 + \frac{s}{\pi} \int ds' \frac{\rho_{\text{QTB}}(s')}{s'(s' - s - i\epsilon)} \equiv \rho_{\text{QTB-DISP}}(s)
\end{array} \right.$$
Fit to data

Dispersive models look better

A. Pilloni – The pole hunter
Analytic continuation

\[ \rho_{QTB}(s) \propto \frac{1}{s} \int_{4m^2}^{(\sqrt{s} - m_\pi)^2} d\sigma_1 f^{(II)}(\sigma_1) f^{(I)}(\sigma_1) \frac{\sqrt{\lambda_1 \lambda_{s1}}}{\sigma_1} \]

Going to complex \( s \) means making the integration path complex.

When the integration boundary hits the \( \rho \) pole in \( f(\sigma_1) \), the woolly cut opens.
Analytic continuation

\[ \rho_{QT\bar{B}}(s) \propto \frac{1}{s} \int_{4m_{\pi}^2}^{(\sqrt{s} - m_{\pi})^2} d\sigma_1 f^{(II)}(\sigma_1) f^{(I)}(\sigma_1) \frac{\sqrt{\lambda_1 \lambda_{s1}}}{\sigma_1} \]

Going to complex \( s \) means making the integration path complex.

When the integration boundary hits the \( \rho \) pole in \( f(\sigma_1) \), the woolly cut opens.
An additional pole appears, almost hidden under the woolly cut. It can be traced back to the $1/s$ factor of the phase space.
Systematics

<table>
<thead>
<tr>
<th>#</th>
<th>Fit studies</th>
<th>QTB-DISP $\chi^2$/n.d.f.</th>
<th>SYMM-DISP $\chi^2$/n.d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$s &lt; 2 \text{ GeV}^2$</td>
<td>53/62</td>
<td>81/62</td>
</tr>
<tr>
<td>2</td>
<td>$R' = 3 \text{ GeV}^{-1}$</td>
<td>18/100</td>
<td>83/100</td>
</tr>
<tr>
<td>3</td>
<td>$m'<em>\rho = m</em>\rho + 10 \text{ MeV}$</td>
<td>37/100</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$m'<em>\rho = m</em>\rho - 10 \text{ MeV}$</td>
<td>30/100</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$m'<em>\rho = m</em>\rho - 20 \text{ MeV}$</td>
<td>66/100</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$\Gamma'<em>\rho = \Gamma</em>\rho + 5 \text{ MeV}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$\Gamma'<em>\rho = \Gamma</em>\rho - 30 \text{ MeV}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ m_p^{(a_1(1260))} = (1209 \pm 4^{+12}_{-9}) \text{ MeV}, \quad \Gamma_p^{(a_1(1260))} = (576 \pm 11^{+80}_{-20}) \text{ MeV} \]
Conclusions

We perform a coupled-channel analysis to the $\eta^{(')}\pi$ COMPASS data.

We can describe data with a model which generates a single stable pole in the relevant region of the $P$-wave.

The pole position is sufficiently stable upon changes of the model.

We also extract the resonant parameters of $a_2^{(')}$.

<table>
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</tbody>
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We perform the analysis of $\tau \rightarrow 3\pi\nu$ ALEPH data.

We consider a simplified quasi 2-body model, with a reduced unitarity equation easier to handle.

The $a_1(1260)$ pole position is determined.

$m_p^{(a_1(1260))} = (1209 \pm 4^{+12}_{-9})$ MeV

$\Gamma_p^{(a_1(1260))} = (576 \pm 11^{+80}_{-20})$ MeV
Joint Physics Analysis Center

- We aim at developing new theoretical tools, to get insight on QCD using first principles of QFT (unitarity, analyticity, crossing symmetry, low and high energy constraints,...) to extract the physics out of the data.

