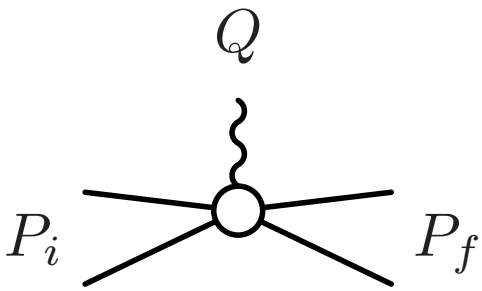
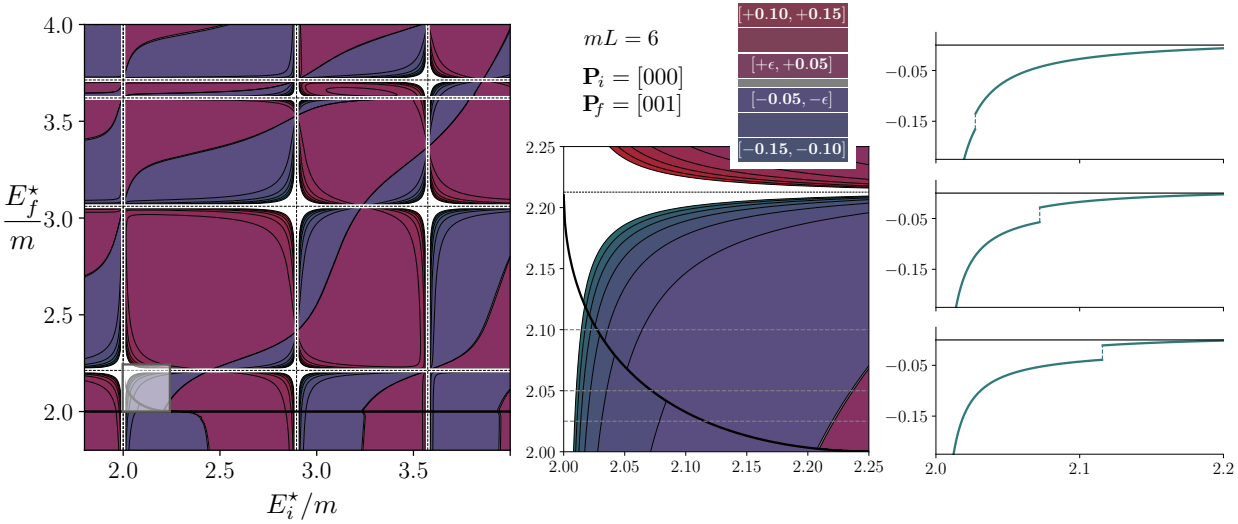
Two-hadron form-factors

2+J $\longrightarrow$ 2 Finite Volume effects





[arXiv:1812.10504]

Felipe Ortega-Gama

Cake Seminar - JLab

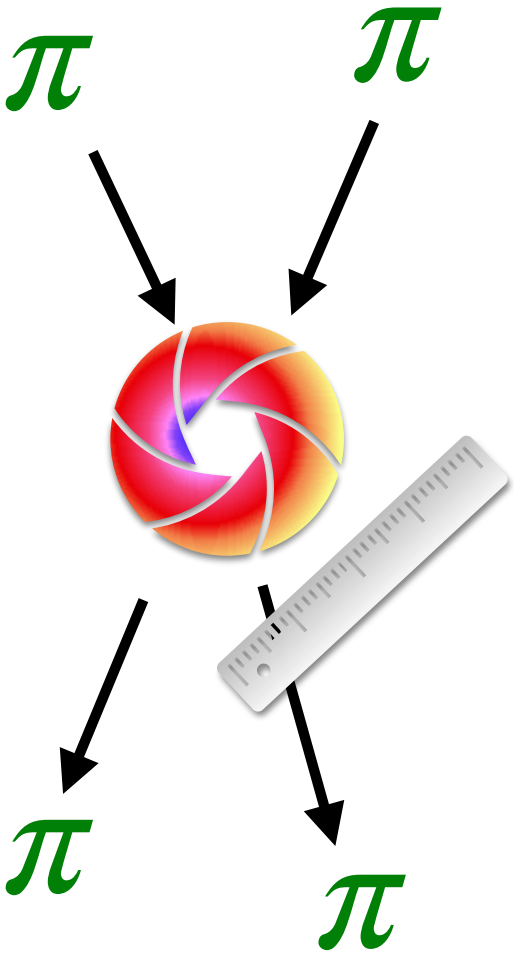
2019-March-27th

felortga@jlab.org

WILLIAM & MARY
CHARTERED 1693

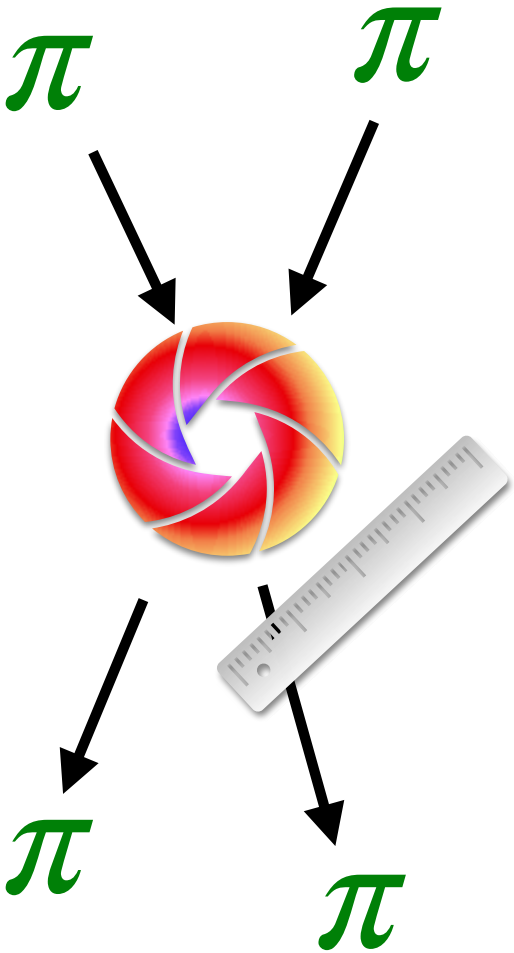
Characterization of resonances

- Mass and width: well constrained experimental results of most of them.
- Less constrained and challenging: elastic/transition form factors, glue content, etc.
- To completely understand the nature of these states we would like to apply full QCD.



Characterization of resonances

- Mass and width: well constrained experimental results of most of them.
- Less constrained and challenging: elastic/transition form factors, glue content, etc.
- To completely understand the nature of these states we would like to apply full QCD.

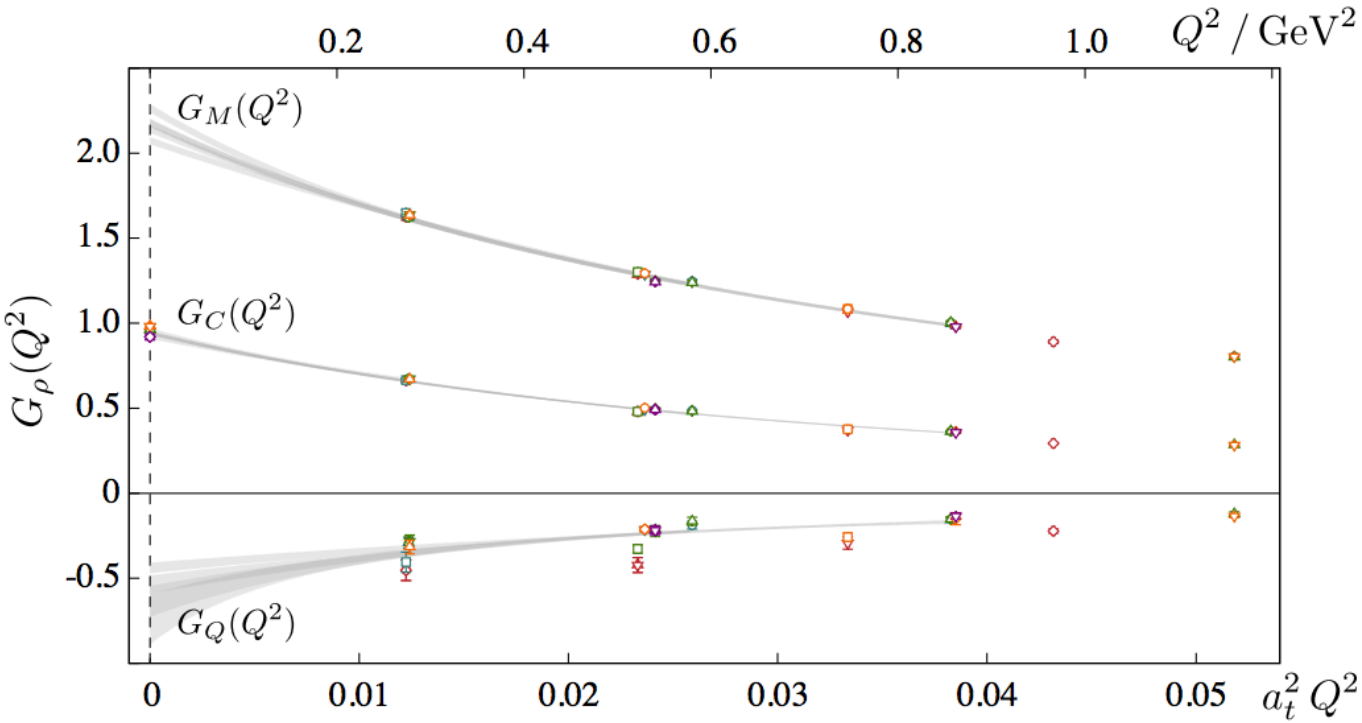


What if we only had it in a box?

Vector-meson electric form factors in a box

[Phys. Rev. D **91**, 114501] by the HadSpec Collaboration.

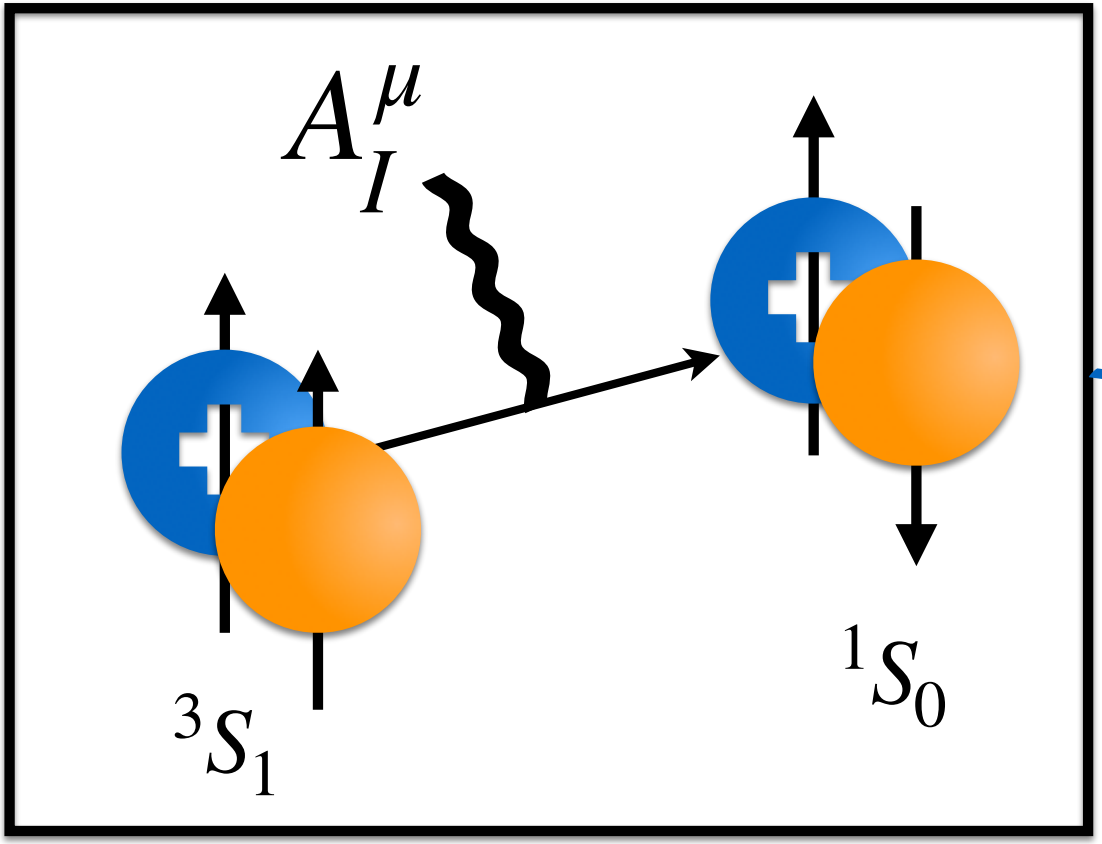
$$\langle \rho(p') | \mathcal{J}^\mu(0) | \rho(p) \rangle = \sum_i C_i^\mu G_i(Q^2)$$



$SU(3)$ point, $m_\pi \sim 700$ MeV, $m_\rho \sim 1$ GeV (stable)

Nuclear processes in a box

[Phys. Rev. Lett. **119**, 062002] by the NPLQCD Collaboration.



pionless EFT

Unphysical mass: deeply bound states

$B \sim 30$ MeV

$$pp \to d + e^+ \bar{\nu}_e$$

Collaborators

[arXiv:1812.10504]

Raúl Briceño

ODU/JLab

Alessandro Baroni

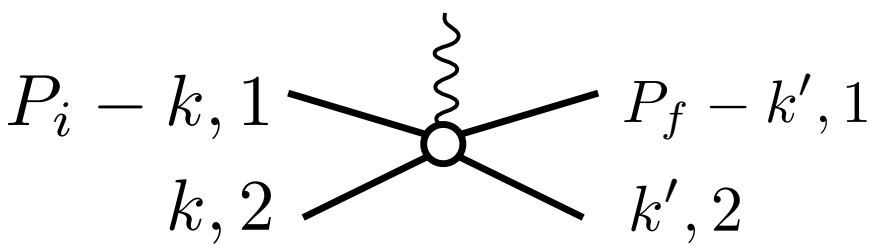
University of South Carolina

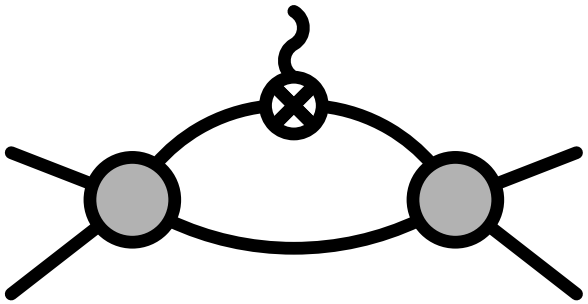
Maxwell Hansen

CERN

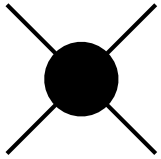
Outline

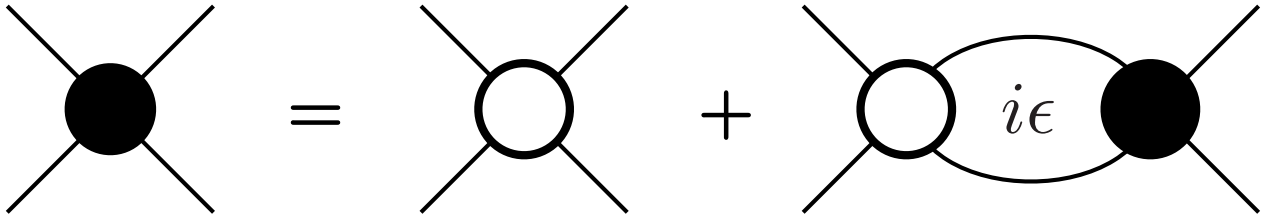
1. Watson's theorem (Infinite Volume).
2. Finite Volume effects.
3. LQCD steps towards two-hadron form factors.
4. Calculation of new geometrical function (G-function).
   - ✓ $P_i = P_f$ case.
   - ✓ $P_i \neq P_f$ case: Triangle singularity.
5. Outlook.



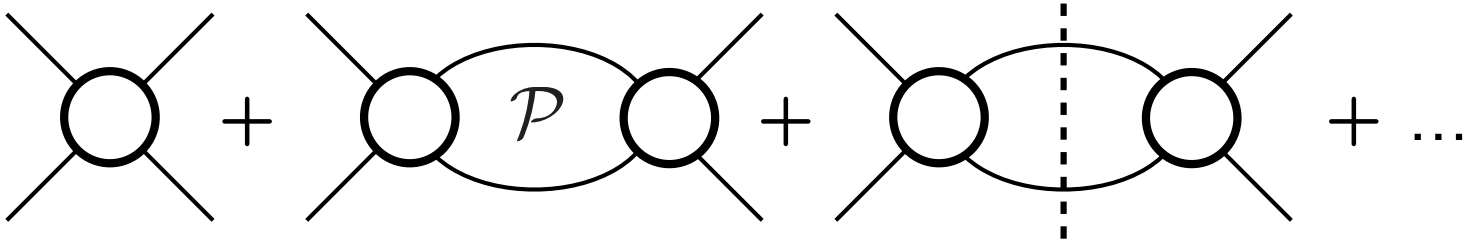


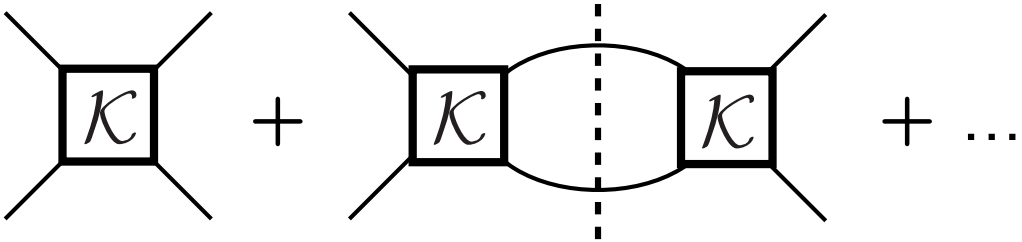
Infinite volume scattering amplitude

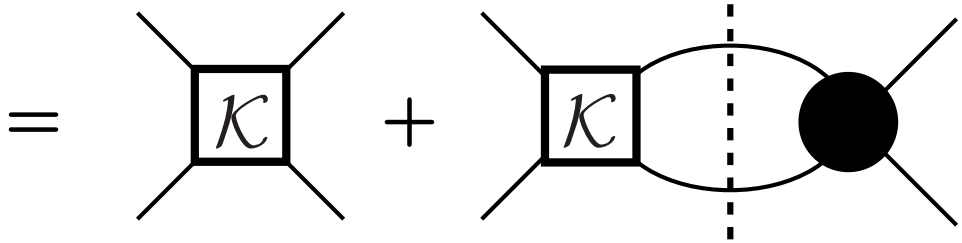
$\mathcal{M} =$ 



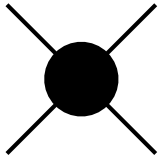
Below next multiparticle threshold

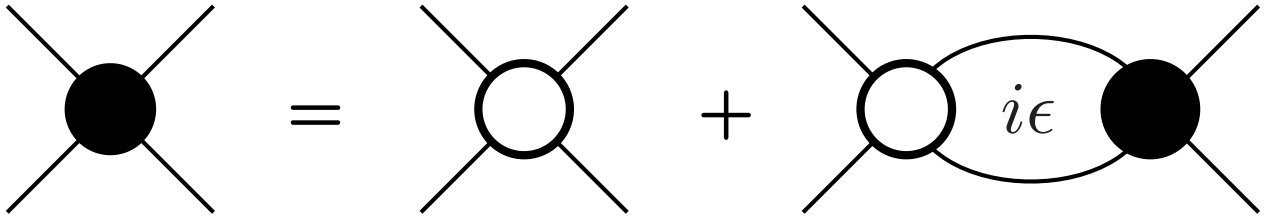




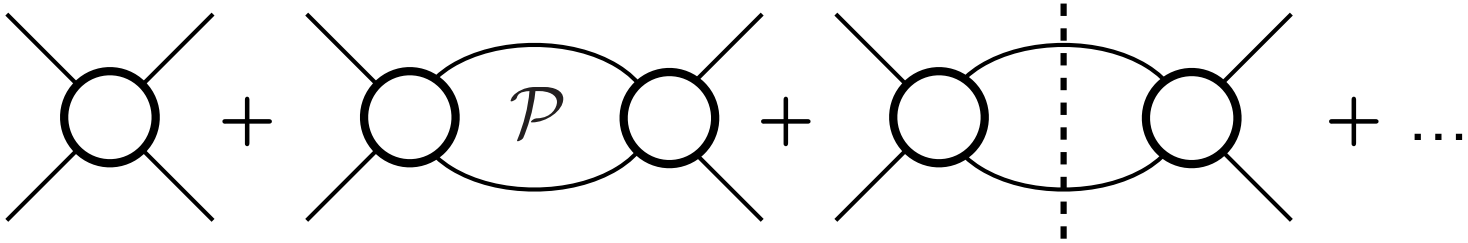


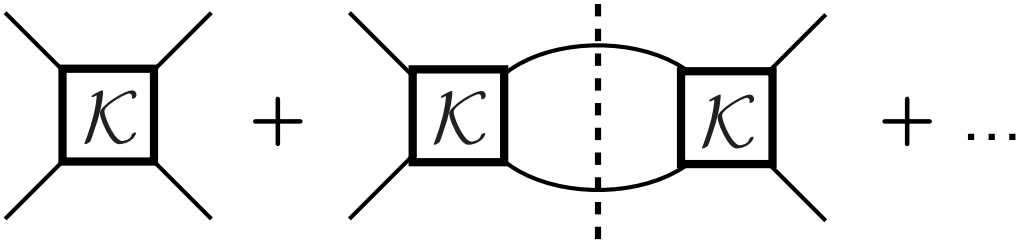
Infinite volume scattering amplitude

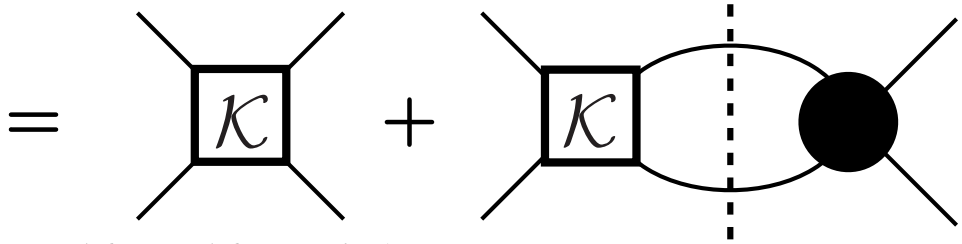
$\mathcal{M} =$ 



Below next multiparticle threshold



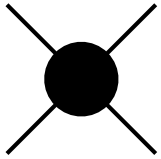


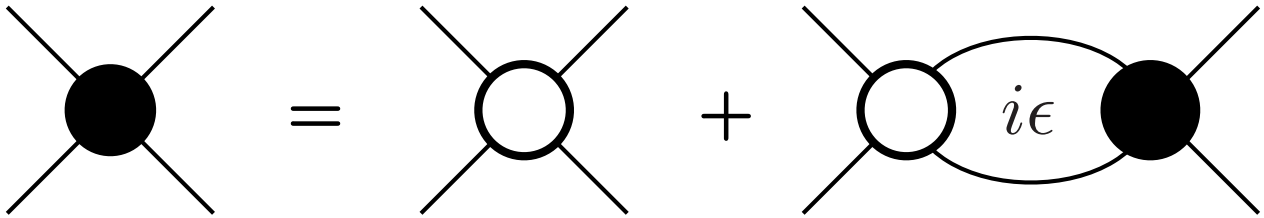


$$\mathcal{M} = \mathcal{K} + \mathcal{K}\, i\rho\, \mathcal{M}$$

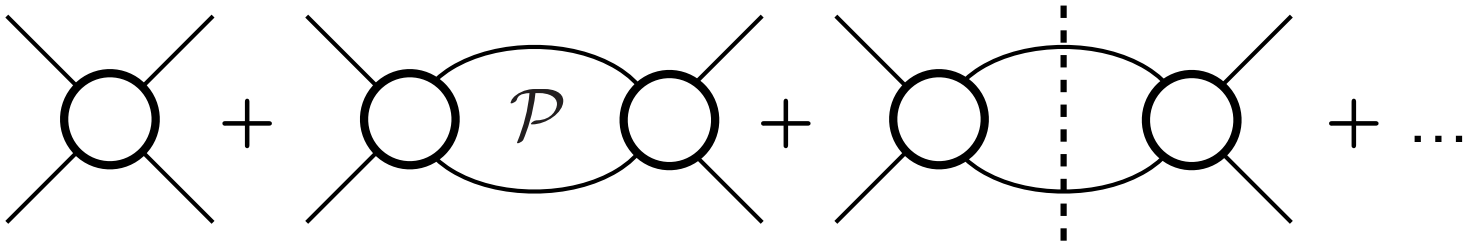
$$= \frac{1}{\mathcal{K}^{-1} - i\rho}$$