- Many other ongoing projects (both for meson and baryon spectroscopy, and for high energy observables), with a particular attention to producing complete reaction models for the golden channels in exotic meson searches.
Formalism

- Process is at fixed $s_{tot}$, and integrated $t$. Interested in resonances in $s$
- Recoil proton kinematically decouples from final state $\eta \pi$
- Expand amplitude into partial waves

$$A_{\mu' \mu}(s_{tot}, s, t, s_1, t_1) = \sum_{LM\epsilon} a_{LM, \mu' \mu}^{\epsilon}(s_{tot}, t, s) Y_{LM}^{\epsilon}(\theta, \phi)$$
Analytic continuation

\[ \rho_{\text{INT}}(s) = \frac{1}{2\pi(8\pi)^2 s} \int_{4m^2_\pi}^{\sigma_{1\text{im}}} d\sigma_1 \int_{\sigma_3^-/(s,\sigma_1)}^{\sigma_3^+(s,\sigma_1)} d\sigma_3 \frac{f^*_\rho(\sigma_1)}{\sqrt{\sigma_1 - 4m^2_\pi}} \frac{f_\rho(\sigma_3)}{\sqrt{\sigma_3 - 4m^2_\pi}} \times \frac{W(\sqrt{s}, \sqrt{\sigma_1}, \sqrt{\sigma_3})}{((\sqrt{s} + \sqrt{\sigma_1})^2 - m^2_\pi)((\sqrt{s} + \sqrt{\sigma_3})^2 - m^2_\pi)}. \]
Correlations

Denominator parameters not very correlated with the numerator ones

A. Pilloni – The pole hunter
Denominator parameters uncorrelated between \(P\)- and \(D\)-wave \(\checkmark\)
Formalism

The differential cross section is

\[
\frac{d\sigma}{ds} = \frac{1}{2(4\pi)^4 \sqrt{s}} \left( \frac{\hbar c}{m_N P_{lab}} \right)^2 \frac{1}{2} \sum_{LM\epsilon} \int_{t_-}^{t_+} dt |p| \sum_{\mu\mu'} |a_{LM,\mu'\mu}^\epsilon(s_{tot}, t, s)|^2
\]

\[
\equiv \frac{\mathcal{N}}{\sqrt{s}} \sum_{LM\epsilon} \mathcal{I}_{LM}^\epsilon(s_{tot}, s)
\]

where the intensity distribution is defined

\[
\mathcal{I}_{LM}^\epsilon(s_{tot}, s) = \int_{t_-}^{t_+} dt |p| \sum_{\mu\mu'} |a_{LM,\mu'\mu}^\epsilon(s_{tot}, t, s)|^2
\]

Model will be compared to intensity distributions given by COMPASS
## Systematic studies

<table>
<thead>
<tr>
<th>Systematic</th>
<th>Poles</th>
<th>Mass (MeV)</th>
<th>Deviation (MeV)</th>
<th>Width (MeV)</th>
<th>Deviation (MeV)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Variation of the function $\rho N(s')$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s_R = 0.8 \text{ GeV}^2$</td>
<td>$a_2(1320)$</td>
<td>1306.4</td>
<td>0.4</td>
<td>115.0</td>
<td>0.6</td>
</tr>
<tr>
<td></td>
<td>$a_2'(1700)$</td>
<td>1720</td>
<td>-3</td>
<td>272</td>
<td>26</td>
</tr>
<tr>
<td></td>
<td>$\pi_1$</td>
<td>1532</td>
<td>-33</td>
<td>484</td>
<td>-8</td>
</tr>
<tr>
<td>$s_R = 1.8 \text{ GeV}^2$</td>
<td>$a_2(1320)$</td>
<td>1305.6</td>
<td>-0.4</td>
<td>113.2</td>
<td>-1.2</td>
</tr>
<tr>
<td></td>
<td>$a_2'(1700)$</td>
<td>1743</td>
<td>21</td>
<td>254</td>
<td>7</td>
</tr>
<tr>
<td></td>
<td>$\pi_1$</td>
<td>1528</td>
<td>-36</td>
<td>410</td>
<td>-82</td>
</tr>
<tr>
<td>Systematic assigned</td>
<td>$a_2(1320)$</td>
<td>0.0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$a_2'(1700)$</td>
<td>21</td>
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# Systematic studies