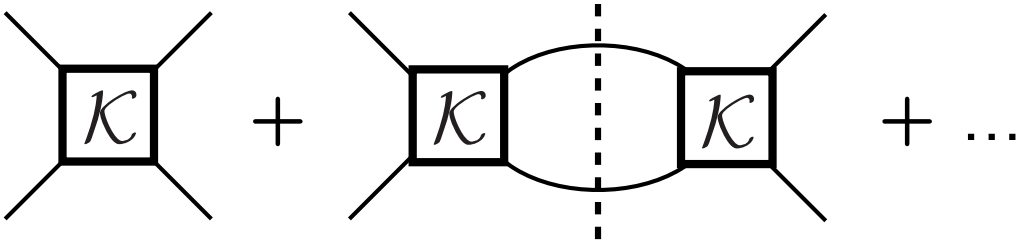
Infinite volume scattering amplitude

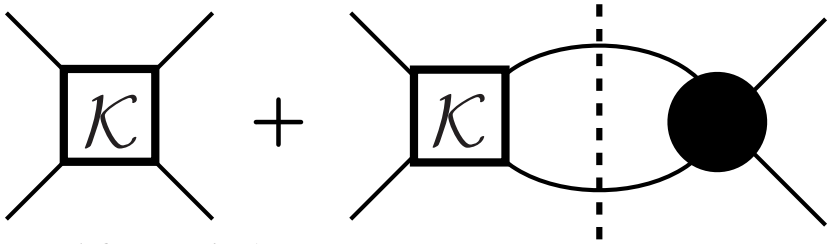
$\mathcal{M} =$ 



Below next multiparticle threshold





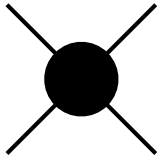


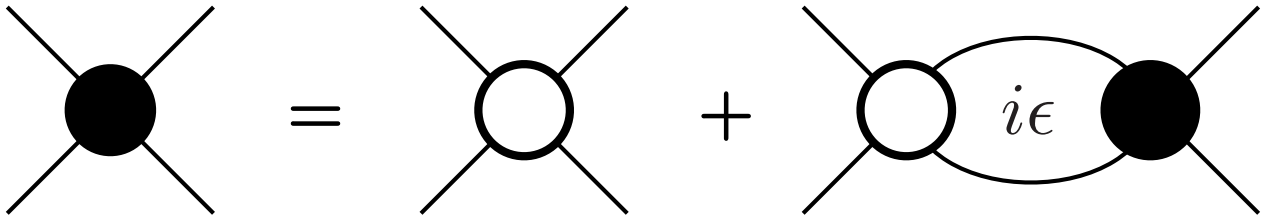
$$\mathcal{M} = \mathcal{K} + \mathcal{K}\, i\rho\, \mathcal{M}$$

$$= \frac{1}{\mathcal{K}^{-1} - i\rho}$$

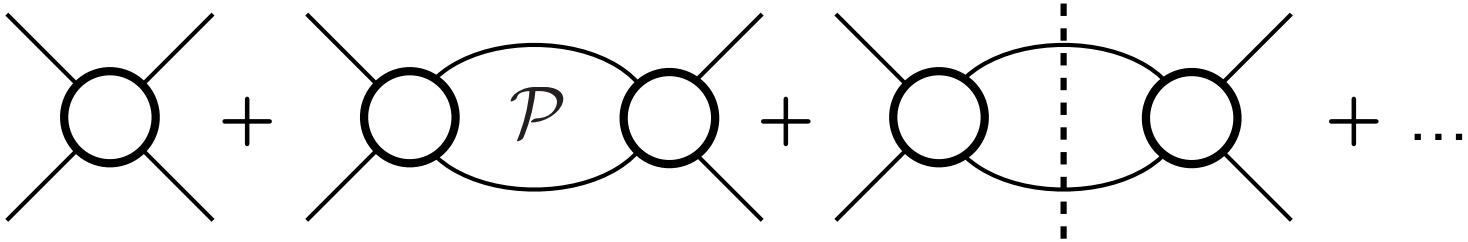
$$\mathcal{K}^{-1} = \rho \cot \delta$$

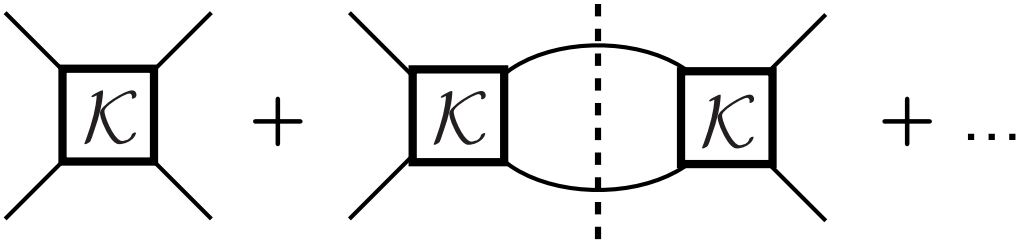
Infinite volume scattering amplitude

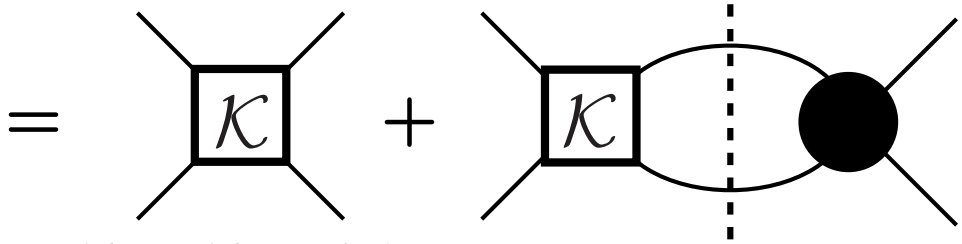
$\mathcal{M} =$ 



Below next multiparticle threshold







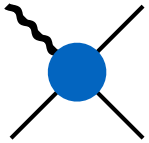
$$\mathcal{M} = \mathcal{K} + \mathcal{K}\, i\rho\, \mathcal{M}$$

$$= \frac{1}{\mathcal{K}^{-1} - i\rho}$$

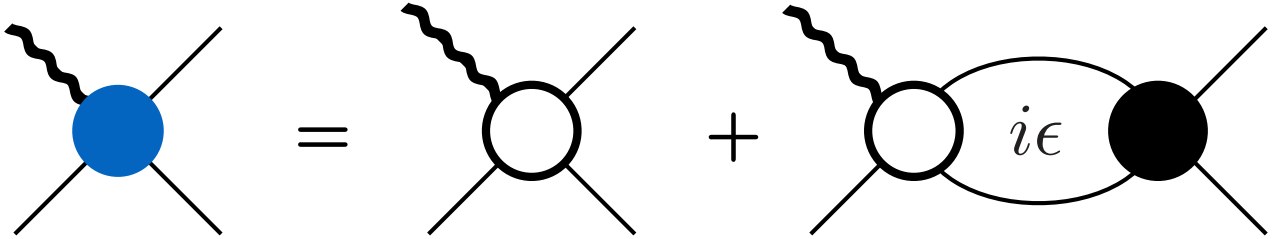
$$\mathcal{K}^{-1} = \rho \cot \delta$$

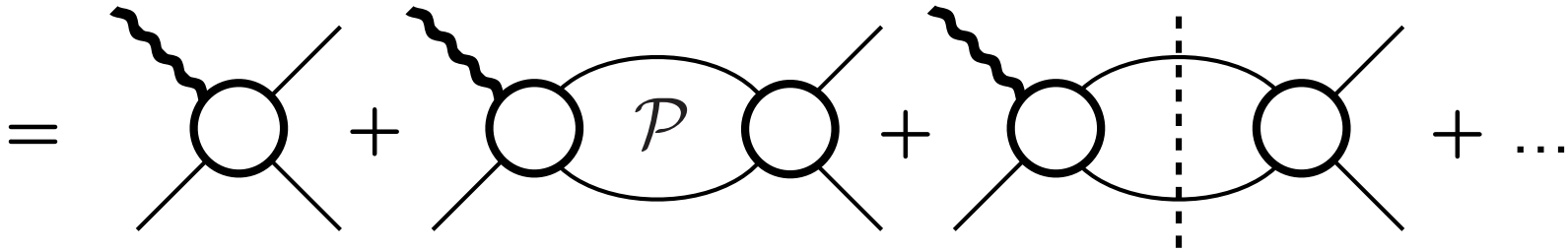
$$\mathcal{M} = |\mathcal{M}| e^{i\delta}$$

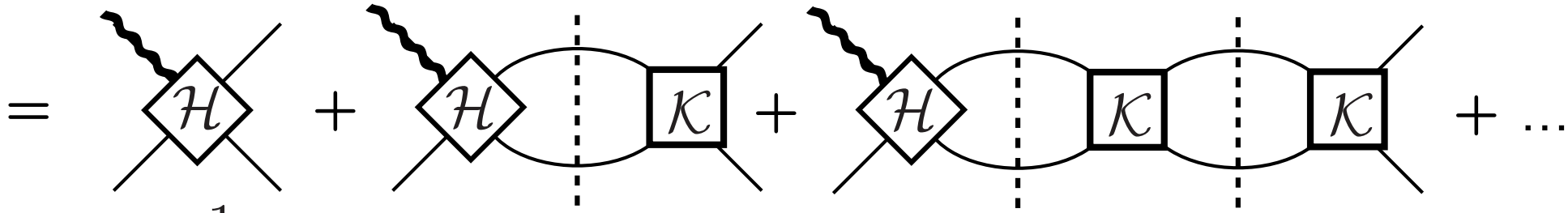
Watson's theorem 1+J $\longrightarrow$ 2 : $\mathcal{A}$ =



[Phys. Rev. D **91**, 034501] by Briceño, Hansen, Walker-Loud.



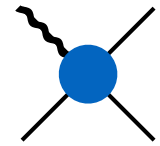




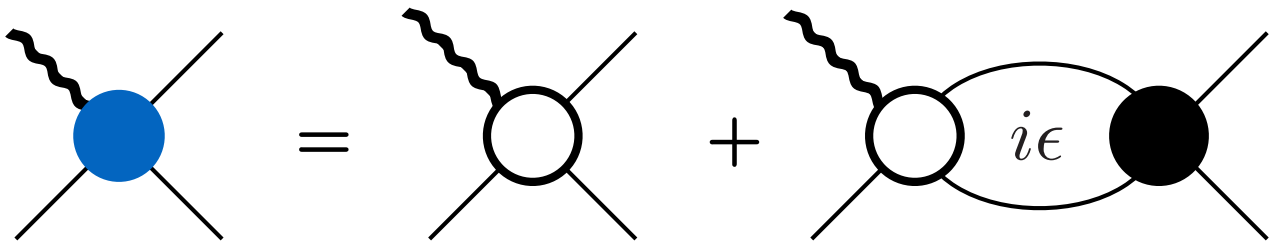
$$\mathcal{A} = \mathcal{H}\frac{1}{1 - i\rho\mathcal{K}}$$

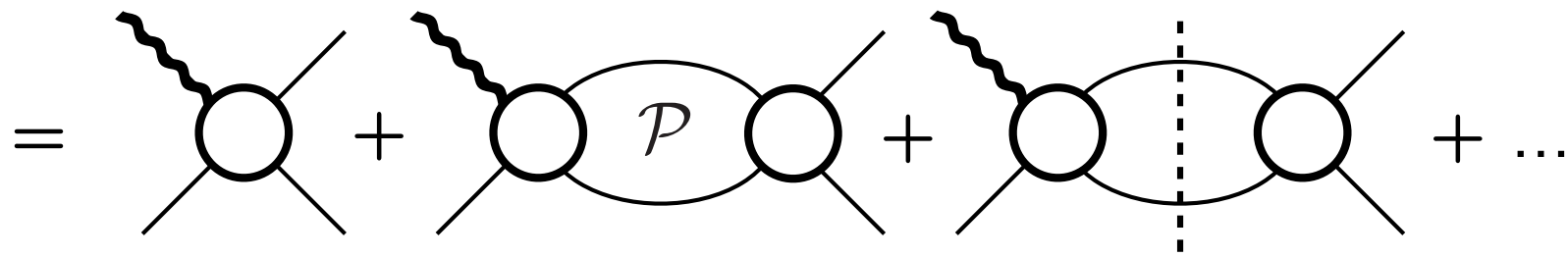
$$= \mathcal{H}\mathcal{K}^{-1}\frac{1}{\mathcal{K}^{-1} - i\rho}$$

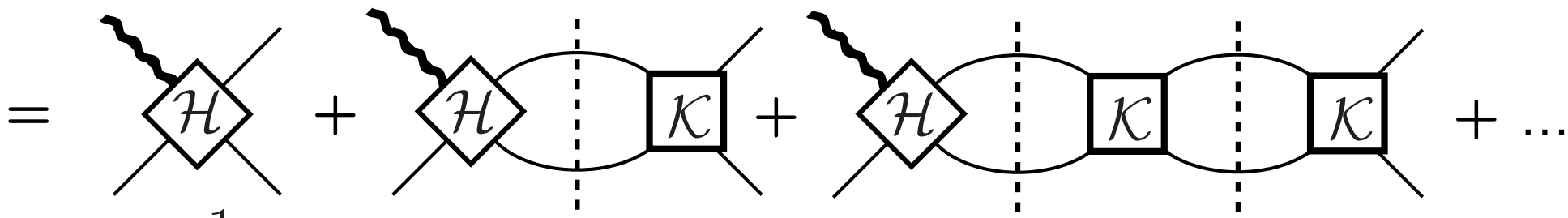
Watson's theorem 1+J $\longrightarrow$ 2 : $\mathcal{A}$ =



[Phys. Rev. D **91**, 034501] by Briceño, Hansen, Walker-Loud.





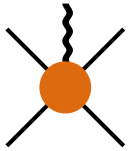


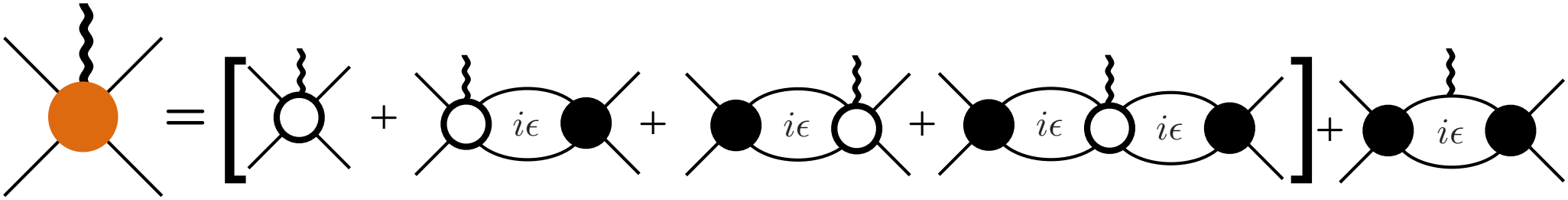
$$\mathcal{A} = \mathcal{H}\frac{1}{1 - i\rho\mathcal{K}}$$

$$= \mathcal{H}\mathcal{K}^{-1}\frac{1}{\mathcal{K}^{-1} - i\rho}$$

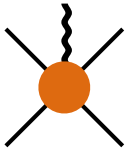
$$\mathcal{K}^{-1} = \rho\cot\delta$$

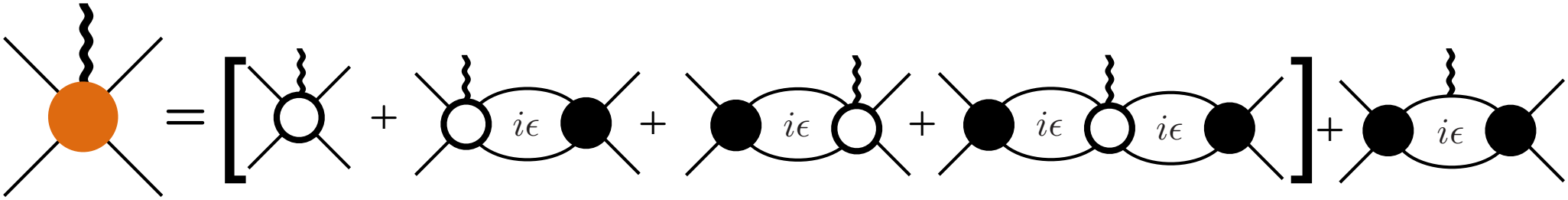
$$\mathcal{A} = |\mathcal{A}|e^{i\delta}$$

Infinite Volume Transition 2+J $\longrightarrow$ 2: $\mathcal{W} =$ 



$$= \left[\frac{1}{1-\mathcal{K}\, i\rho_f} \mathcal{J} \frac{1}{1-i\rho_i\, \mathcal{K}} \right] + \frac{1}{1-\mathcal{K}\, i\rho_f} \mathcal{K}\mathcal{G}\mathcal{K} \frac{1}{1-i\rho_i\, \mathcal{K}}$$

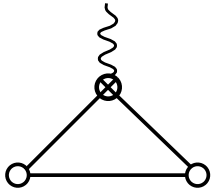
Infinite Volume Transition 2+J $\longrightarrow$ 2: $\mathcal{W} =$ 



$$= \left[\frac{1}{1-\mathcal{K}\, i\rho_f} \mathcal{J} \frac{1}{1-i\rho_i\, \mathcal{K}} \right] + \frac{1}{1-\mathcal{K}\, i\rho_f} \mathcal{K}\mathcal{G}\mathcal{K} \frac{1}{1-i\rho_i\, \mathcal{K}}$$

Dynamical dependence

(real function)

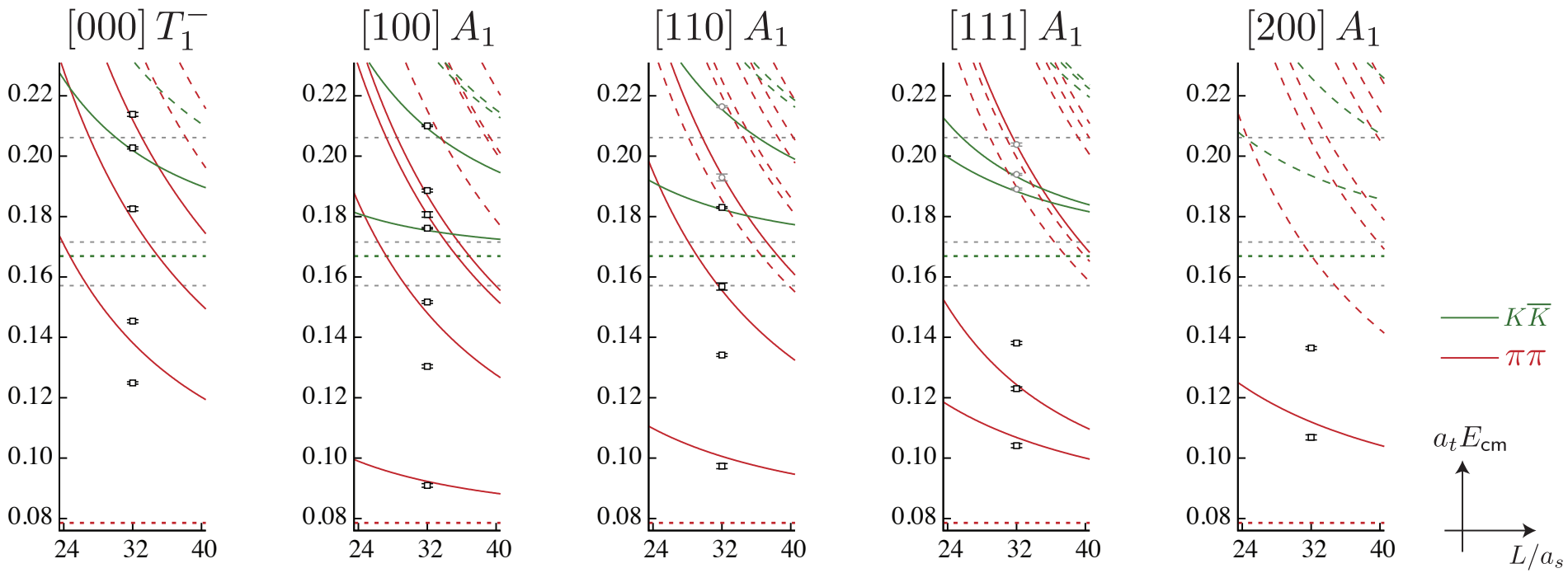


- Watson's theorem is recovered when the current only couples to multiparticle states.

Is Finite Volume an advantage or a disadvantage?

[Phys. Rev. D **92**, 094502] by the HadSpec Collaboration.

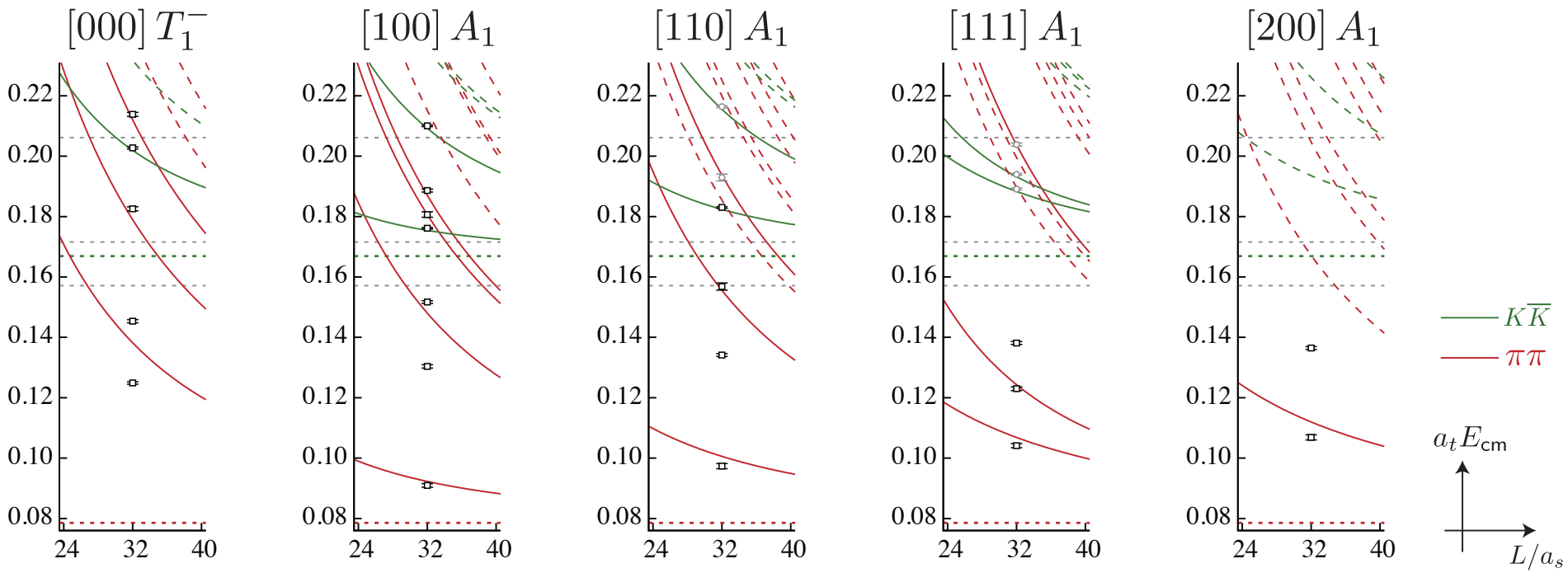
isospin-1 spectrum:



Is Finite Volume an advantage or a disadvantage?

[Phys. Rev. D **92**, 094502] by the HadSpec Collaboration.

isospin-1 spectrum:

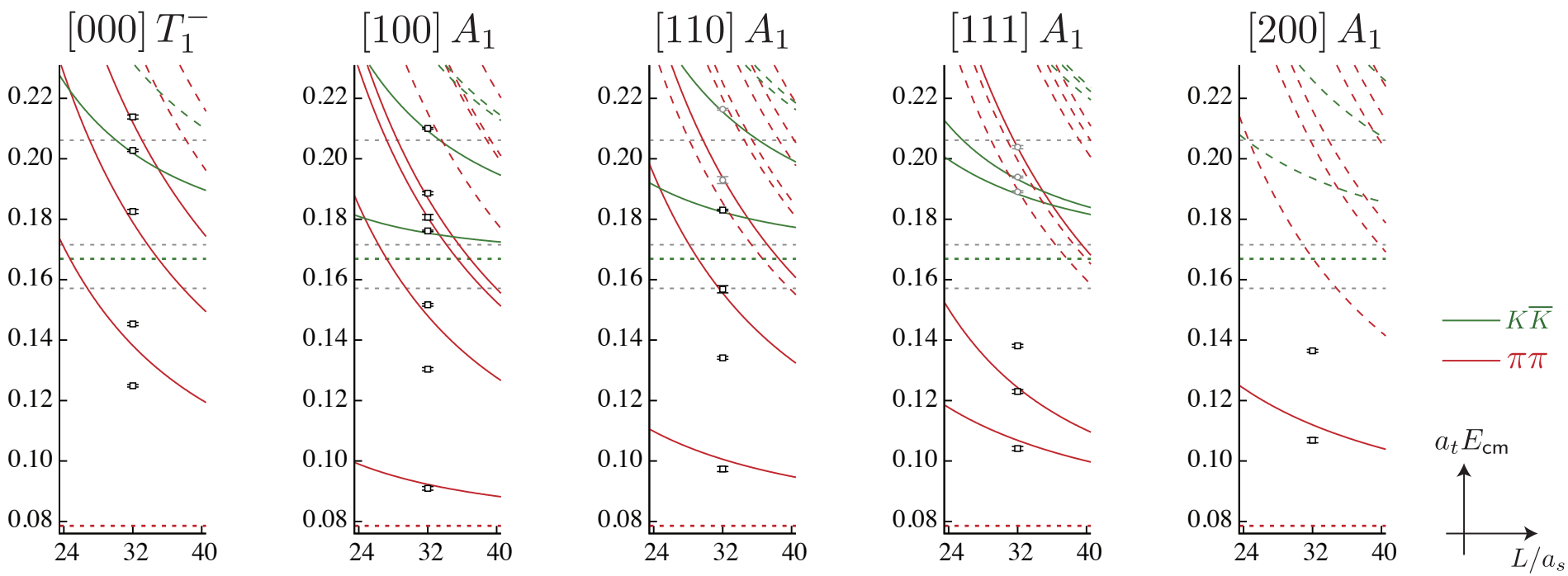


Short range interactions manifest as deviations from the free two-body spectrum in a box.

Is Finite Volume an advantage or a disadvantage?

[Phys. Rev. D **92**, 094502] by the HadSpec Collaboration.

isospin-1 spectrum:



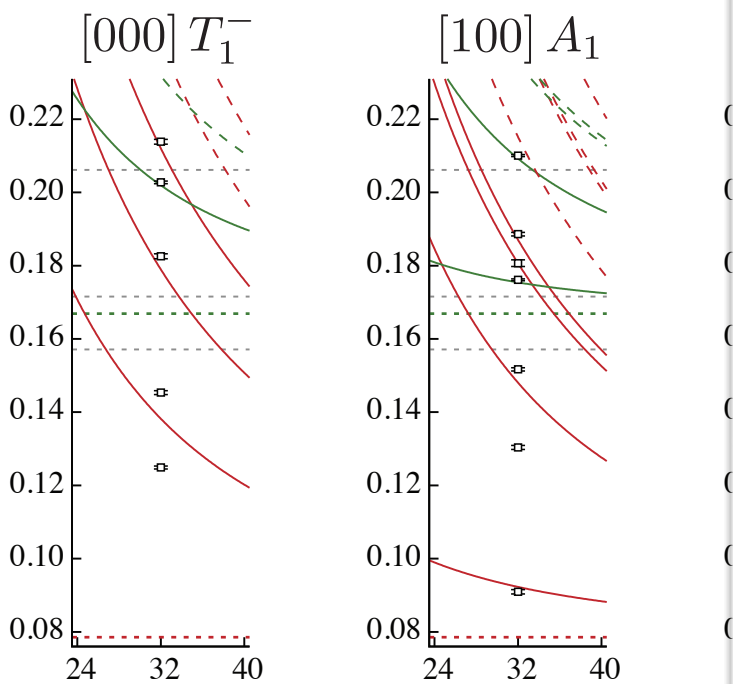
Short range interactions manifest as deviations from the free two-body spectrum in a box.

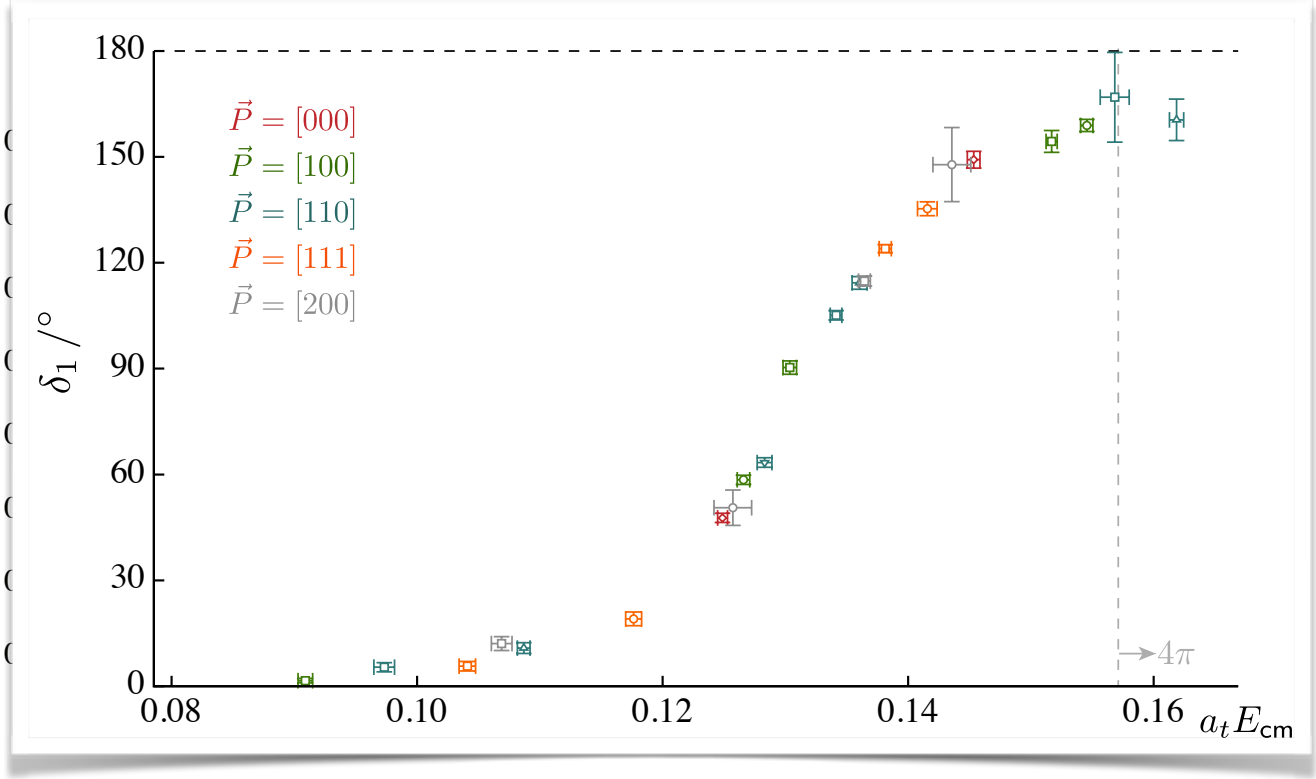
$$\det\left(F^{-1}(s,L) + \mathcal{M}(s)\right) = 0$$

Is Finite Volume an advantage or a disadvantage?

[Phys. Rev. D **92**, 094502] by the HadSpec Collaboration.

isospin-1 spectrum:





Short range interactions manifest as deviations from the free two-body spectrum in a box.

$$\det\left(F^{-1}(s,L) + \mathcal{M}(s)\right) = 0$$

Poisson summation formula

$$\sum_{x=-\infty}^{\infty} f(x) = \sum_{n=-\infty}^{\infty} \tilde{f}(n) \qquad \text{(Fourier transform)}$$

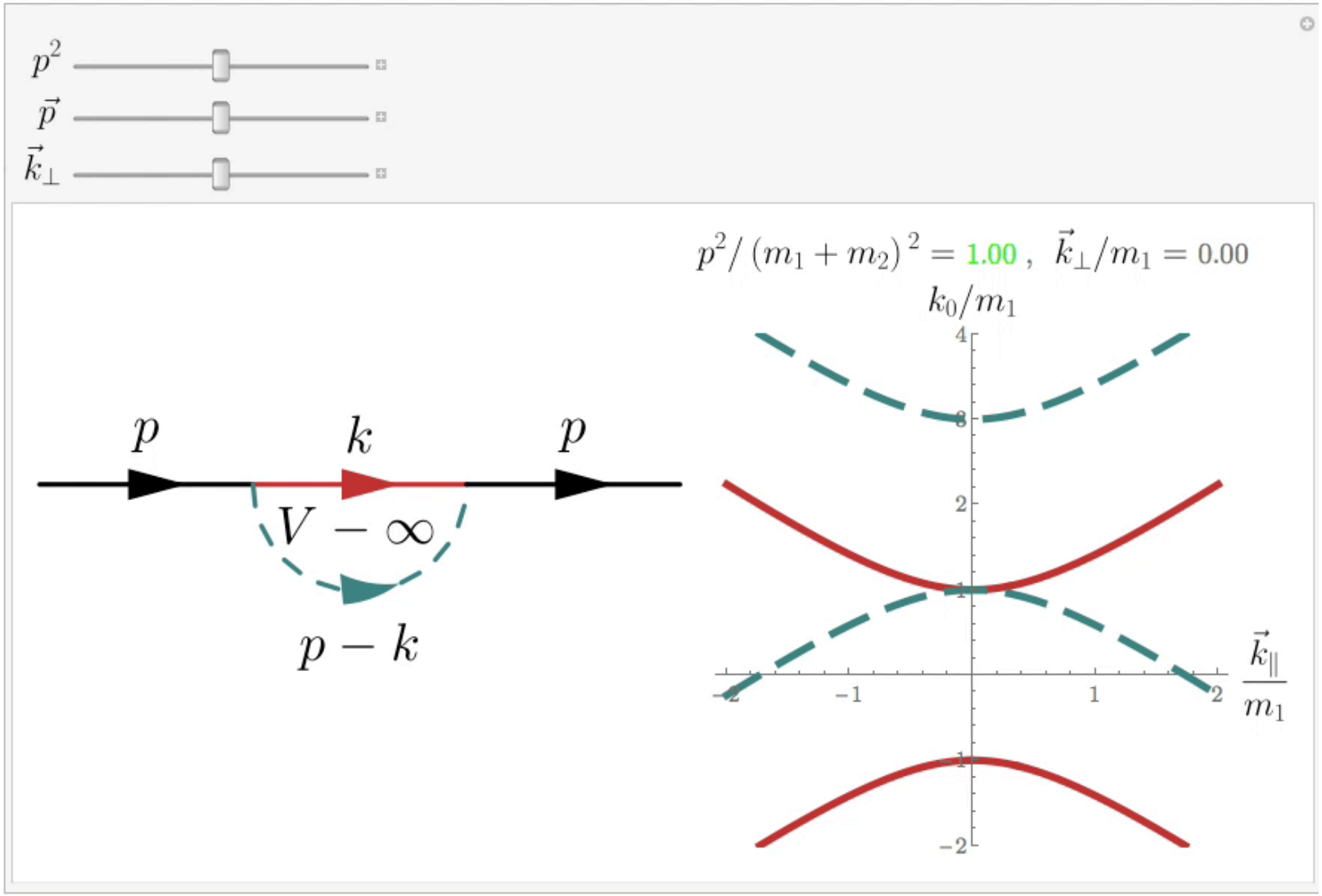
$$= \int_{-\infty}^{\infty} dx f(x) + \sum_{n\neq 0} \tilde{f}(n)$$

$$\left(\sum_x - \int dx\right) f(x) = \sum_{n\neq 0} \tilde{f}(n) \sim e^{-m/\Delta x} \qquad (\Delta x \to 0)$$

characteristic scale in f

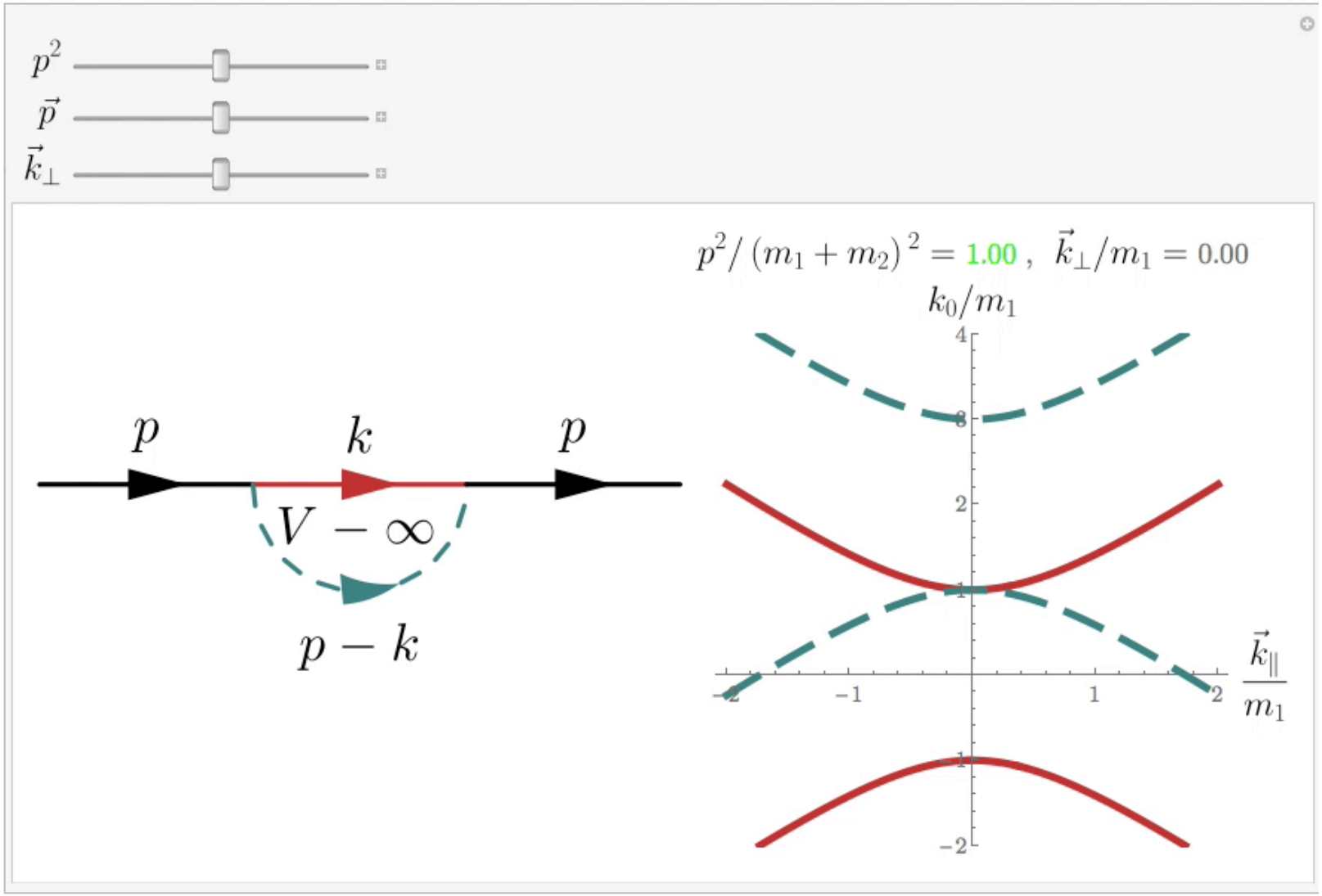
As long as f is a smooth function.

Smooth intregands: exponentially suppressed



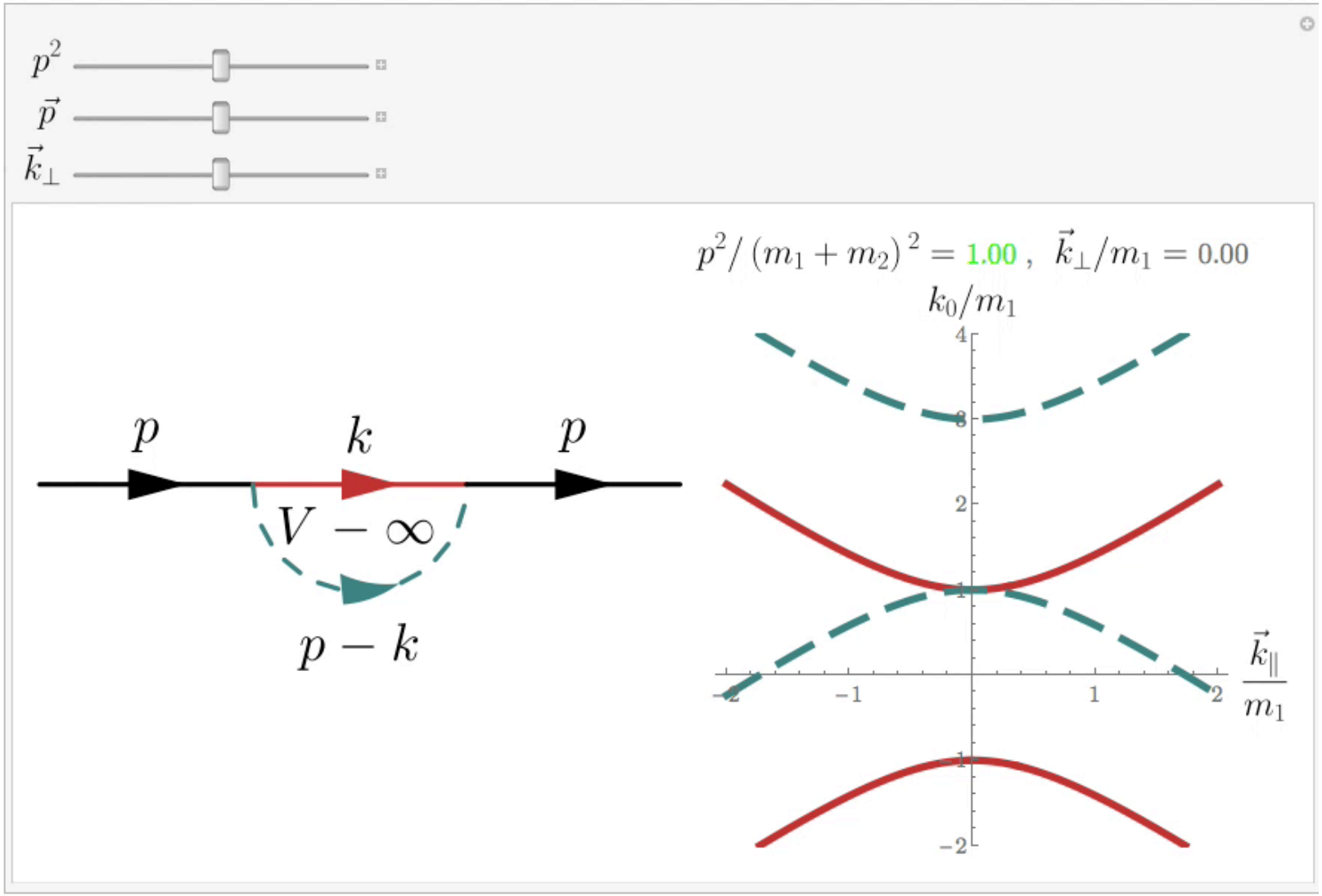
$$\left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \int \frac{dk_0}{2\pi} \frac{i}{k^2 - m_1^2 + i\epsilon} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon}$$

Smooth intregands: exponentially suppressed



$$\left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \int \frac{dk_0}{2\pi} \frac{i}{k^2 - m_1^2 + i\epsilon} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon} = \left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \frac{1}{2\omega_{k2}} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon}\Bigg|_{k_0=\omega_{k2}}$$

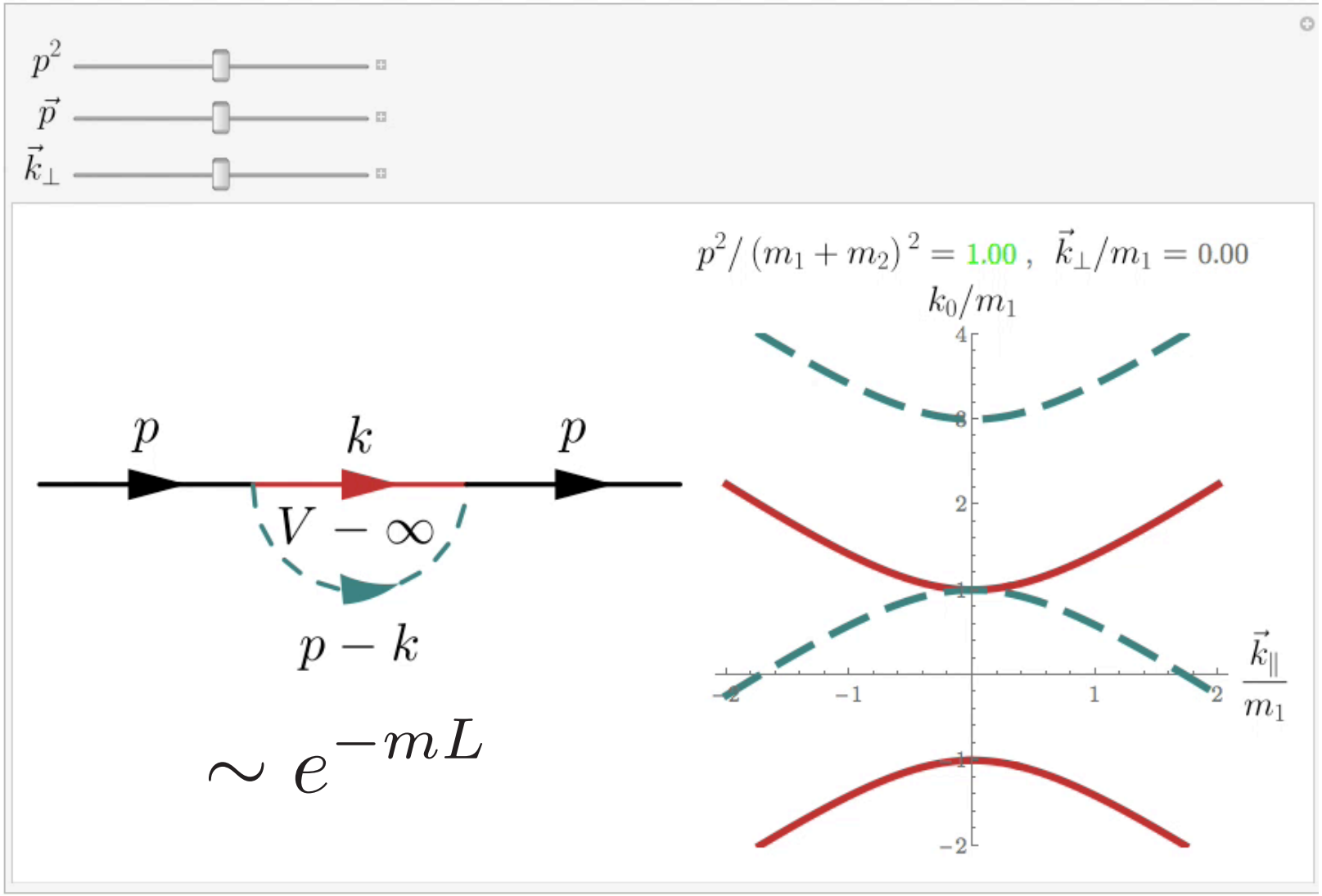
Smooth intregands: exponentially suppressed



$$\left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \int \frac{dk_0}{2\pi} \frac{i}{k^2 - m_1^2 + i\epsilon} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon} = \left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \frac{1}{2\omega_{k2}} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon}\Bigg|_{k_0=\omega_{k2}}$$

Smooth intregands: exponentially suppressed

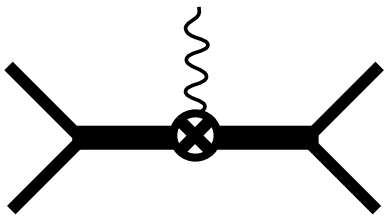
Stable particle



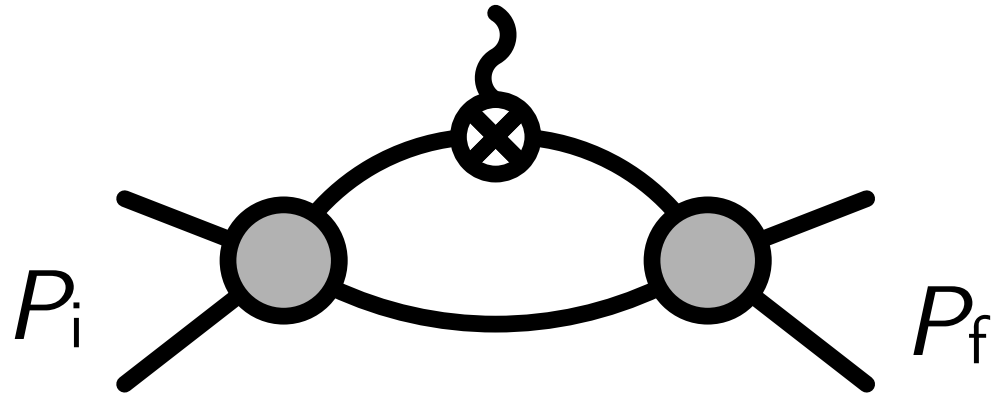
$$\left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \int \frac{dk_0}{2\pi} \frac{i}{k^2 - m_1^2 + i\epsilon} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon} = \left(\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right) \frac{1}{2\omega_{k2}} \frac{i}{(p-k)^2 - m_2^2 + i\epsilon}\Bigg|_{k_0=\omega_{k2}}$$

Singular functions give rise to power law corrections

two-hadron form factors



$\ni$

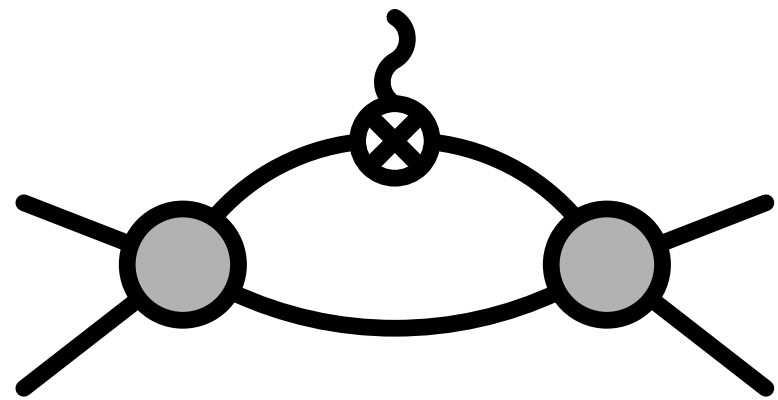


- Calculate resonance Form Factors in a generic QFT (EFT).
- Diagrams with multiple particles on-shell in intermediate states give rise to singularities, FV corrections are no longer suppressed.
- Calculation of FV corrections rigorously allows to constrain the dynamics of resonances using LQCD.