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Formalism

- $\pi p \rightarrow \eta \pi p$ is high-energy peripheral process $\implies$ pomeron dominated exchange
- Factorize pomeron-nuclear vertex
- Pomeran has effective mass $\sqrt{-t}$
- Denote $p = |p|$ the momentum of the $\eta \pi$ system, and $q = |q|$ the momentum of the $\pi \bar{P}$ system
Searching for resonances in $\eta \pi$.
Signatures as $J^{PC} = 1^{-+}$ are not allowed in the quark model, Coulomb gauge QCD and flux tube predict glue excitation to be a quasi-particle with $J^{PC} = 1^{+-}$, $q\bar{q}g$ states expected. Need some constraint to draw robust conclusions about the existence of exotic states.
Recap: single channel $\eta\pi$

The denominator $D(s)$ contains all the FSI constrained by unitarity → universal

$$D(s) = (K^{-1})(s) - \frac{s}{\pi} \int_{s_i}^{\infty} \frac{\rho(s')N(s')}{s' (s' - s)} ds'$$

$$K(s) = \sum_{R} \frac{g_R^2}{M_R^2 - s} \quad \text{OR} \quad K^{-1}(s) = c_0 - c_1 s + \sum_{i} \frac{c_i}{M_i^2 - s}$$

$$\rho(s)N(s) = g \frac{\lambda^{(2l+1)/2} (s, m_{\pi}^2, m_{\eta}^2)}{(s + s_R)^7} \quad n(s) = \sum_{n} a_n T_n \left( \frac{s}{s + s_0} \right)$$
Recap: single channel $\eta\pi$

The denominator $D(s)$ contains all the FSI constrained by unitarity $\rightarrow$ universal

$$D(s) = (K^{-1})(s) - \frac{s}{\pi} \int_{s_i}^{\infty} \frac{\rho(s')N(s')}{s'(s' - s)} ds'$$

$$K(s) = \sum_R \frac{g_R^2}{M_R^2 - s} \quad \text{K-matrix}$$

more QFT motivated

poles on the 1st sheet unlikely

OR

$$K^{-1}(s) = c_0 - c_1 s + \sum_i \frac{c_i}{M_i^2 - s} \quad \text{CDD parameterization}$$

more $S$-matrix motivated

poles on the 1st sheet impossible
Recap: single channel $\eta\pi$

The denominator $D(s)$ contains all the FSI constrained by unitarity $\rightarrow$ universal

$$D(s) = \left(K^{-1}\right)(s) - \frac{s}{\pi \int_{s_i}^{\infty} \frac{\rho(s')N(s')}{s' \left(s' - s\right)} ds'$$

Numerator functions know about crossed channel dynamics unconstrained, we use a smooth model

$$\rho(s)N(s) = g \frac{\lambda^{(2l+1)/2}}{(s + s_R)^7} \left(s, m_{\pi}^2, m_{\eta}^2\right)$$

$$n(s) = \sum_{n} a_n T_n \left\{ \frac{s}{s + s_0} \right\}$$
Searching for resonances in $\eta\pi$

$m(a_2) = (1307 \pm 1 \pm 6) \text{ MeV}$ \quad $m(a'_2) = (1720 \pm 10 \pm 60) \text{ MeV}$

$\Gamma(a_2) = (112 \pm 1 \pm 8) \text{ MeV}$ \quad $\Gamma(a'_2) = (280 \pm 10 \pm 70) \text{ MeV}$

- The coupled channel analysis involving the exotic $P$-wave is ongoing, as well as the extension to the GlueX production mechanism and kinematics.
Correlations

Polynomial parameters uncorrelated between $P$- and $D$-wave $\checkmark$
## Correlations

| $m_{D,2}^2$ | 0  | -19 | 28  | 27  | -22 | 21  | -2  | -17 | 51  | 7   | 22  | 10  | 84  | 23  | 100 | 23  |
| $g_{\eta'\pi}^{D,2}$ | -5 | -3  | 5   | -14 | -18 | 14  | -17 | -20 | 54  | 14  | 30  | 13  | 30  | 100 | 23  |
| $g_{\eta \pi}^{D,2}$ | -1 | -5  | 5   | -0  | -16 | 16  | -17 | -20 | 49  | 2   | 34  | 1   | 100 | 30  | 84  |
| $m_{D,1}^2$ | -22 | 14  | -24 | -67 | 17  | 14  | -10 | -3  | -6  | 99  | -23 | 100 | 1   | 13  | 10  |
| $g_{\eta'\pi}^{D,1}$ | 14 | -6  | -0  | 9   | -11 | 6   | -12 | -16 | 24  | -20 | 100 | -23 | 34  | 30  | 22  |
| $g_{\eta \pi}^{D,1}$ | -33 | 13  | -19 | -74 | 19  | 18  | -10 | -2  | -7  | 100 | -20 | 99  | 2   | 14  | 7   |
| $m_{P,1}^2$ | 5   | -9  | 21  | 16  | -38 | 0   | -26 | -51 | 100 | -7  | 24  | -6  | 49  | 54  | 51  |
| $g_{\eta'\pi}^{P,1}$ | -2 | -2  | 4   | -1  | 11  | 1   | 16  | 100 | -51 | -2  | -16 | -3  | -20 | -20 | -17 |
| $g_{\eta \pi}^{P,1}$ | 5   | -16 | 34  | 13  | -22 | 14  | 16  | -26 | -10 | -12 | -10 | -17 | -17 | -2  |
| $\Gamma_{\pi}$ | -8  | -17 | 43  | -20 | 8   | 100 | 14  | 1   | 0   | 18  | 6   | 14  | 16  | 14  | 21  |
| $m_{\pi}$ | -13 | 5   | -13 | -25 | 100 | 8   | -22 | 11  | -38 | 19  | -11 | 17  | -16 | -18 | -22 |
| $\Gamma_{\pi'}$ | 30  | -14 | 16  | 100 | -25 | -20 | 13  | -1  | 16  | -74 | 9   | -67 | -0  | -14 | 27  |
| $m_{\pi'}$ | 2   | -41 | 100 | 16  | -13 | 43  | 34  | 4   | 21  | -19 | -0  | -24 | 5   | 5   | 28  |
| $\Gamma_{\pi''}$ | 0   | $100$ | -41 | -14 | 5   | -17 | -16 | -2  | -9  | 13  | -6  | 14  | -5  | -3  | -19 |
| $m_{\pi''}$ | $100$ | 0   | 2   | 30  | -13 | 8   | 5   | 2   | 5   | -33 | 14  | -22 | -1  | -5  | 0   |

A. Pilloni – The pole hunter
For the best fit solution, we look at the closest Riemann sheet in the complex plane. We see a limited amount of poles in the relevant region. We also see some close-to-threshold poles, which are artifacts of the model for $N(s)$. How to distinguish the two?
For the best fit solution, we look at the closest Riemann sheet in the complex plane. We see a limited amount of poles in the relevant region. We also see some close-to-threshold poles, which are artifacts of the model for $N(s)$.

How to distinguish the two?
\[ \pi_1(1600) \rightarrow \rho \pi \rightarrow \pi \pi \pi \]

The strength of the Deck effect depends on the momentum transferred \( t \), but the precise estimates rely on the model for the Deck amplitude.
**Pole hunting**

More complicated structure when more thresholds arise: two sheets for each new threshold

- **III sheet**: usual resonances
- **IV sheet**: cusps (virtual states)

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A. Pilloni – The pole hunter
For the best fit solution, we look at the closest Riemann sheet in the complex plane.