Singular functions in 2 + J → 2 : $\mathcal{W}_{\mu_1 \cdots \mu_n}$

a) Loops where two or more particles can go on-shell.

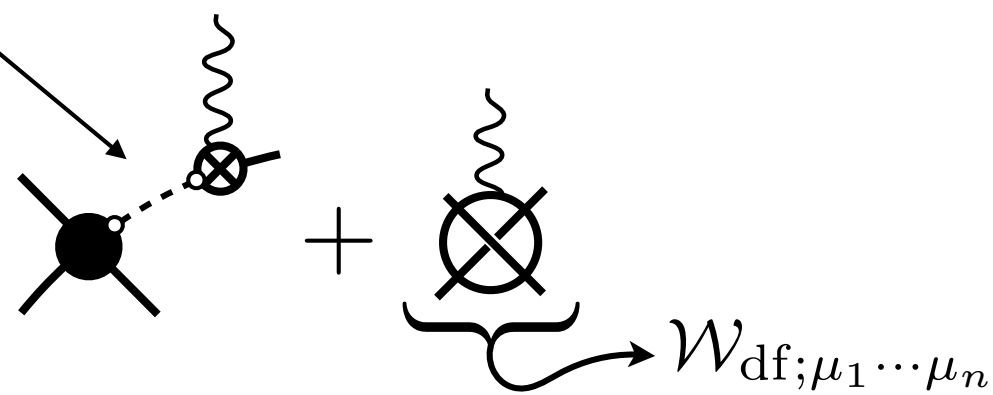
b) Current coupled to long-lived states.



Work strategy:

a) ME without the known long-distance singularities.

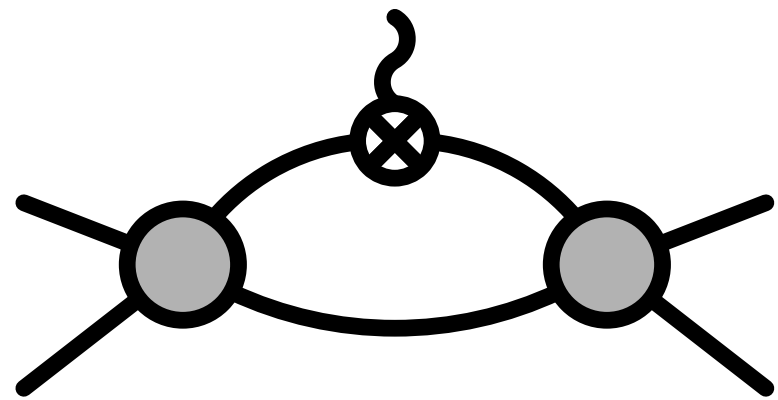
b) Focus on a kinematic window below 3-particle threshold.



Singular functions in 2 + J → 2 : $\mathcal{W}_{\mu_1 \cdots \mu_n}$

a) Loops where two or more particles can go on-shell.

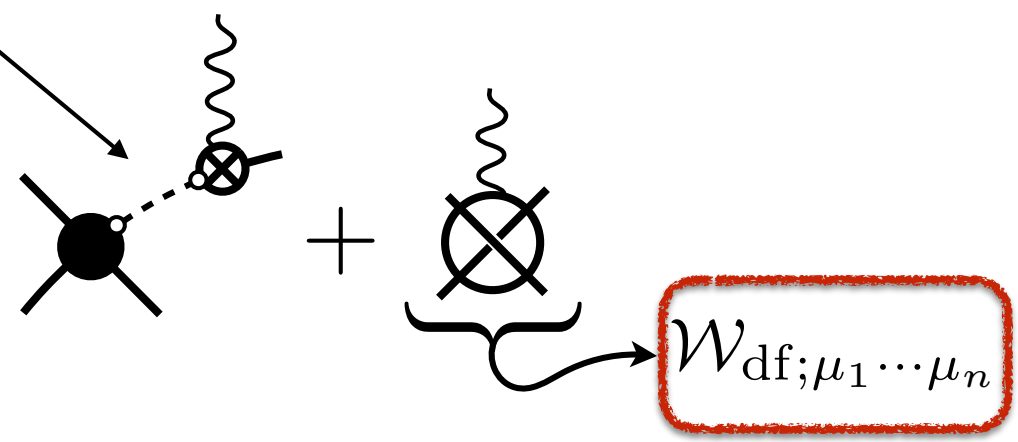
b) Current coupled to long-lived states.



Work strategy:

a) ME without the known long-distance singularities.

b) Focus on a kinematic window below 3-particle threshold.

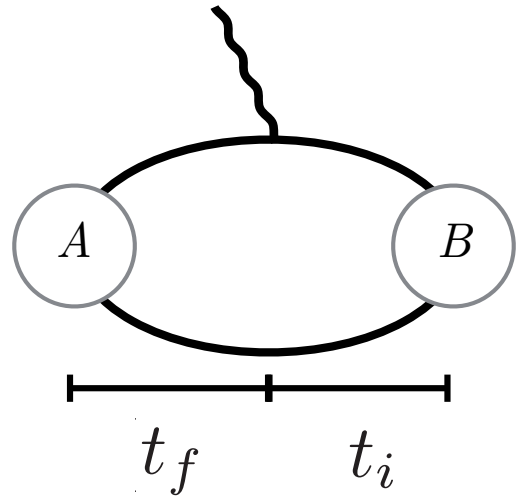


The (euclidean) lattice calculation

[arXiv:1509.08507]

R.Briceño, M. Hansen

$C^{3\mathrm{pt}}(t_f, t_i) =$



$\propto \langle m|\mathcal{J}(0)|n\rangle\, e^{-E_n t_i - E_m t_f}$

m, n are the finite volume energy eigenstates

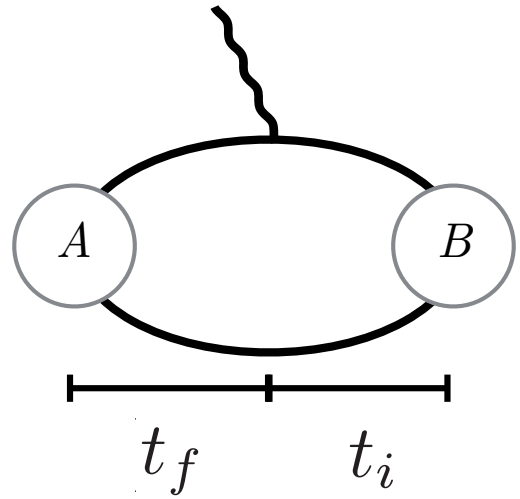
$$|\langle m|\mathcal{J}(0)|n\rangle|^2 = \frac{1}{L^6}\mathcal{R}_n \mathcal{W}_{L,\mathrm{df}} \mathcal{R}_m \mathcal{W}_{L,\mathrm{df}}$$

The (euclidean) lattice calculation

[arXiv:1509.08507]

R.Briceño, M. Hansen

$C^{3\mathrm{pt}}(t_f, t_i) =$



$\propto \langle m|\mathcal{J}(0)|n\rangle\, e^{-E_n t_i - E_m t_f}$

m, n are the finite volume energy eigenstates

$$\boxed{|\langle m|\mathcal{J}(0)|n\rangle|^2 = \frac{1}{L^6}\mathcal{R}_n \mathcal{W}_{L,\mathrm{df}} \mathcal{R}_m \mathcal{W}_{L,\mathrm{df}}}$$

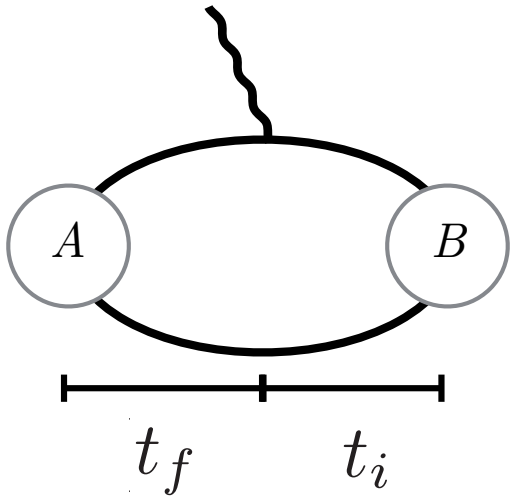
$$\mathcal{R}_n \equiv \lim_{E\to E_n} \frac{E - E_n}{F^{-1} + \mathcal{M}}$$

The (euclidean) lattice calculation

[arXiv:1509.08507]

R.Briceño, M. Hansen

$C^{3\mathrm{pt}}(t_f, t_i) =$



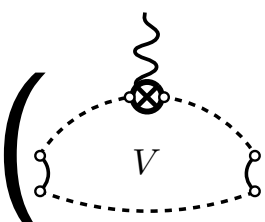
$\propto \langle m|\mathcal{J}(0)|n\rangle\, e^{-E_n t_i - E_m t_f}$

$m,\ n$ are the finite volume energy eigenstates

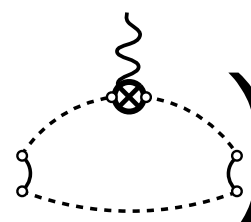
$$\boxed{|\langle m|\mathcal{J}(0)|n\rangle|^2 = \frac{1}{L^6}\mathcal{R}_n \mathcal{W}_{L,\mathrm{df}} \mathcal{R}_m \mathcal{W}_{L,\mathrm{df}}}$$

$$\mathcal{R}_n \equiv \lim_{E\to E_n} \frac{E - E_n}{F^{-1} + \mathcal{M}}$$

$\mathcal{W}_{L,\mathrm{df}} = \mathcal{W}_{\mathrm{df}} + \mathcal{M} \cdot \Big($



$-$

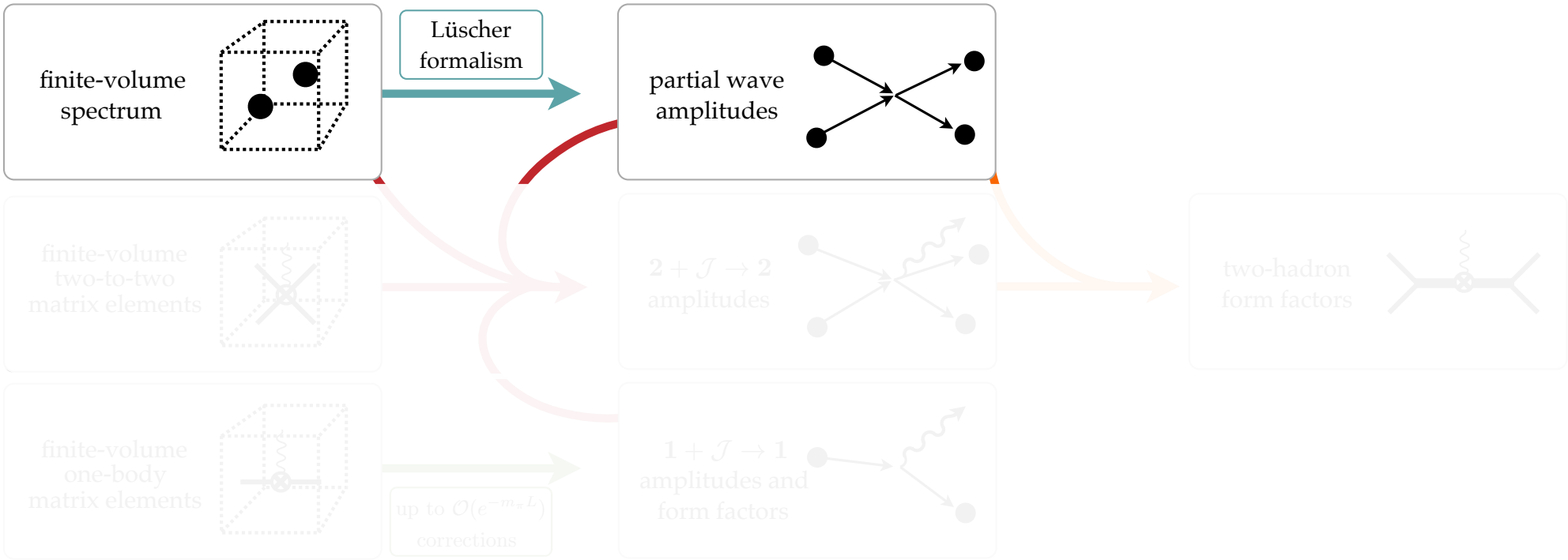


$\Big) \cdot \mathcal{M}$

Infinite volume observables

[arXiv:1509.08507]

R.Briceño, M. Hansen

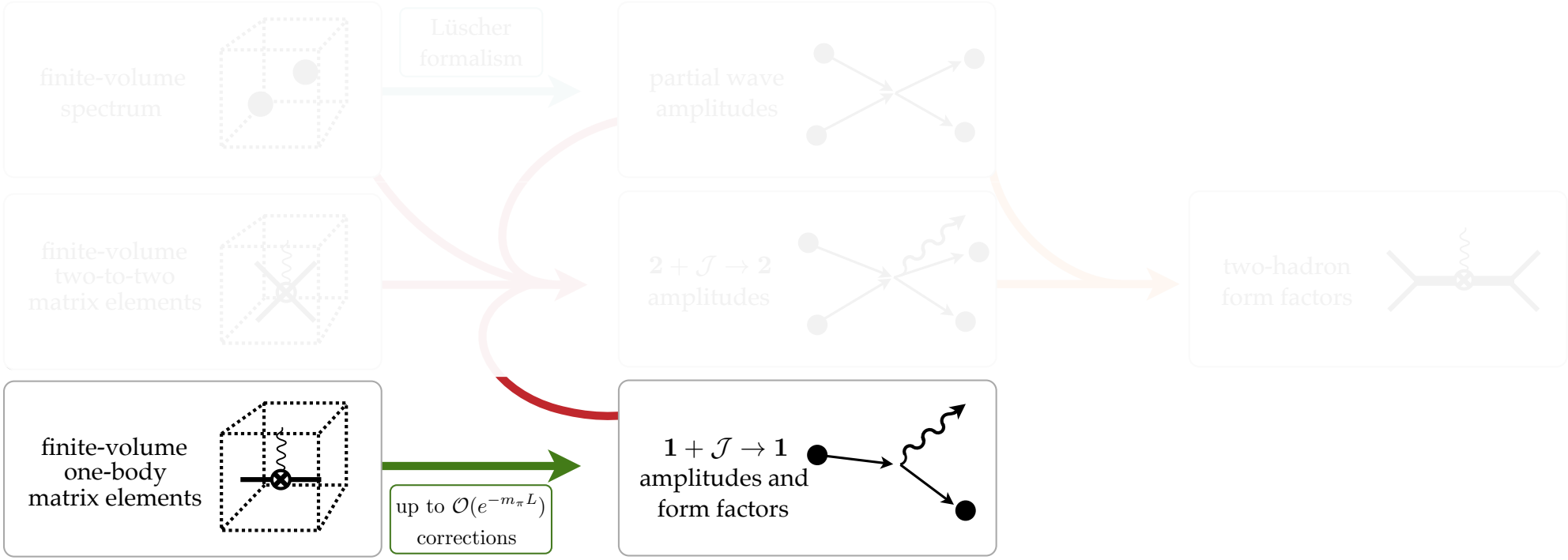


- $\det\left(F^{-1} + \mathcal{M}\right) = 0$

Infinite volume observables

[arXiv:1509.08507]

R.Briceño, M. Hansen

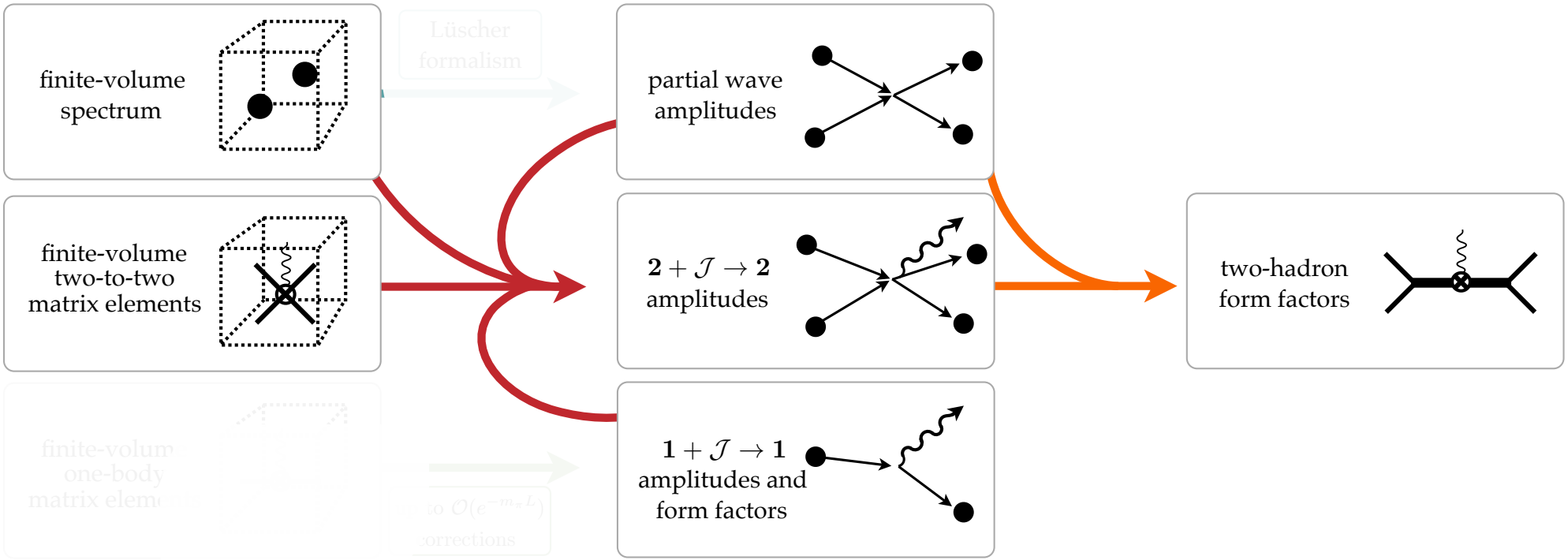


- $\omega(Q^2) = \sum_i \tilde{C}_i f^{(i)}(Q^2)$

Infinite volume observables

[arXiv:1509.08507]

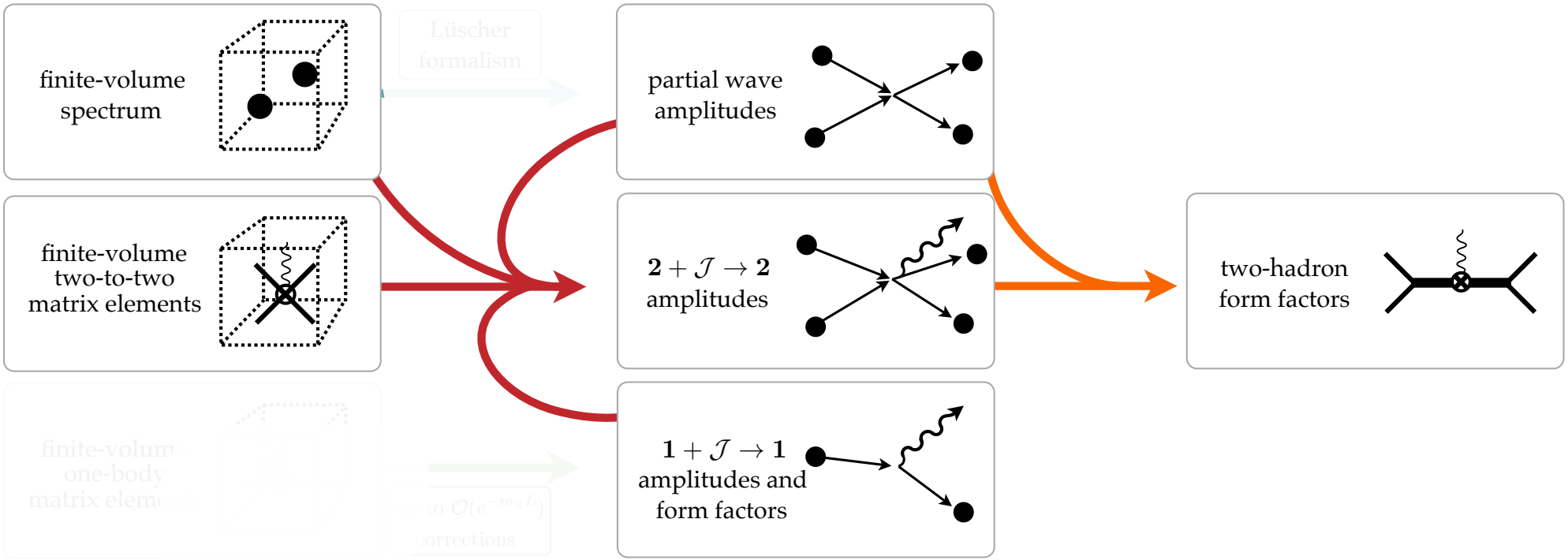
R.Briceño, M. Hansen



Infinite volume observables

[arXiv:1509.08507]

R.Briceño, M. Hansen

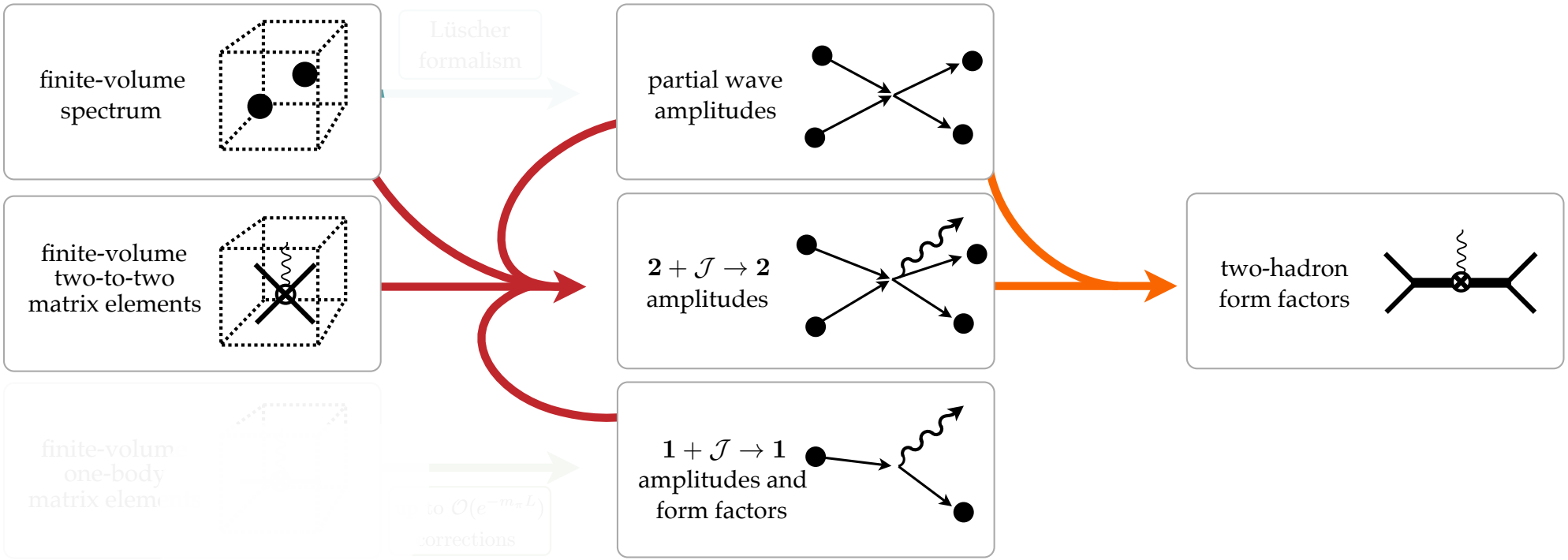


- $$\mathcal{W}_{L,\mathrm{df}} - \mathcal{W}_{\mathrm{df}} = \sum_j C^{(j)} \, f^{(j)} \, \mathcal{M}(s_f) \, G(P_i, P_f, L) \, \mathcal{M}(s_i)$$