We see a limited amount of poles in the relevant region.

We also see some close-to-threshold poles, which are artifacts of the model for \( N(s) \).

How to distinguish the two?
Regge exchange

Resonances are poles in $s$ for fixed $l$
reggeons are poles in $l$ for fixed $s$
dominate low energy region

dominate high energy region

$$A_l \sim \frac{g_1 g_2}{s_p - s}$$

$$A \sim \sum s^l \sim g_1(t) g_2(t) s^{\alpha_{\pm}(t)} \left( -\frac{e^{i\pi \alpha_{\pm}(t)} \pm 1}{\sin \pi \alpha_{\pm}(t)} \right)$$

Diagram:

- Beam
- Target
- Exchanged particle
- Rapidity gap
- "Fast" and "Slow"
- Resonance production
- Reggeon exchange

A. Pilloni – Challenges in the analysis of meson-spectroscopy data: Theory
Finite energy sum rules

\[ s_1 = m(\pi \pi)^2 \]

\[ \nu = \frac{s - u}{2} \]

\[ \frac{1}{\Lambda^n} \int_{\nu_{th}}^{\Lambda} \text{Im} A(\nu, t) \nu^n d\nu = \frac{\beta(t) \Lambda^{\alpha(t)+1}}{\alpha(t) + n + 1} \]

**aim**: first systematic analysis of peripheral production using FESR

A. Pilloni – Challenges in the analysis of meson-spectroscopy data: Theory
**$\eta\pi$ production**

$m(\eta\pi) < 3 \,(\text{GeV/c}^2)^2$

$m(\eta\pi) \in [5-6] \,(\text{GeV/c}^2)^2$

COMPASS coll. (2015)

A. Pilloni – Challenges in the analysis of meson-spectroscopy data: Theory
Model dependence and physics

Understanding the model dependence is mandatory: models with similar fit qualities can lead to dramatically different physical interpretations.

E.g. $e^+e^- \rightarrow J/\psi \pi\pi$ and the $Z_c(3900)$

AP et al. (JPAC), PLB772, 200
Recipes to build an amplitude

M. Mikhasenko, AP, J. Nys et al. (JPAC), EPJC78, 3, 229

The literature abounds with discussions on the optimal approach to construct the amplitudes for the hadronic reactions

- Helicity formalism
  Jacob, Wick, Annals Phys. 7, 404 (1959)

- Covariant tensor formalisms
  Chung, PRD48, 1225 (1993)
  Chung, Friedrich, PRD78, 074027 (2008)
  Anisovich, Sarantsev, EPJA30, 427 (2006)

The common lore is that the former one is nonrelativistic, especially when expressed in terms of LS couplings and the latter takes into account the proper relativistic corrections.
How helicity formalism works

- Helicity formalism enforces the constraints about rotational invariance
- It allows us to fix the angular dependence of the amplitude
- What about energy dependence?

Example: $B \rightarrow \psi K^* \rightarrow \pi K$

$$
\mathcal{M}_{\Delta \lambda, \mu}^{K*} \equiv \sum_{n} \sum_{\lambda_{K^*}} \sum_{\lambda_{\psi}} \mathcal{H}_{\lambda_{K^*}, \lambda_{\psi}}^{B \rightarrow K_n^* \psi} \delta_{\lambda_{K^*}, \lambda_{\psi}}
$$

\begin{align*}
\mathcal{H}_{K_n^* \rightarrow K \pi} & D_{\lambda_{K^*}, 0}^{J_{K_n^*}}(\phi_K, \theta_{K^*}, 0)^* \\
R_{K_n^* n} (m_{K^*}) & D_{1}^{\lambda_{\psi}, \Delta \mu}(\phi_{\mu}, \theta_{\psi}, 0)^*
\end{align*}

Each set of angles is defined in a different reference frame
How tensor formalism works

The method is based on the construction of explicitly covariant expressions.

- To describe the decay $a \rightarrow bc$, we first consider the polarization tensor of each particle, $\varepsilon_{\mu_1...\mu_i}^j (p_i)$
- We combine the polarizations of $b$ and $c$ into a “total spin” tensor $S_{\mu_1...\mu_S}(\varepsilon_b, \varepsilon_c)$
- Using the decay momentum, we build a tensor $L_{\mu_1...\mu_L}(p_{bc})$ to represent the orbital angular momentum of the $bc$ system, orthogonal to the total momentum of $p_a$
- We contract $S$ and $L$ with the polarization of $a$

Tensor $\times R_X(m)$ which contain resonances and form factors
What do we know?

- Energy dependence is not constrained by symmetry
- Still, there are some known properties one can enforce

\[ R_X(m) = B'_{LX} (p, p_0, d) \left( \frac{p}{M_{\Lambda_0^b}} \right)^{L_X^0_{\Lambda_0^b}} \]

\[ BW(m|M_{0X}, \Gamma_{0X}) B'_{LX} (q, q_0, d) \left( \frac{q}{M_{0X}} \right)^{L_X} \]

- **Kinematical singularities**: e.g. barrier factors (known)
- **Left hand singularities** (need model, e.g. Blatt-Weisskopf)
- **Right hand singularities** = resonant content (Breit Wigner, K-matrix...)
Kinematics

- Kinematical singularities appear because of the spin of the external particle involved
- We can write the most general covariant parametrization of the amplitude as tensor of external polarizations $\otimes$ scalar amplitudes
- Scalar amplitudes must be kinematical singularities free
- They can be matched to the helicity amplitudes
- We can get the minimal energy dependent factor
- Any other additional energy factor would be model-dependent
$B \rightarrow \psi \pi K$

To consider the effect of spin, let’s consider $B \rightarrow \psi \pi K$
We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics

$p = \text{incoming 3-momentum in the COM} = \frac{\lambda_{12}^{1/2}}{2\sqrt{s}}$

$$\sqrt{s - (m_1 + m_2)^2} \left[ s - (m_1 - m_2)^2 \right] \frac{1}{2\sqrt{s}}$$

A. Pilloni – The pole hunter
$B \rightarrow \psi \pi K$

To consider the effect of spin, let’s consider $B \rightarrow \psi \pi K$

We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics

$q = \text{outgoing 3-momentum in the COM} = \frac{\frac{\lambda_{34}^{1/2}}{2\sqrt{s}}}{2\sqrt{s}}$

$= \frac{\sqrt{s - (m_3 + m_4)^2} [s - (m_3 - m_4)^2]}{2\sqrt{s}}$
$B \rightarrow \psi \pi K$

To consider the effect of spin, let’s consider $B \rightarrow \psi \pi K$
We focus on the parity violating amplitude for the $K^*$ isobars, scattering kinematics

$$z_s = \text{cosine of the scatt. angle in the COM}$$

$$= \frac{s(t-u) + (m_1^2 - m_2^2)(m_3^2 - m_4^2)}{\lambda_{12}^{1/2} \lambda_{34}^{1/2}} = \frac{\text{polynomial}}{\lambda_{12}^{1/2} \lambda_{34}^{1/2}}$$
Helicity amplitudes

\[ A_\lambda = \frac{1}{4\pi} \sum_{j=|\lambda|}^{\infty} (2j + 1) A^j_\lambda(s) d^j_{\lambda_0}(z_s) \]

\[ d^j_{\lambda_0}(z_s) = \hat{d}^j_{\lambda_0}(z_s) \xi_{\lambda_0}(z_s), \quad \xi_{\lambda_0}(z_s) = \left( \sqrt{1 - z_s^2} \right)^{\lambda} \]

\( \hat{d}^j_{\lambda_0}(z_s) \) is a polynomial of order \( j - |\lambda| \) in \( z_s \).