Infinite volume observables

[arXiv:1509.08507]

R.Briceño, M. Hansen

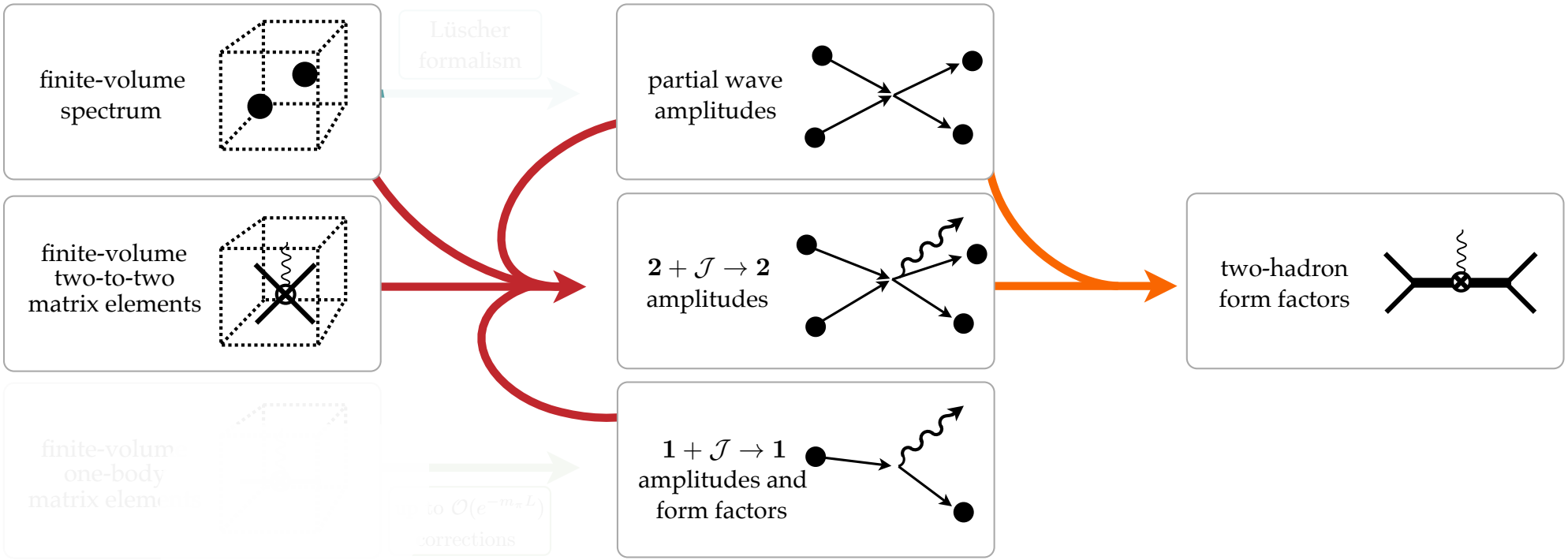


- $$\mathcal{W}_{L,\mathrm{df}} - \mathcal{W}_{\mathrm{df}} = \sum_j \underbrace{C^{(j)}}_{\text{Kinematic factors}} f^{(j)} \, \mathcal{M}(s_f) \, G(P_i, P_f, L) \, \mathcal{M}(s_i)$$

Infinite volume observables

[arXiv:1509.08507]

R.Briceño, M. Hansen

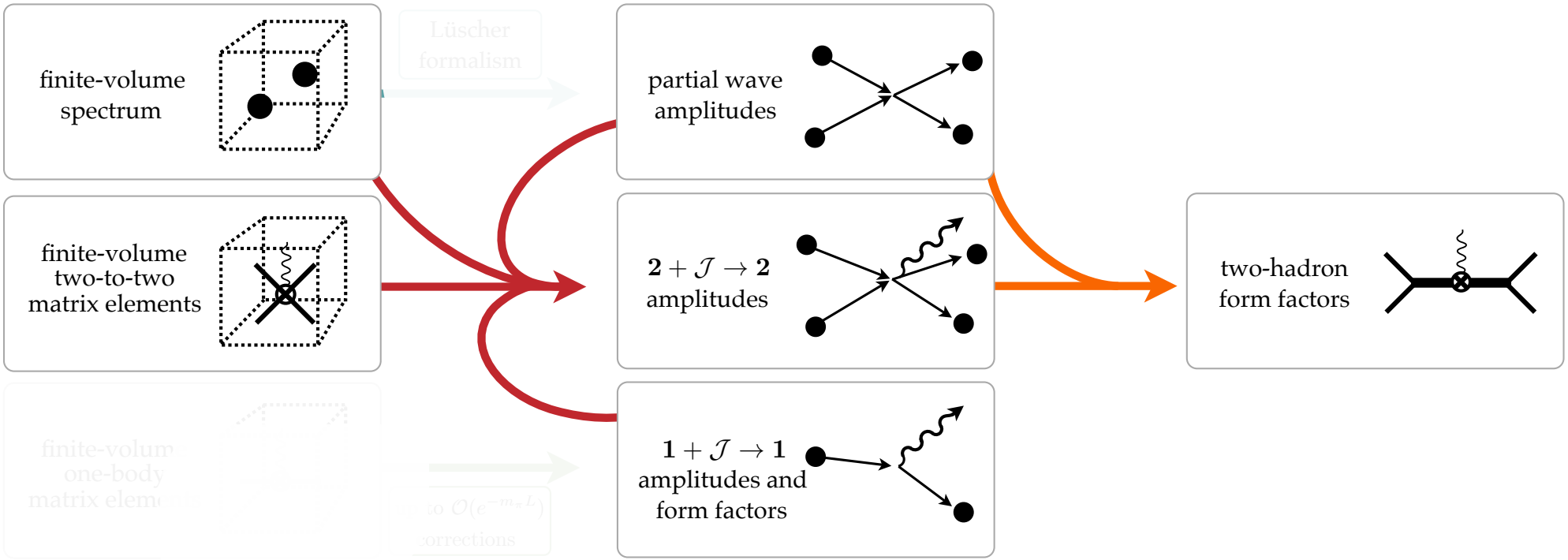


- $$\mathcal{W}_{L,\mathrm{df}} - \mathcal{W}_{\mathrm{df}} = \sum_j \underbrace{C^{(j)}}_{\text{Kinematic factors}} \overbrace{f^{(j)}}^{\text{Single particle form-factors}} \mathcal{M}(s_f)\, G(P_i, P_f, L)\, \mathcal{M}(s_i)$$

Infinite volume observables

[arXiv:1509.08507]

R.Briceño, M. Hansen



Single particle form-factors

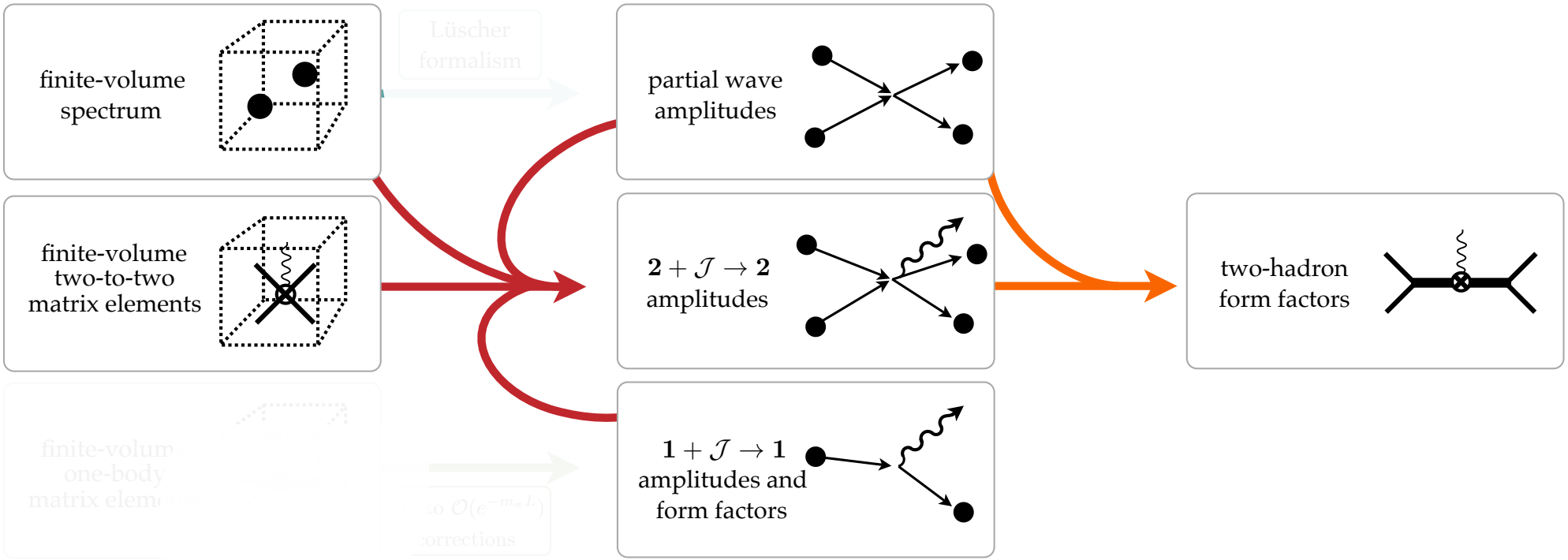
- $$\mathcal{W}_{L,\mathrm{df}} - \mathcal{W}_{\mathrm{df}} = \sum_j \underbrace{C^{(j)}}_{\text{Kinematic factors}} \overbrace{f^{(j)}} \mathcal{M}(s_f)\, G(P_i, P_f, L)\, \mathcal{M}(s_i)$$

Scattering amplitude

Infinite volume observables

[arXiv:1509.08507]

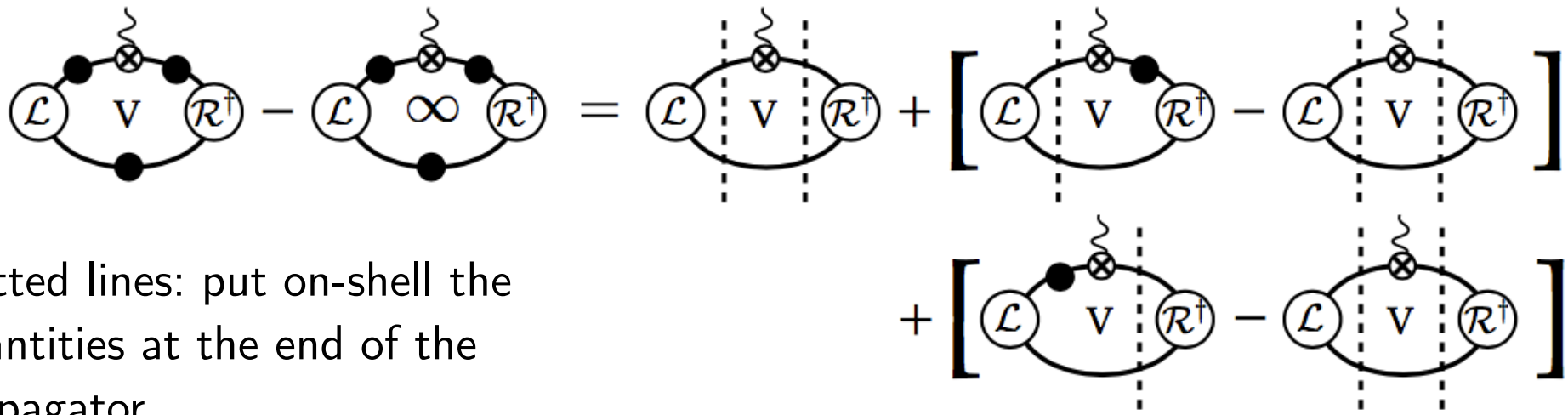
R.Briceño, M. Hansen



- $$\mathcal{W}_{L,\mathrm{df}} - \mathcal{W}_{\mathrm{df}} = \sum_j \underbrace{C^{(j)}}_{\text{Kinematic factors}} \overbrace{f^{(j)}}^{\text{Single particle form-factors}} \mathcal{M}(s_f)\, \overbrace{G(P_i, P_f, L)}^{\text{New Geometric function!}}\, \mathcal{M}(s_i)$$

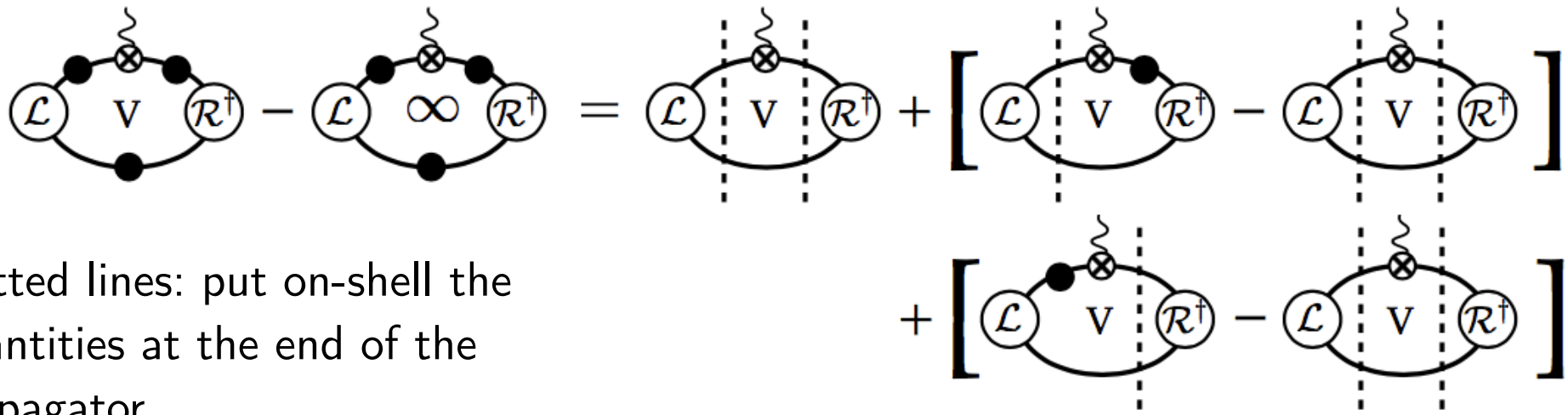
Scattering amplitude

The loop function of interest

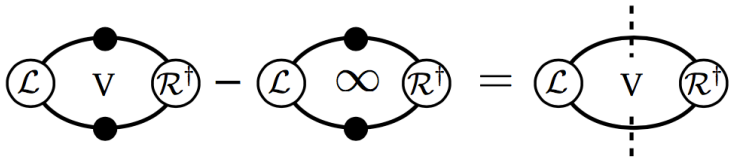


Dotted lines: put on-shell the quantities at the end of the propagator.

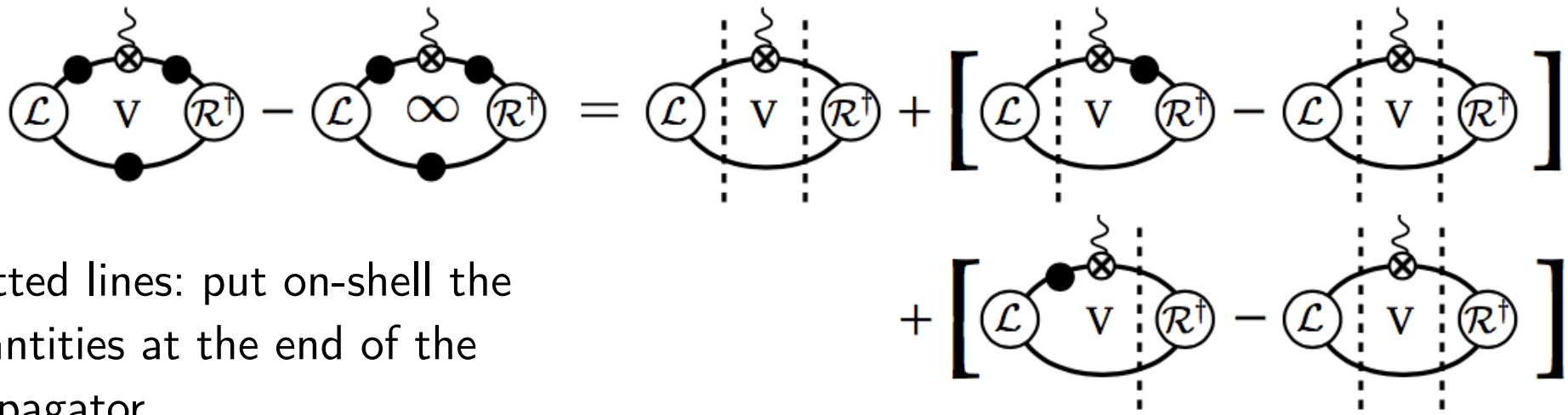
The loop function of interest



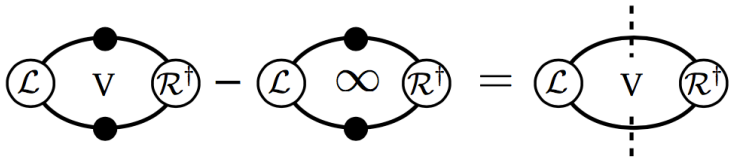
Dotted lines: put on-shell the quantities at the end of the propagator.



The loop function of interest

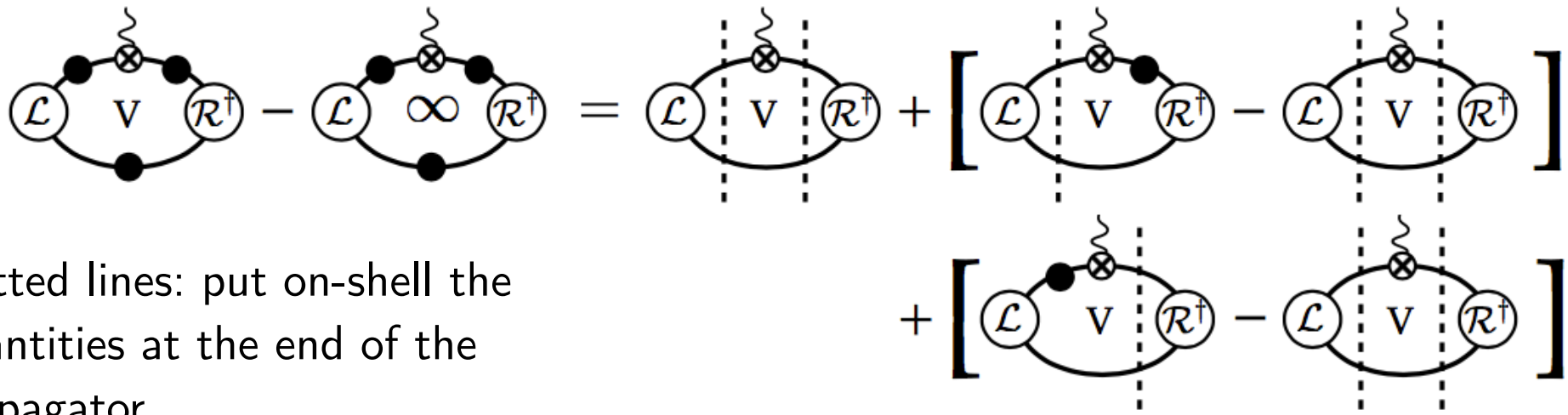


Dotted lines: put on-shell the quantities at the end of the propagator.



$$\mathcal{L}\, V\, \mathcal{R} - \mathcal{L}\, \infty\, \mathcal{R} = -\mathcal{L}_{\rm on}\, F\, \mathcal{R}_{\rm on}$$

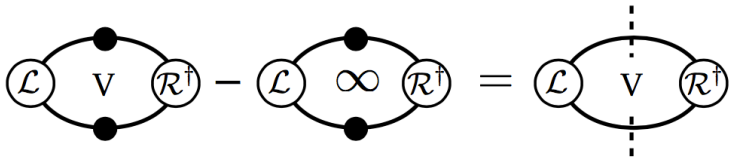
The loop function of interest



Dotted lines: put on-shell the quantities at the end of the propagator.

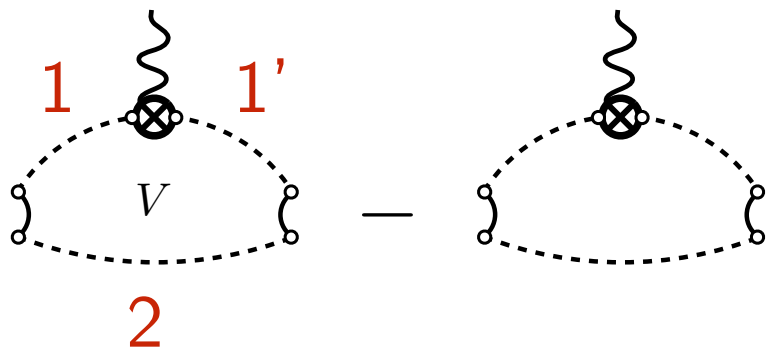
$$\mathcal{L}\,\tilde{V}\,\mathcal{R} - \mathcal{L}\,\tilde{\infty}\,\mathcal{R} = \mathcal{L}_{\rm on}\,(f_{\rm on}\cdot G)\,\mathcal{R}_{\rm on} - [\mathcal{L}_{\rm on}\,F\,(\omega\Delta\mathcal{R})_{\rm on,df}]$$

$$- [(\mathcal{L}\Delta\omega)_{\rm on,df}\,F\,\mathcal{R}_{\rm on}]$$



$$\mathcal{L}\,V\,\mathcal{R} - \mathcal{L}\,\infty\,\mathcal{R} = -\mathcal{L}_{\rm on}\,F\,\mathcal{R}_{\rm on}$$

G function

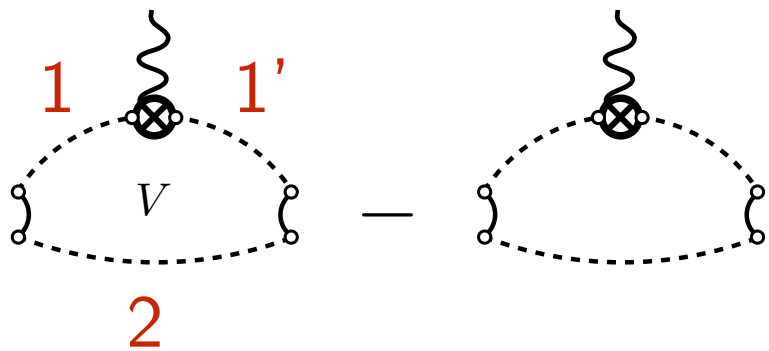


$$\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star)\, D(m,\mathbf{k}) K^{\omega}_{\mu_1\cdots\mu_n}(m,\mathbf{k})\, \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

Temporal integral puts p2 on-shell

+ smooth terms. $\left(k_0 = \sqrt{k^2 + m_2^2}\right)$

G function

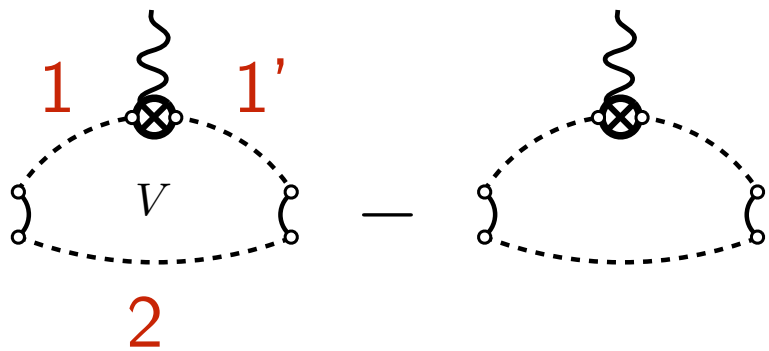


$$\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star)\, D(m,\mathbf{k}) \overbrace{K^{\omega}_{\mu_1\cdots\mu_n}(m,\mathbf{k})}^{\text{Single FF tensor decomposition}}\, \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

Temporal integral puts p2 on-shell

+ smooth terms. $\left(k_0 = \sqrt{k^2 + m_2^2}\right)$

G function



Single FF tensor decomposition

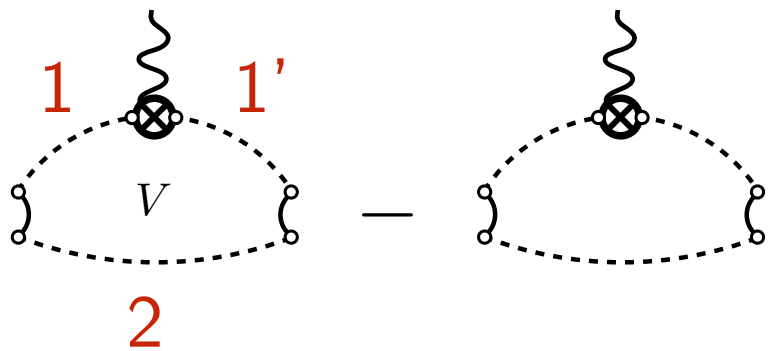
$$\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star)\, D(m,\mathbf{k}) \overbrace{K^{\omega}_{\mu_1\cdots\mu_n}(m,\mathbf{k})}^{\text{Single FF tensor decomposition}}\, \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

End-caps angular momentum projection

Temporal integral puts p2 on-shell + smooth terms.