The kinematical singularities of \( A^j_\lambda(s) \) can be isolated by writing

\[ A^j_0 = \frac{m_1}{p\sqrt{s}} (pq)^j \hat{A}^j_0 \quad \text{for } j \geq 1, \]

\[ A^j_\pm = q (pq)^{j-1} \hat{A}^j_\pm \quad \text{for } j \geq 1, \]

\[ A^0_0 = \frac{p\sqrt{s}}{m_1} \hat{A}^0_0 \quad \text{for } j = 0, \]
Identify covariants

Two helicity couplings $\rightarrow$ two independent covariant structures

**Important**: we are not imposing any intermediate isobar

\[
A_\lambda(s, t) = \varepsilon_\mu(\lambda, p_1) \left[ (p_3 - p_4)^\mu - \frac{m_3^2 - m_4^2}{s} (p_3 + p_4)^\mu \right] C(s, t) \\
+ \varepsilon_\mu(\lambda, p_1) (p_3 + p_4)^\mu B(s, t)
\]

\[
C(s, t) = \frac{1}{4\pi \sqrt{2}} \sum_{j>0} (2j + 1)(pq)^{-1} \hat{A}^j_+(s) \hat{d}^j_{10}(z_s)
\]

\[
B(s, t) = \frac{1}{4\pi} \hat{A}^0_0 + \frac{1}{4\pi} \frac{4m_1^2}{\lambda_{12}} \sum_{j>0} (2j + 1)(pq)^j \left[ \hat{A}^j_0(s) \hat{d}^j_{00}(z_s) + \frac{s + m_1^2 - m_2^2}{\sqrt{2}m_1^2} \hat{A}^j_+(s) z_s \hat{d}^j_{10}(z_s) \right]
\]

Everything looks fine **but** the $\lambda_{12}$ in the denominator
The brackets must vanish at $\lambda_{12} = 0 \Rightarrow s = s_\pm = (m_1 \pm m_2)^2$, $\hat{A}^j_+$ and $\hat{A}^j_0$ cannot be independent
General expression and comparison

\[ \hat{A}^j_+ = \langle j - 1, 0; 1, 1|j, 1 \rangle g_j(s) + f_j(s) \]
\[ \hat{A}^j_0 = \langle j - 1, 0; 1, 0|j, 0 \rangle \frac{s + m_1^2 - m_2^2}{2m_1^2} g'_j(s) + f'_j(s) \]

\( g_j(s_{\pm}) = g'_j(s_{\pm}) \), and \( f_j(s), f'_j(s) \sim O(s - s_{\pm}) \)
All these four functions are free of kinematic singularity.

Comparison with tensor formalisms \((j = 1)\)

\[ g_1 = g'_1 = \frac{4\pi}{3} g_S, \quad f_1 = \frac{2\pi \lambda_{12}}{3s} g_D, \quad f'_1 = -\frac{4\pi \lambda_{12}}{3s} \frac{s + m_1^2 - m_2^2}{m_1^2} g_D. \]

If the \( g_S, g_D \) are the usual Breit-Wigner, the \( g', f' \) are fine

There is no unique recipe to build the right amplitude, but one can ensure the right singularities to be respected
General expression and comparison

We consider the example of two intermediate $K^*(892)$ and $K^*(1410)$
We set $g_S(s) = 0$ and $g_D(s) = \text{sum of Breit-Wigner}$
For the plot on the right we multiply the amplitudes by the Blatt-Weisskopf barrier factors
Interactive tools

- Completed projects are fully documented on interactive portals
- These include description on physics, conventions, formalism, etc.
- The web pages contain source codes with detailed explanation how to use them. Users can run codes online, change parameters, display results.

http://www.indiana.edu/~jpac/