$\left(k_0 = \sqrt{k^2 + m_2^2}\right)$

G function



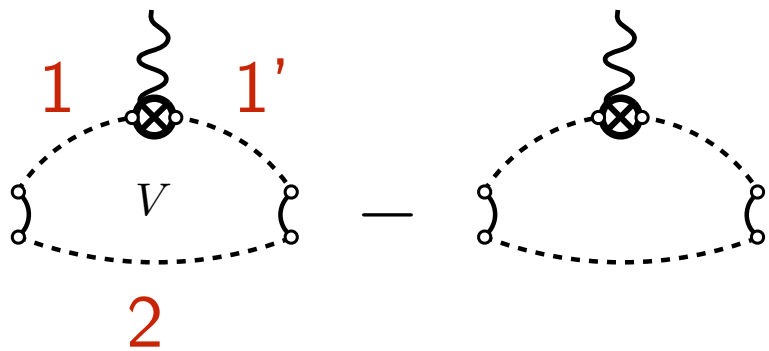
$$\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star)\, \overbrace{D(m,\mathbf{k})}^{\text{Propagators}}\, \overbrace{K^{\omega}_{\mu_1\cdots\mu_n}(m,\mathbf{k})}^{\text{Single FF tensor decomposition}}\, \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

End-caps angular momentum projection

Temporal integral puts p2 on-shell + smooth terms.

$\left(k_0 = \sqrt{k^2 + m_2^2}\right)$

G function



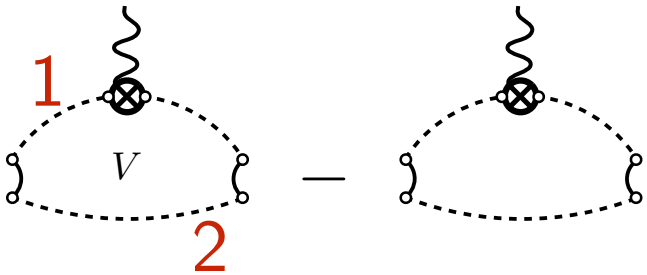
$$\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star)\, \overbrace{D(m,\mathbf{k})}^{\text{Propagators}}\, \overbrace{K^{\omega}_{\mu_1\cdots\mu_n}(m,\mathbf{k})}^{\text{Single FF tensor decomposition}}\, \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

End-caps angular momentum projection

Temporal integral puts p2 on-shell + smooth terms. $\left(k_0 = \sqrt{k^2 + m_2^2}\right)$

How to evaluate this 3D sum - integral without losing the rigor?

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)



scalar 1-particle FF $\qquad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} \frac{1}{\left((P-k)^2 - m_1^2 + i\epsilon\right)^2}\Bigg|_{k_0=\omega_{k2}}$$

b) a)

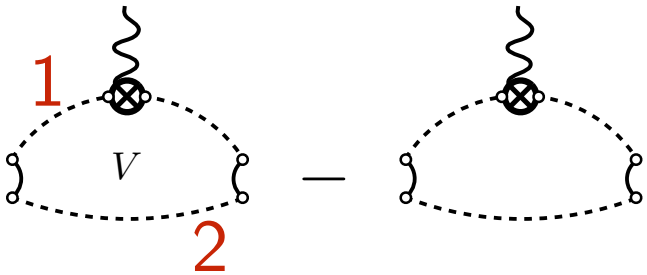
Expand functions around the pole: $f(\mathbf{k}^2) = \sum_{n=-2}^{\infty} a_n(\mathbf{q}^2)\left(\mathbf{q}^2 - \mathbf{k}^2\right)^n$

CM relative momenta of p1-p2.

a)

b)

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)



scalar 1-particle FF $\qquad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} \frac{1}{\left((P-k)^2 - m_1^2 + i\epsilon\right)^2}\Bigg|_{k_0=\omega_{k2}}$$

b) a)

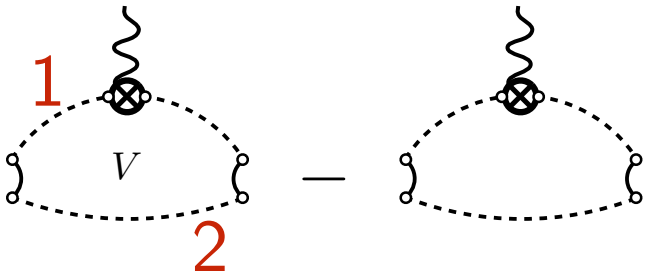
Expand functions around the pole: $f(\mathbf{k}^2) = \sum_{n=-2}^{\infty} a_n(\mathbf{q}^2)\left(\mathbf{q}^2 - \mathbf{k}^2\right)^n$

CM relative momenta of p1-p2.

a) $$\frac{\omega_{q2}^2}{E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2} - \frac{1}{2E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)} + \mathcal{O}\Big(\left(\mathbf{q}^2 - \mathbf{k}^2\right)^0\Big)$$

b)

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)



scalar 1-particle FF $\qquad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} \frac{1}{\left((P-k)^2 - m_1^2 + i\epsilon\right)^2}\Bigg|_{k_0=\omega_{k2}}$$

b) a)

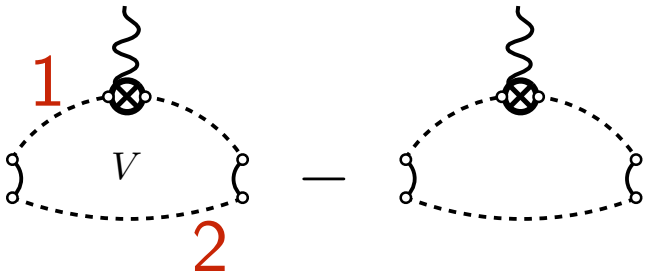
Expand functions around the pole: $f(\mathbf{k}^2) = \sum_{n=-2}^{\infty} a_n(\mathbf{q}^2)\left(\mathbf{q}^2 - \mathbf{k}^2\right)^n$

CM relative momenta of p1-p2.

a) $$\frac{\omega_{q2}^2}{E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2} - \frac{1}{2E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)} + \mathcal{O}\left(\left(\mathbf{q}^2 - \mathbf{k}^2\right)^0\right)$$

b) $$\frac{1}{2\omega_{q2}} - \left(\mathbf{q}^2 - \mathbf{k}^2\right)\frac{\partial}{\partial \mathbf{q}^2}\frac{1}{2\omega_{q2}}$$

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)



scalar 1-particle FF $\qquad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} \frac{1}{\left((P-k)^2 - m_1^2 + i\epsilon\right)^2}\Bigg|_{k_0=\omega_{k2}}$$

b) a)

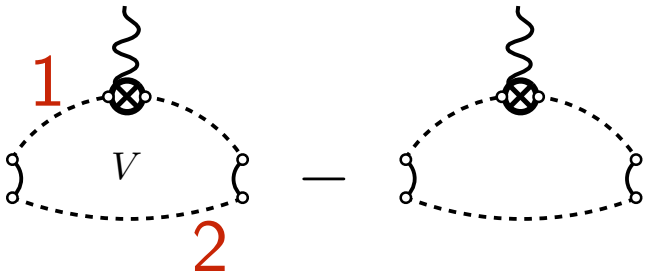
Expand functions around the pole: $f(\mathbf{k}^2) = \sum_{n=-2}^{\infty} a_n(\mathbf{q}^2)\left(\mathbf{q}^2 - \mathbf{k}^2\right)^n$

CM relative momenta of p1-p2.

a) $$\frac{\omega_{q2}^2}{E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2} - \frac{1}{2E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)} + \mathcal{O}\left(\left(\mathbf{q}^2 - \mathbf{k}^2\right)^0\right)$$

b) $$\frac{1}{2\omega_{q2}} - \left(\mathbf{q}^2 - \mathbf{k}^2\right)\frac{\partial}{\partial \mathbf{q}^2}\frac{1}{2\omega_{q2}} + \ldots$$

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)



scalar 1-particle FF $\qquad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_{k2}} \frac{1}{\left((P-k)^2 - m_1^2 + i\epsilon\right)^2}\Bigg|_{k_0=\omega_{k2}}$$

b) a)

Expand functions around the pole: $f(\mathbf{k}^2) = \sum_{n=-2}^{\infty} a_n(\mathbf{q}^2)\left(\mathbf{q}^2 - \mathbf{k}^2\right)^n$

CM relative momenta of p1-p2.

a) $$\frac{\omega_{q2}^2}{E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2} - \frac{1}{2E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)} + \mathcal{O}\left(\left(\mathbf{q}^2 - \mathbf{k}^2\right)^0\right)$$

b) $$\frac{1}{2\omega_{q2}} - \left(\mathbf{q}^2 - \mathbf{k}^2\right)\frac{\partial}{\partial \mathbf{q}^2}\frac{1}{2\omega_{q2}} + \ldots$$

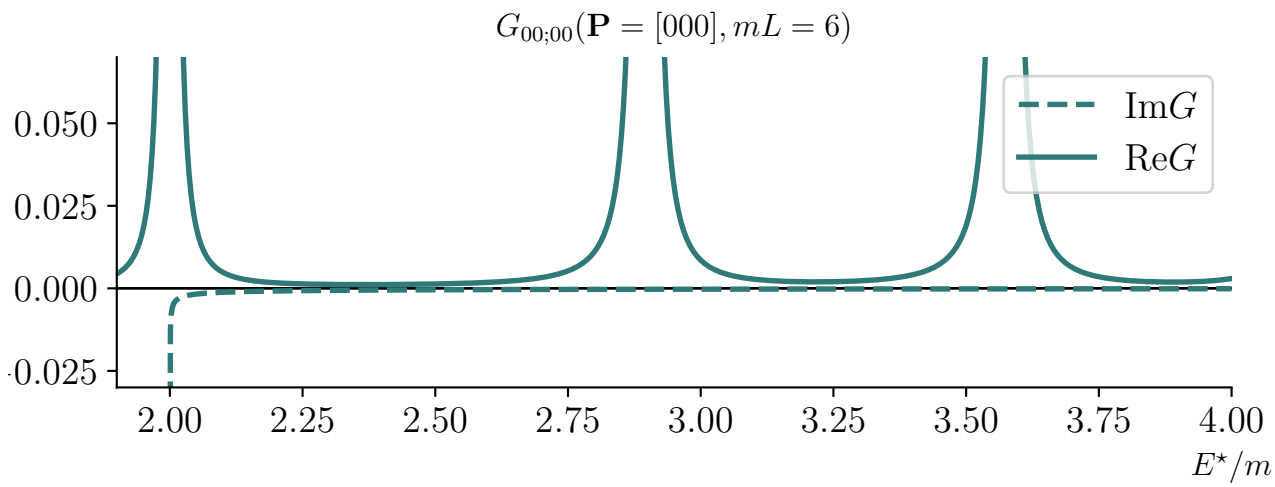
$$\frac{\omega_{q2}}{2E^2}\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2}$$

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)

scalar 1-particle FF $\quad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \frac{\omega_{q2}}{2E^2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2} + \mathcal{O}(e^{-mL})$$

- The integral can be evaluated analytically. $\quad \propto \dfrac{i}{|\mathbf{q}|}$ Inverse sqrt singularity
- The sum can be quickly saturated.

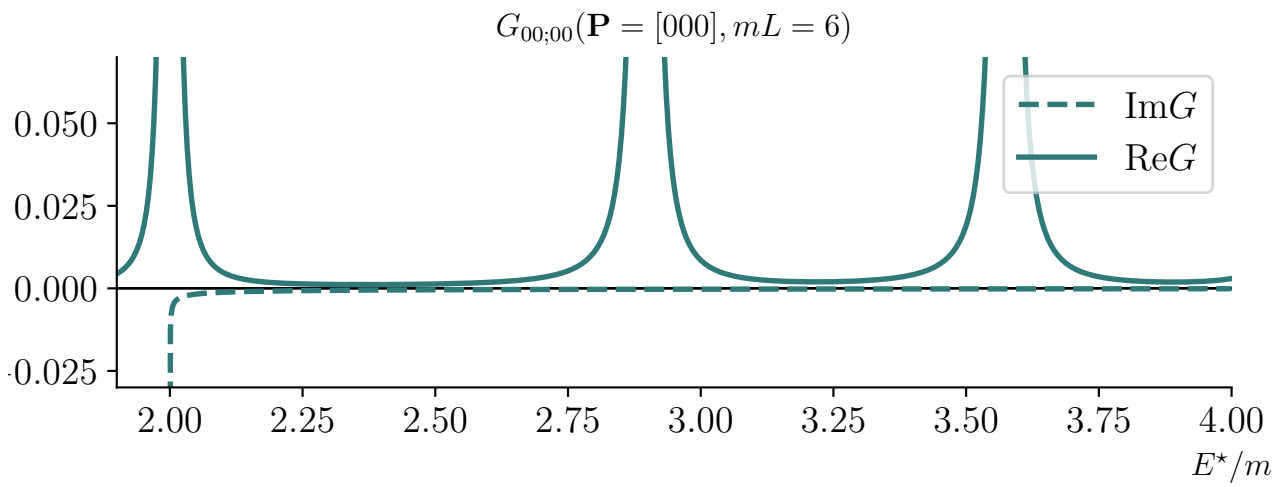


G-function when $P_\mathrm{i} = P_\mathrm{f}$ (one pole)

scalar 1-particle FF $\quad \mathbf{P} = \mathbf{0}$

$$G_{\ell_f=0,\ell_i=0}(P) = \frac{\omega_{q2}}{2E^2}\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right]\frac{1}{(\mathbf{q}^2 - \mathbf{k}^2 + i\epsilon)^2} + \mathcal{O}(e^{-mL})$$

- The integral can be evaluated analytically. $\quad \propto \dfrac{i}{|\mathbf{q}|}$ Inverse sqrt singularity
- The sum can be quickly saturated.



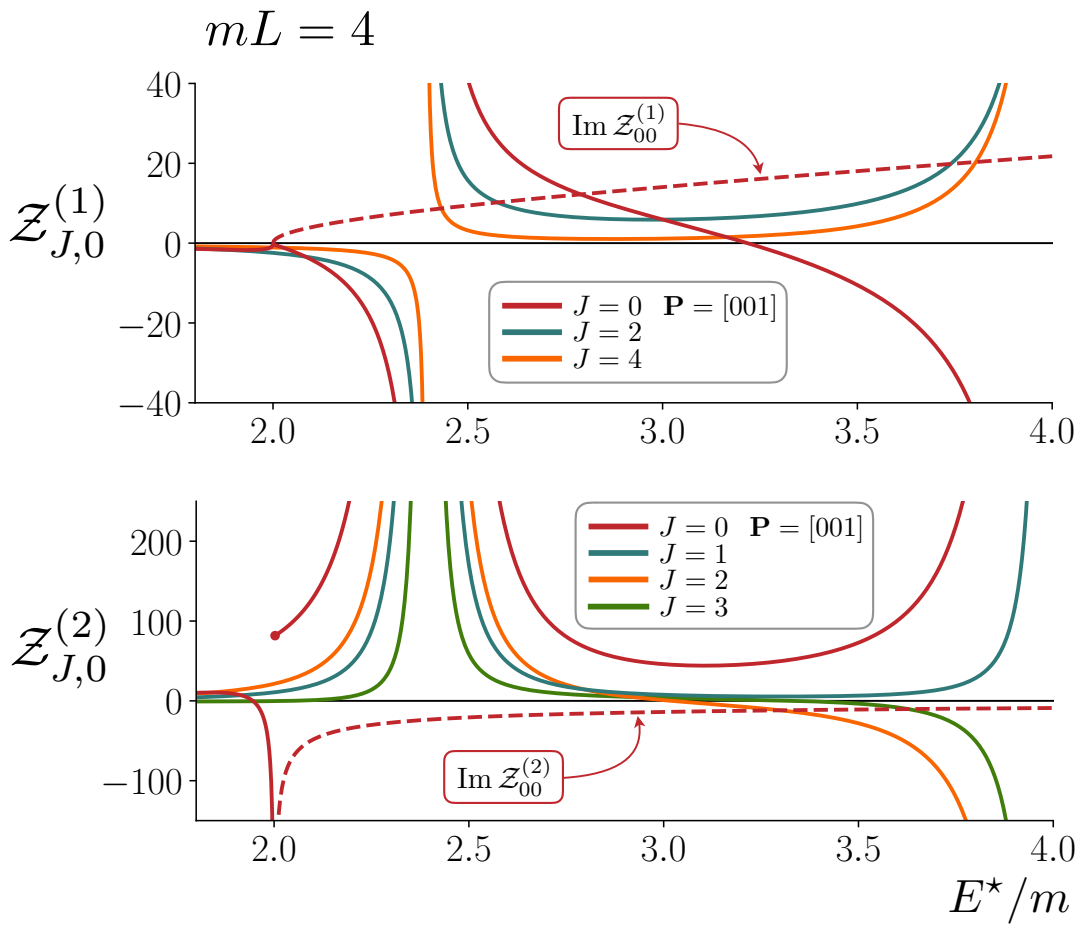
General one pole case

$$G_\sigma = \sum_{nJM}^{n\leq 2}\left[\frac{1}{L^3}\sum_{\mathbf{k}}\frac{\omega^\star_{k2}}{\omega_{k2}} - \int\frac{d^3\mathbf{k}^\star}{(2\pi)^3}\right]\frac{A_{n;\mu_1\cdots\mu_n}(q^\star)}{\left(q^{\star 2} - k^{\star 2}\right)^n}k^{\star J}Y_{JM}(\hat{\mathbf{k}}^\star)$$

Evaluated in CM

G-function when $P_\mathrm{i} = P_\mathrm{f}$

- The G-function is a linear combination of generalized Lüscher Z-functions.



$$\mathcal{Z}^{(n)}_{JM}(P, L, \alpha) = \left[\sum_{\mathbf{r}} \frac{\omega^\star_{k2}}{\omega_{k2}} - \int d^3\mathbf{r}^\star\right] \frac{\sqrt{4\pi}\, r^{\star J} Y_{JM}(\hat{\mathbf{r}}^\star)}{(x^2 - r^{\star 2} + i\epsilon)^n} e^{-\alpha(r^{\star 2} - x^2)^n}\,.$$

Regularize with a cut-off function.

General case

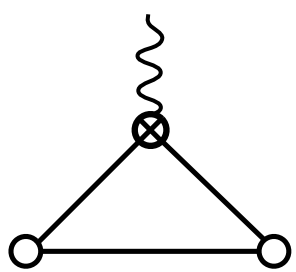
$$D(m,\mathbf{k}) \equiv \frac{1}{2\omega_{k2}} \frac{1}{(P_f-k)^2-m_1^2+i\epsilon} \frac{1}{(P_i-k)^2-m_1^2+i\epsilon}\Bigg|_{k_0=\omega_{k2}}$$

a) The integrand contains poles at two-different locations (different cut-off function H needed, sum is straight forward).

b) The G-function will feature both finite-and infinite-volume analytic structure:

1. Poles at free finite volume energies.

2. Threshold/Triangle singularities at the Landau conditions.



The integral evaluation

or

Keep the rigor and avoid brute force

$$D_c(m,k) = \frac{i}{k^2 - m_2^2 + i\epsilon} \frac{1}{(P_f - k)^2 - m_1^2 + i\epsilon} \frac{1}{(P_i - k)^2 - m_1^2 + i\epsilon}$$

$$\int \frac{dk^0}{2\pi} D_c(m,k) = D(m,\mathbf{k}) + \underbrace{D_r(m,\mathbf{k})}_{\text{Smooth function}}$$

The integral evaluation

or

Keep the rigor and avoid brute force

$$D_c(m,k) = \frac{i}{k^2 - m_2^2 + i\epsilon} \frac{1}{(P_f - k)^2 - m_1^2 + i\epsilon} \frac{1}{(P_i - k)^2 - m_1^2 + i\epsilon}$$

$$\int \frac{dk^0}{2\pi} D_c(m,k) = D(m,\mathbf{k}) + \underbrace{D_r(m,\mathbf{k})}_{\text{Smooth function}}$$

- Can be generalized to an expression with general tensor FF.

$$\underbrace{\int \frac{dk^0}{2\pi} D_c(m,k) K_{\nu_1 \cdots \nu_N}(k)} = D(m,\mathbf{k}) K^{\omega}_{\nu_1 \cdots \nu_N}(m,\mathbf{k}) + \underbrace{\mathcal{K}_{r;\nu_1 \cdots \nu_N}(m,\mathbf{k})}$$

The integral evaluation

or

Keep the rigor and avoid brute force

$$D_c(m,k) = \frac{i}{k^2 - m_2^2 + i\epsilon}\frac{1}{(P_f - k)^2 - m_1^2 + i\epsilon}\frac{1}{(P_i - k)^2 - m_1^2 + i\epsilon}$$

$$\int \frac{dk^0}{2\pi} D_c(m,k) = D(m,\mathbf{k}) + \underbrace{D_r(m,\mathbf{k})}_{\text{Smooth function}}$$

- Can be generalized to an expression with general tensor FF.

$$\underbrace{\int \frac{dk^0}{2\pi} D_c(m,k) K_{\nu_1\ldots\nu_N}(k)}_{\mathcal{I}_{\mathcal{A};\sigma}} = D(m,\mathbf{k}) K^{\omega}_{\nu_1\ldots\nu_N}(m,\mathbf{k}) + \underbrace{\mathcal{K}_{r;\nu_1\ldots\nu_N}(m,\mathbf{k})}$$

The integral evaluation

or

Keep the rigor and avoid brute force

$$D_c(m,k) = \frac{i}{k^2 - m_2^2 + i\epsilon}\frac{1}{(P_f - k)^2 - m_1^2 + i\epsilon}\frac{1}{(P_i - k)^2 - m_1^2 + i\epsilon}$$

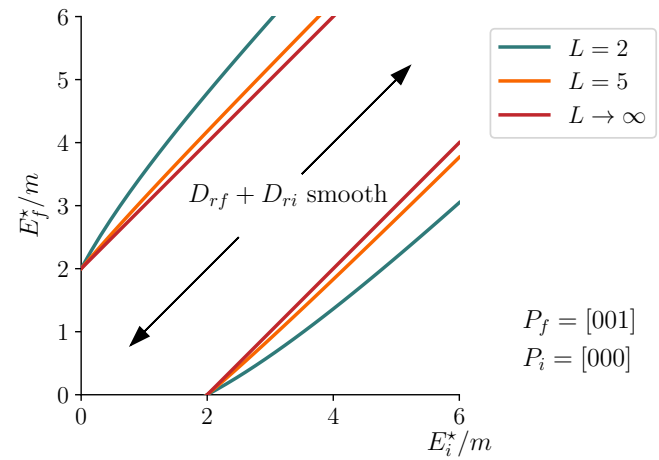
$$\int \frac{dk^0}{2\pi} D_c(m,k) = D(m,\mathbf{k}) + \underbrace{D_r(m,\mathbf{k})}_{\text{Smooth function}}$$

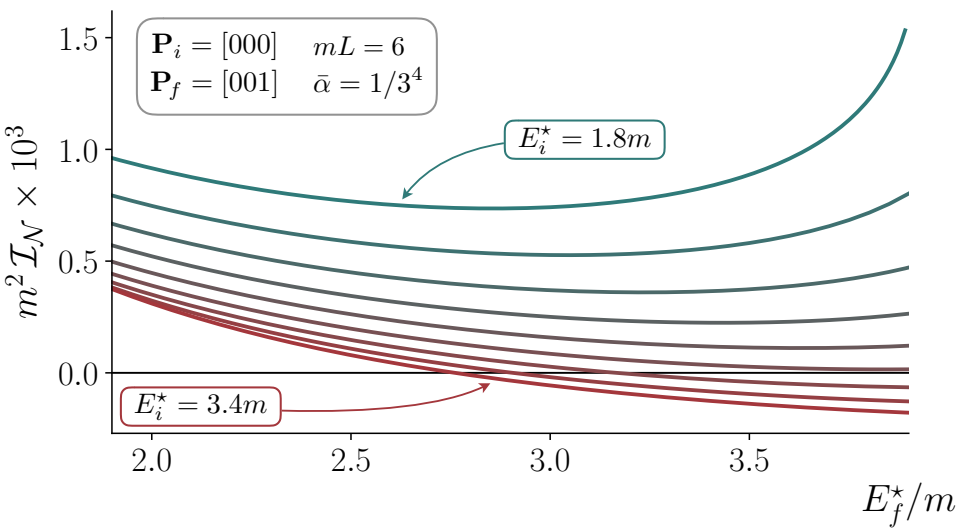
- Can be generalized to an expression with general tensor FF.

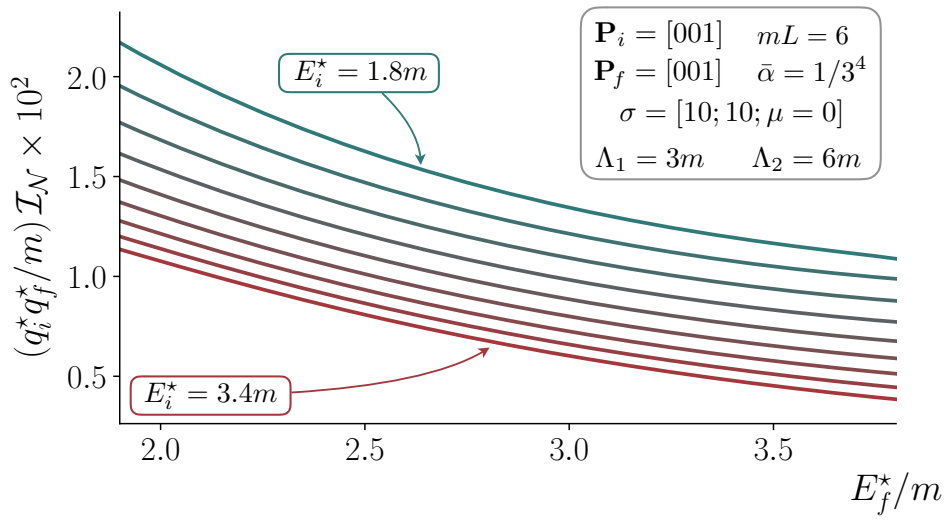
$$\underbrace{\int \frac{dk^0}{2\pi} D_c(m,k) K_{\nu_1\cdots\nu_N}(k)}_{\mathcal{I}_{\mathcal{A};\sigma}} = D(m,\mathbf{k}) K^{\omega}_{\nu_1\cdots\nu_N}(m,\mathbf{k}) + \underbrace{\mathcal{K}_{r;\nu_1\cdots\nu_N}(m,\mathbf{k})}_{\mathcal{I}_{\mathcal{N};\sigma}}$$

$\mathcal{I}_{\mathcal{N};\mu_1\cdots\mu_n;\ell_f m_f;\ell_i m_i}(\bar{\alpha}, \Lambda, P_f, P_i)$

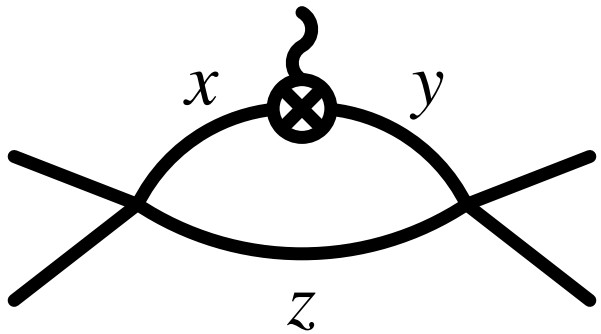
- Smooth, UV-convergent, 3D-integral.
- Lorentz covariant:
  - It can be evaluated in a frame where one of the external 3-momentum is 0, and the other only points in the z-axis.
  - Effectively only a 2D numerical integral is needed.







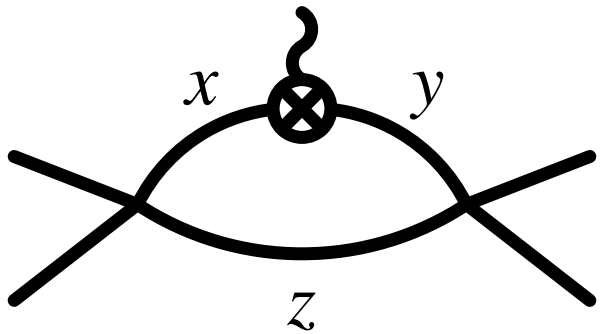
$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i)$



$$\sim \int dxdydz \frac{d^4k}{(2\pi)^4} \frac{\delta(1-x-y-z)}{(k^2-\phi(x,y,z))^3}$$

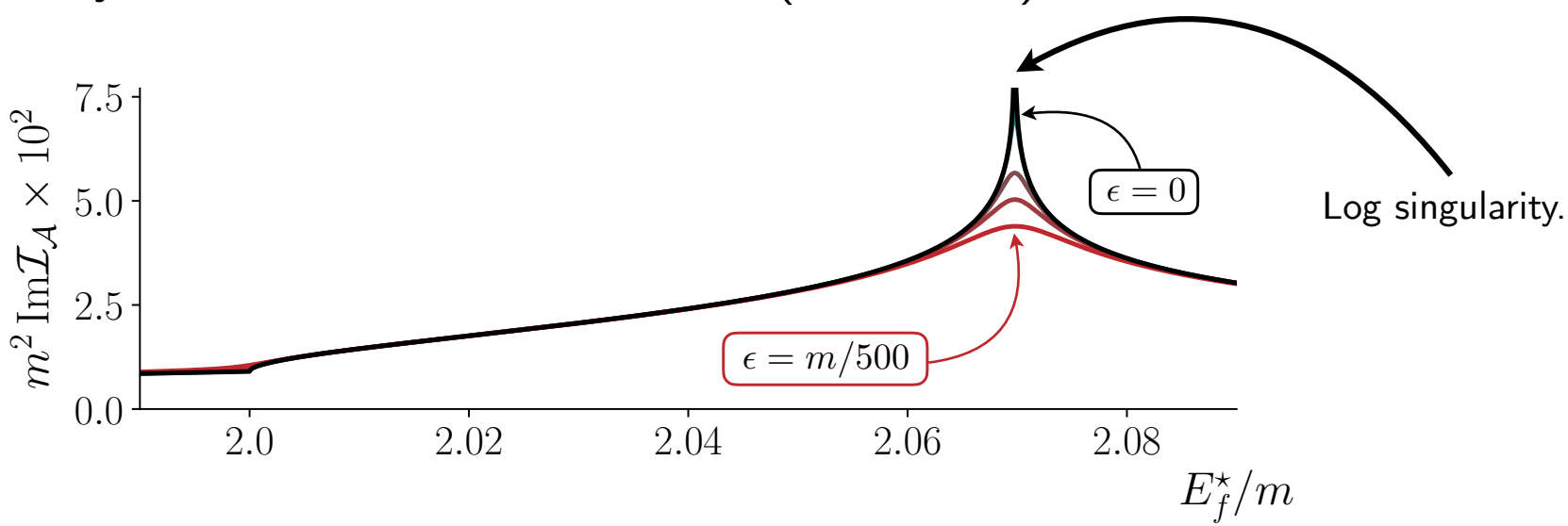
a) The final Feynman integral has to be done numerically, the $i\epsilon$ can be taken to 0, (with care).

$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i)$

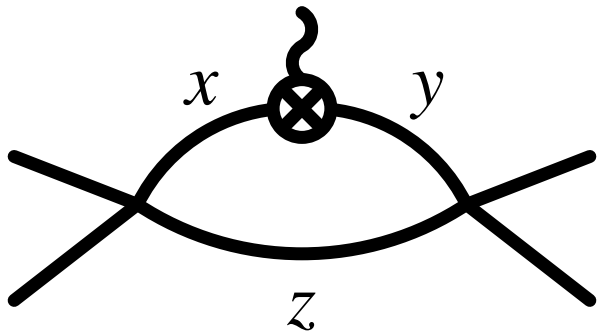


$$\sim \int dx dy dz \frac{d^4k}{(2\pi)^4} \frac{\delta(1-x-y-z)}{(k^2-\phi(x,y,z))^3}$$

a) The final Feynman integral has to be done numerically, the $i\epsilon$ can be taken to 0, (with care).

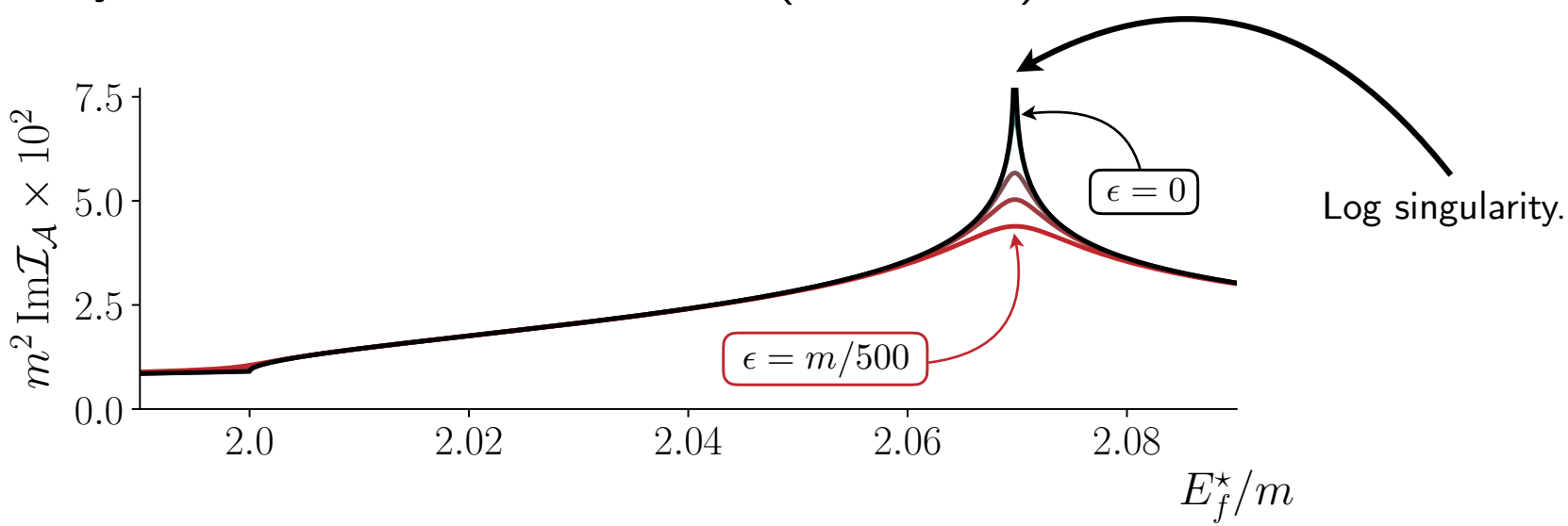


$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i)$



$$\sim \int dx dy dz \frac{d^4k}{(2\pi)^4} \frac{\delta(1-x-y-z)}{(k^2-\phi(x,y,z))^3}$$

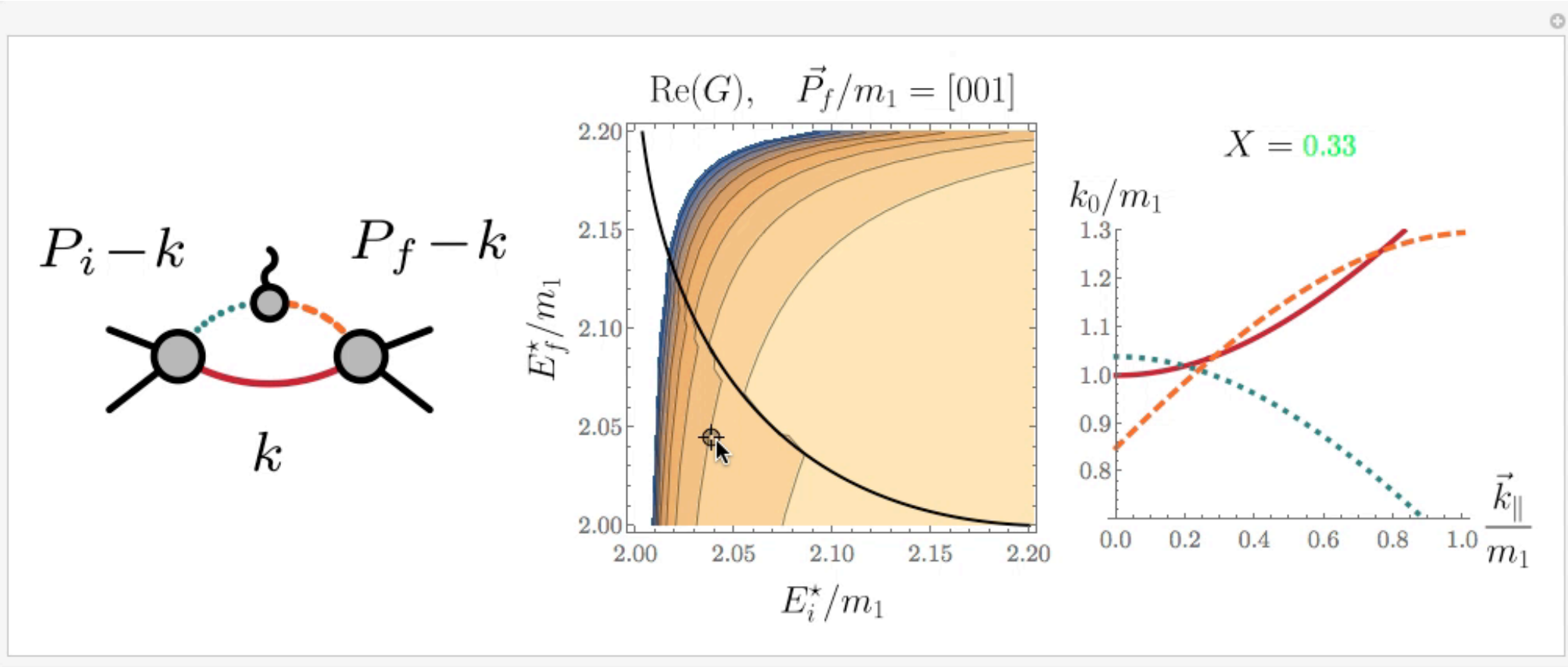
a) The final Feynman integral has to be done numerically, the $i\epsilon$ can be taken to 0, (with care).



Landau conditions on the Feynman parameters: $\frac{\partial\phi}{\partial x} = 0, \quad \frac{\partial\phi}{\partial y} = 0, \quad \phi(x_c, y_c) = 0.$

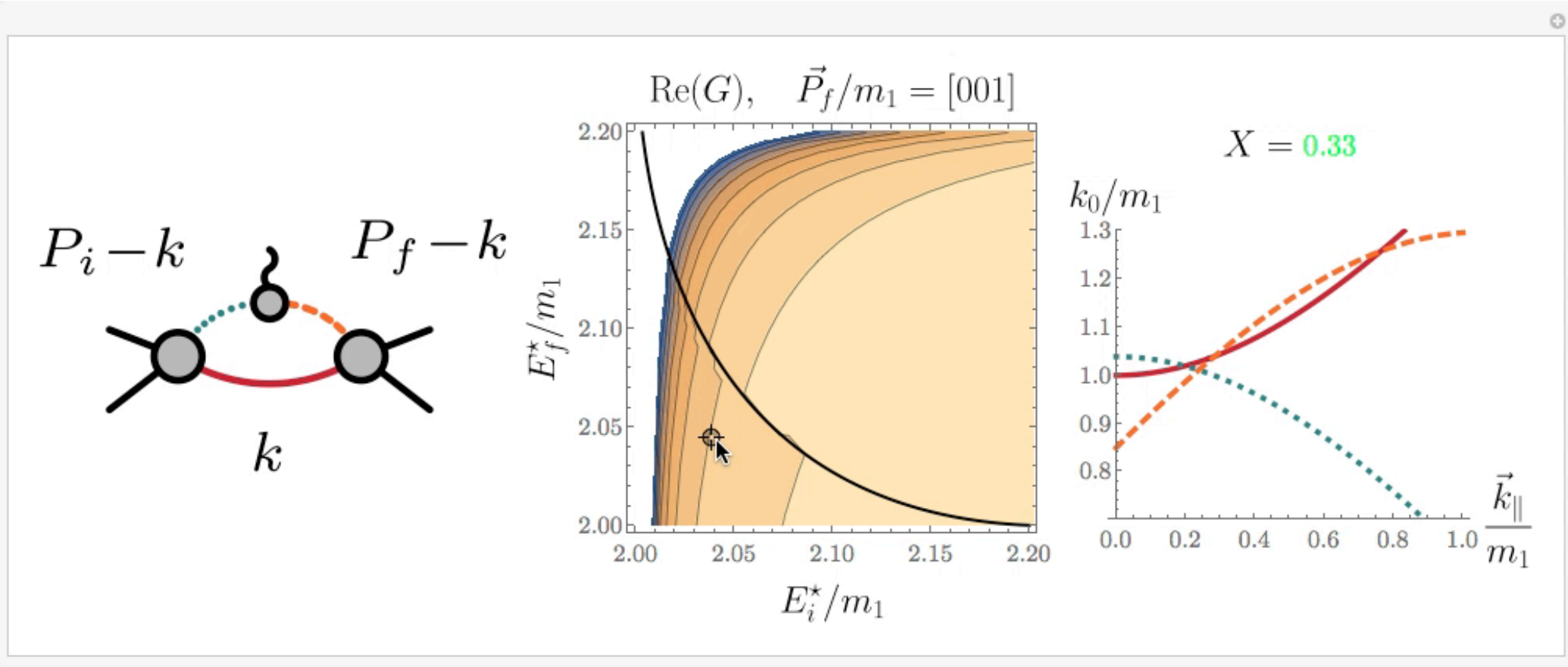
A visualization of the triangle singularities

Landau conditions of triangle singularity $\implies X(s_i, s_f, Q^2) = 0$

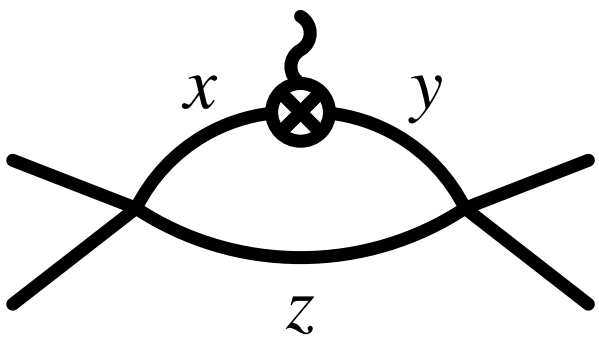


A visualization of the triangle singularities

Landau conditions of triangle singularity $\implies X(s_i, s_f, Q^2) = 0$



Character of the singularity



$$\propto \left(\sqrt{s_0 - s}\right)^{\overset{\text{loops}}{4\ell}+\underset{\text{independent Feynman parameters}}{m}-\overset{\text{propagators}}{2n}} \equiv \left(\sqrt{s_0 - s}\right)^{\mathcal{E}}$$

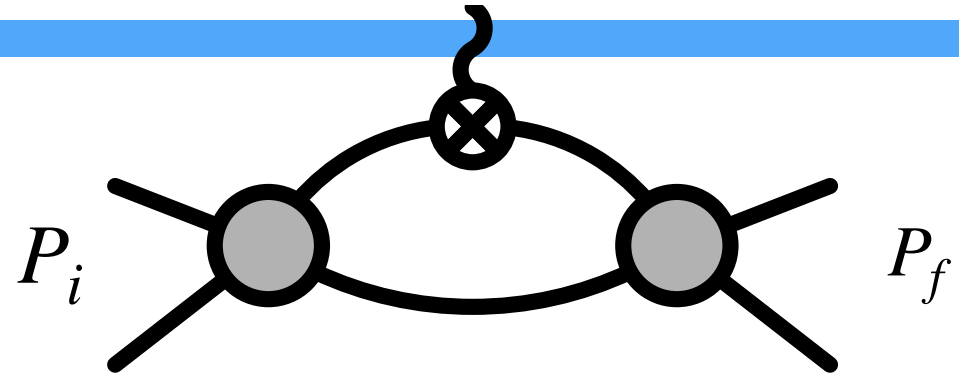
if $\mathcal{E} = 2k \leq 0$ $\quad \propto \left(\sqrt{s_0 - s}\right)^{\mathcal{E}} \log(s_0 - s)$

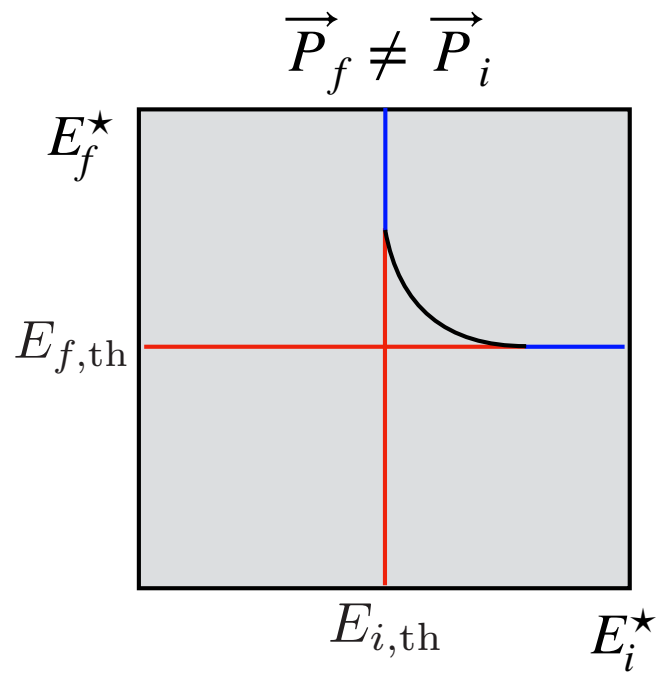
$P_\mathrm{i} = P_\mathrm{f}$ 1 loop, 1 ind. Feynman, 3 prop $\longrightarrow$ inverse sqrt

$P_\mathrm{i}, P_\mathrm{f}$ 1 loop, 2 ind. Feynman, 3 prop $\longrightarrow$ log

$P_\mathrm{x,threshold}$ 1 loop, 1 ind. Feynman, 2 prop $\longrightarrow$ sqrt

$\mathcal{I}_{\mathcal{A}}$ analytical structure

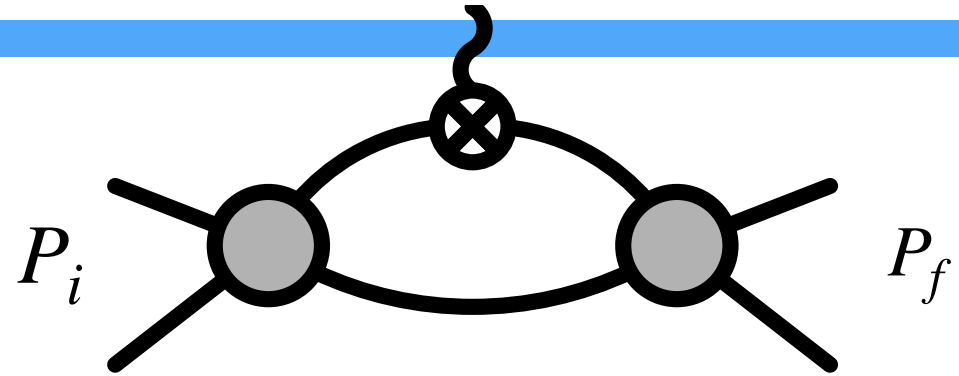


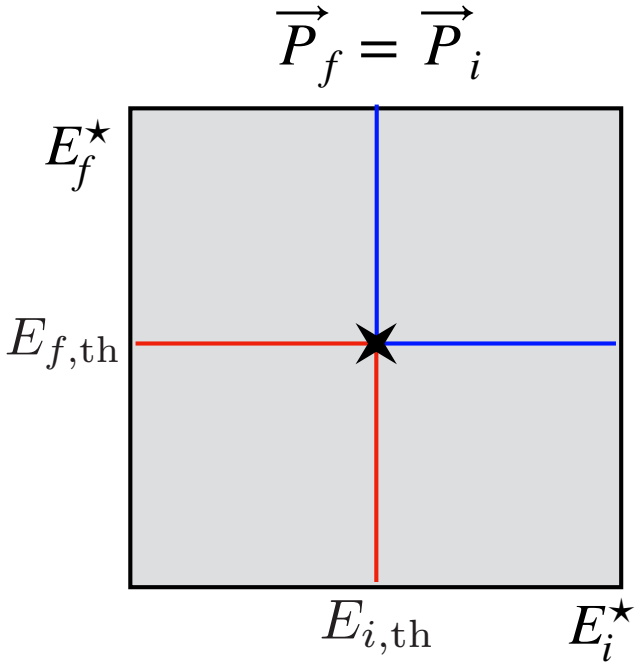


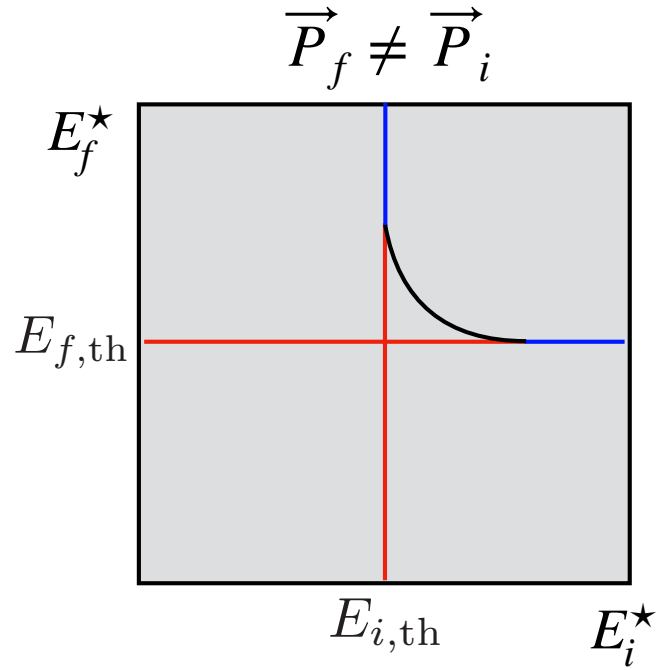
Analytic

$i\sqrt{s - s_{th}}$ $\quad -i\sqrt{s - s_{th}}$ $\quad -i\log\left(s_\Delta - s\right)$

$\mathcal{I}_{\mathcal{A}}$ analytical structure







Analytic

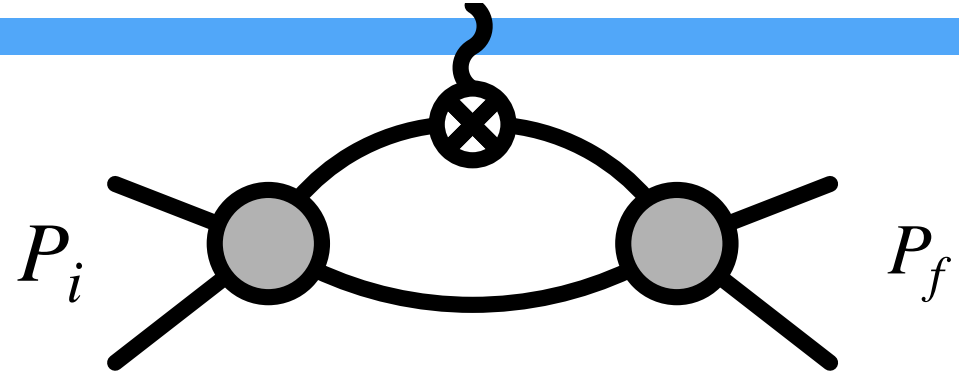
$i\sqrt{s - s_{th}}$

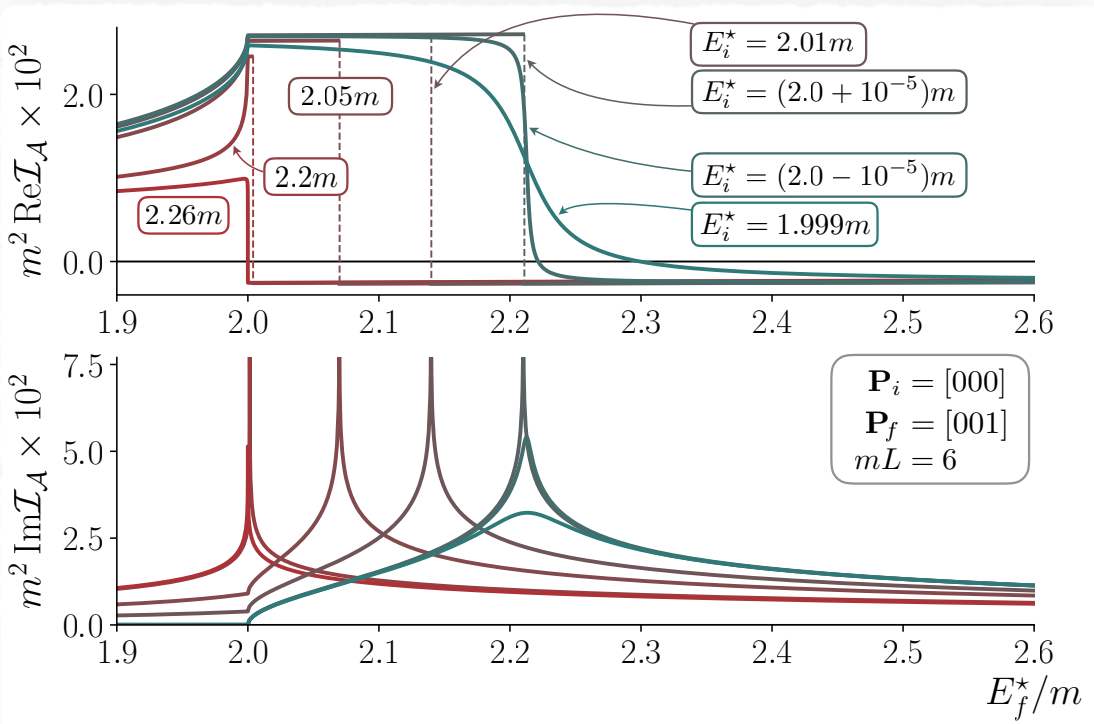
$-i\sqrt{s - s_{th}}$

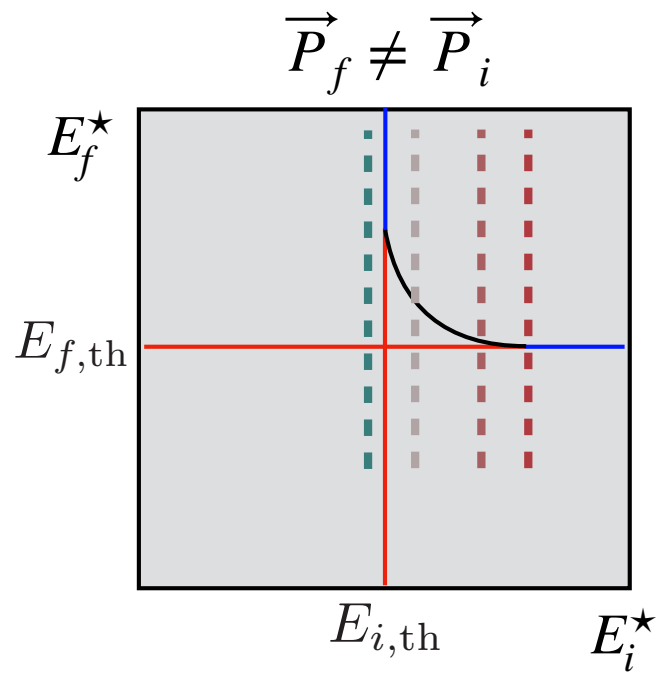
$-i \log\left(s_\Delta - s\right)$

$\dfrac{i}{\sqrt{s - s_{th}}}$

$\mathcal{I}_{\mathcal{A}}$ analytical structure



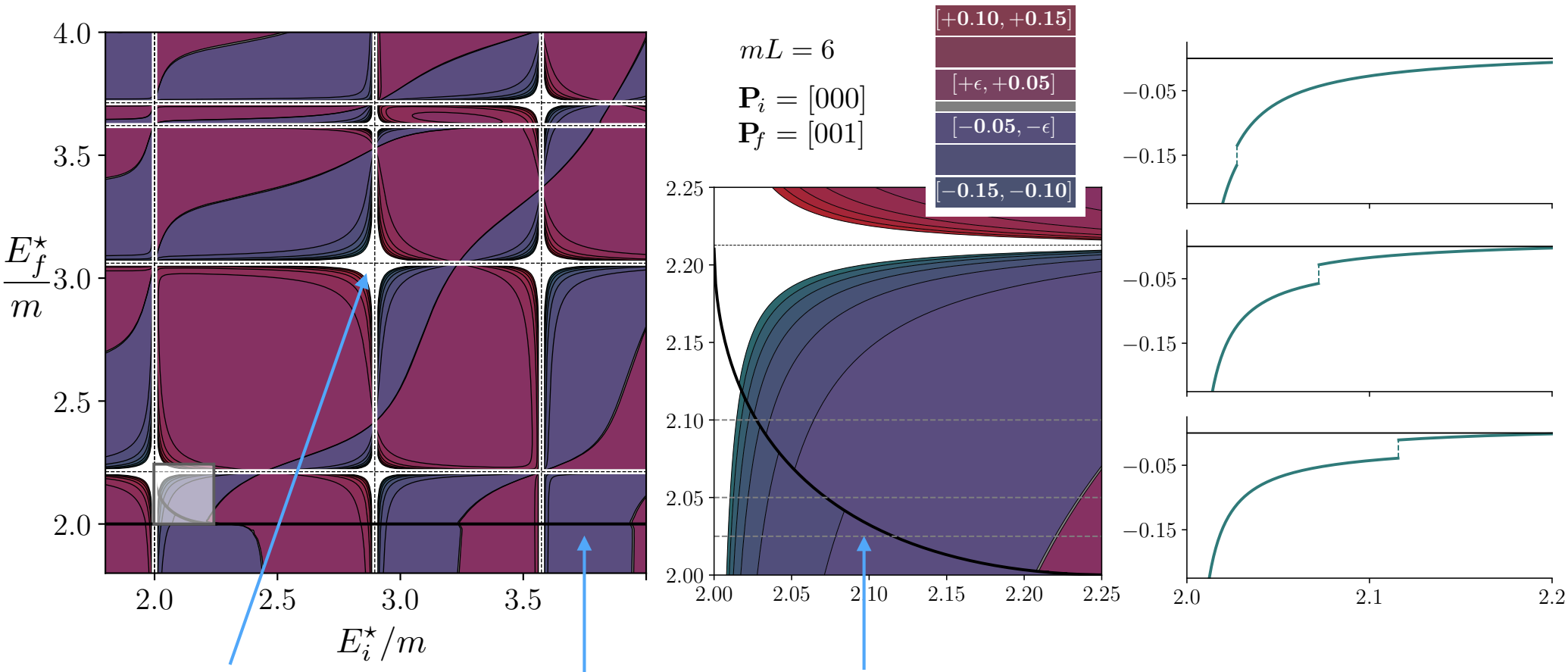




Analytic

$i\sqrt{s - s_{th}}$ $\quad -i\sqrt{s - s_{th}}$ $\quad -i\log\left(s_\Delta - s\right)$ $\quad \dfrac{i}{\sqrt{s - s_{th}}}$

And finally this is how the G-function looks like

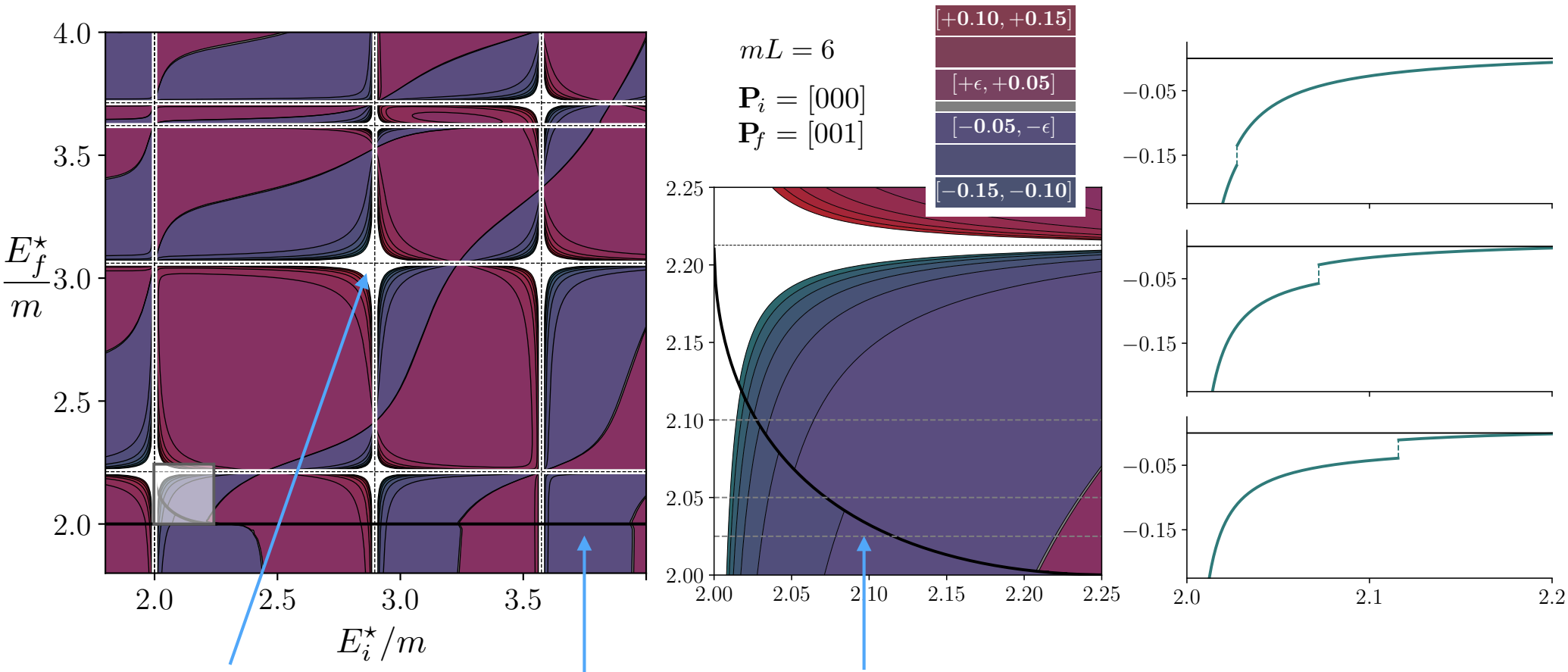


Poles at FV free energies.

Threshold singularities.

Triangle singularities.

And finally this is how the G-function looks like



Poles at FV free energies.

Threshold singularities.

Triangle singularities.

- $$\mathcal{W}_{L,\mathrm{df}} - \mathcal{W}_{\mathrm{df}} = \sum_j C^{(j)}\, f^{(j)}\, \mathcal{M}(s_f)\, G(P_i, P_f, L)\, \mathcal{M}(s_i)$$

Summary and Outlook

- Overview of an efficient evaluation of the geometrical function encoding power-law finite-volume corrections of hadronic matrix elements (*local* currents).
- Kinematic singularities have to be understood in order to parametrize dynamics independently.
- Matching of Lorentz decomposition with p.w. decomposition of W_{df} is undergoing.
- Extension to spin/multichannel (proliferation of indices).
- Future calculation of Resonance/bound states FF.

Additional slides

G-function when $P_\mathrm{i} = P_\mathrm{f}$ (only one pole)

$$G_{\mu_1\cdots\mu_n;\ell_f m_f;\ell_i m_i} = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star)\, D(m,\mathbf{k}) \underbrace{K^\omega_{\mu_1\cdots\mu_n}(m,\mathbf{k})}_{k_{\mu_1}\cdots k_{\mu_n}\big|_{k_0=\sqrt{k^2+m_2^2}}} \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

- Every k 4-vector can be decomposed into SH.
- The numerator is a linear combination of SH.
- Expand the numerator and denominator around the pole and only keep singular terms.

$$k_\nu^\star\Big|_{k_0=\sqrt{k^2+m_2^2}} = \sum_{\ell m<2} T_{\nu;\ell m}(k^\star)\, Y_{\ell m}(\hat{\mathbf{k}}^\star)$$

$$\sqrt{4\pi}\sum_{JM} \mathcal{C}_{\sigma,JM}(\boldsymbol{\beta},k^\star)\, k^{\star J} Y_{JM}(\hat{\mathbf{k}}^\star)$$

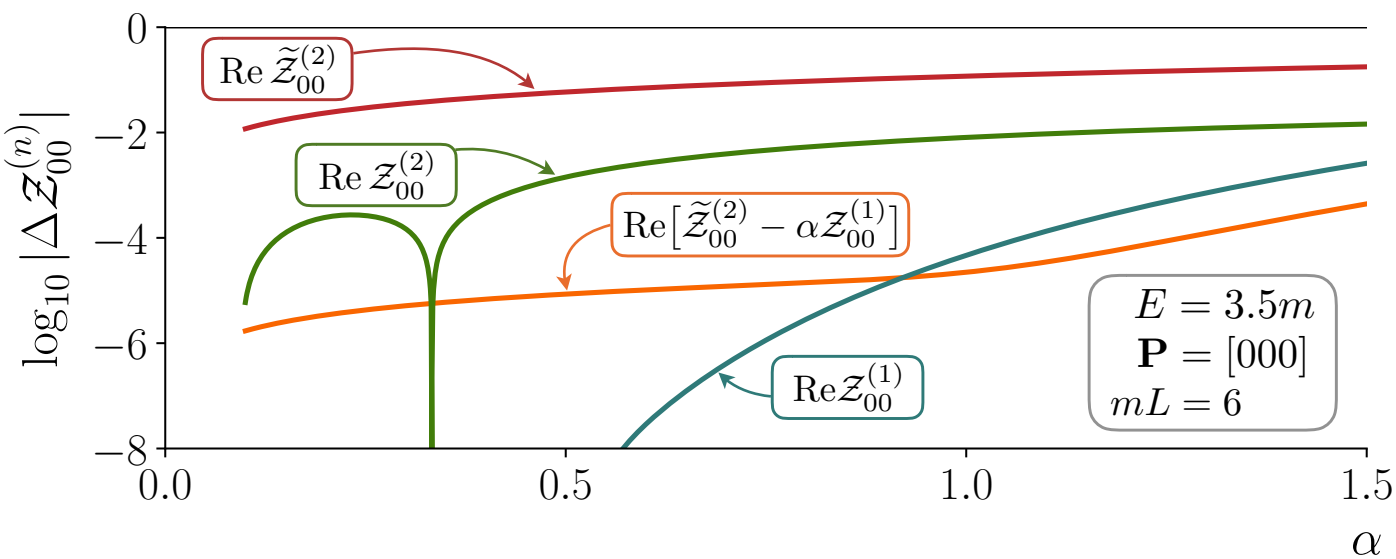
$$f(k^\star) = \sum_n a_n(q^{\star 2})\left(q^{\star 2} - k^{\star 2}\right)^n$$

$$G_\sigma = \sum_{nJM} \frac{A_{n;\mu_1\cdots\mu_n}(q^\star)}{\left(q^{\star 2} - k^{\star 2}\right)^n} k^{\star J} Y_{JM}(\hat{\mathbf{k}}^\star)$$

Cut-off dependence

$$\mathcal{Z}^{(n)}_{JM}(P,L,\alpha) = \left[\sum_{\mathbf{r}} \frac{\omega^\star_{k2}}{\omega_{k2}} - \int d^3\mathbf{r}^\star\right] \frac{\sqrt{4\pi}\, r^{\star J} Y_{JM}(\hat{\mathbf{r}}^\star)}{(x^2 - r^{\star 2} + i\epsilon)^n} e^{-\alpha(r^{\star 2} - x^2)^n} \,.$$

$$\tilde{\mathcal{Z}}^{(n)}_{JM}(P,L,\alpha) = \left[\sum_{\mathbf{r}} \frac{\omega^\star_{k2}}{\omega_{k2}} - \int d^3\mathbf{r}^\star\right] \frac{\sqrt{4\pi}\, r^{\star J} Y_{JM}(\hat{\mathbf{r}}^\star)}{(x^2 - r^{\star 2} + i\epsilon)^n} e^{-\alpha(r^{\star 2} - x^2)} \,.$$



Regularize the Integral

Pauli-Villars regulators

$$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \int \frac{d^4k}{(2\pi)^4} K_{\nu_1\cdots\nu_N}(k) \sum_{j=0}^{n_j} c_j D_c(\Lambda_j, k)\,,$$

$$\begin{aligned}\mathcal{I}_{\mathcal{N};\sigma}(\bar{\alpha}, \Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \Bigg[&\int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[H(\bar{\alpha}, \mathbf{k}) - 1\right] \sum_{j=0}^{n_j} c_j \Big[D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) + \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k})\Big] \\ &- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=1}^{n_j} c_j D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \\ &- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=0}^{n_j} c_j \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Bigg]\,.\end{aligned}$$

SH decomposition

Cut-off function

Regularize the Integral

$$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \int \frac{d^4k}{(2\pi)^4} K_{\nu_1\cdots\nu_N}(k) \sum_{j=0}^{n_j} c_j D_c(\Lambda_j, k),$$

Pauli-Villars regulators

$$\mathcal{I}_{\mathcal{N};\sigma}(\bar{\alpha}, \Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \Bigg[\int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[H(\bar{\alpha}, \mathbf{k}) - 1\right] \sum_{j=0}^{n_j} c_j \left[D(\Lambda_j, \mathbf{k}) K^{\omega}_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) + \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \right]$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=1}^{n_j} c_j D(\Lambda_j, \mathbf{k}) K^{\omega}_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k})$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=0}^{n_j} c_j \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Bigg].$$

SH decomposition

Cut-off function

Regularize the Integral

$$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \int \frac{d^4k}{(2\pi)^4} K_{\nu_1\cdots\nu_N}(k) \sum_{j=0}^{n_j} c_j D_c(\Lambda_j, k),$$

Pauli-Villars regulators

$$\mathcal{I}_{\mathcal{N};\sigma}(\bar{\alpha}, \Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \Bigg[\int \frac{d^3\mathbf{k}}{(2\pi)^3} [H(\bar{\alpha}, \mathbf{k}) - 1] \sum_{j=0}^{n_j} c_j \Big[D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) + \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Big]$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=1}^{n_j} c_j D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k})$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=0}^{n_j} c_j \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Bigg].$$

SH decomposition

Cut-off function

Regularize the Integral

$$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \int \frac{d^4k}{(2\pi)^4} K_{\nu_1\cdots\nu_N}(k) \sum_{j=0}^{n_j} c_j D_c(\Lambda_j, k),$$

Pauli-Villars regulators

$$\mathcal{I}_{\mathcal{N};\sigma}(\bar{\alpha}, \Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \Bigg[\int \frac{d^3\mathbf{k}}{(2\pi)^3} [H(\bar{\alpha}, \mathbf{k}) - 1] \sum_{j=0}^{n_j} c_j \Big[D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) + \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Big]$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=1}^{n_j} c_j D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k})$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=0}^{n_j} c_j \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Bigg].$$

SH decomposition

Cut-off function

Regularize the Integral

$$\mathcal{I}_{\mathcal{A};\sigma}(\Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \int \frac{d^4k}{(2\pi)^4} K_{\nu_1\cdots\nu_N}(k) \sum_{j=0}^{n_j} c_j D_c(\Lambda_j, k),$$

Pauli-Villars regulators

$$\mathcal{I}_{\mathcal{N};\sigma}(\bar{\alpha}, \Lambda, P_f, P_i) \equiv M_\sigma^{\nu_1\cdots\nu_N} \Bigg[\int \frac{d^3\mathbf{k}}{(2\pi)^3} [H(\bar{\alpha}, \mathbf{k}) - 1] \sum_{j=0}^{n_j} c_j \Big[D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) + \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Big]$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=1}^{n_j} c_j D(\Lambda_j, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k})$$

$$- \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) \sum_{j=0}^{n_j} c_j \mathcal{K}_{r;\nu_1\cdots\nu_N}(\Lambda_j, \mathbf{k}) \Bigg].$$

SH decomposition

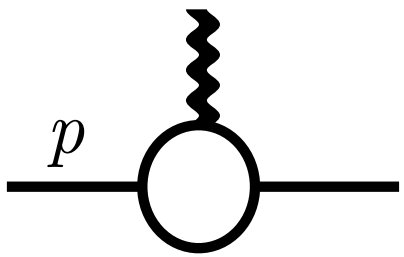
Cut-off function

$$\mathcal{I}_{\mathcal{A};\sigma} + \mathcal{I}_{\mathcal{N};\sigma} = M_\sigma^{\nu_1\cdots\nu_N} \int \frac{d^3\mathbf{k}}{(2\pi)^3} H(\bar{\alpha}, \mathbf{k}) D(m, \mathbf{k}) K^\omega_{\nu_1\cdots\nu_N}$$

- This is UV convergent and what we need to combine with the sum.

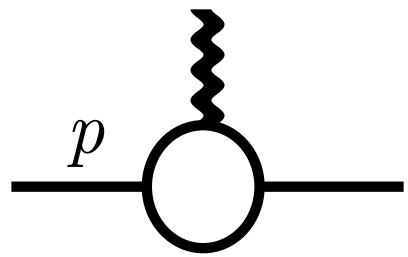
$$\mathcal{S}_\sigma(\bar{\alpha}, P_f, P_i, L) = \frac{1}{L^3} \sum_{\mathbf{k}} H(\bar{\alpha}, \mathbf{k}) \, \mathcal{Y}_{\ell_f m_f}(\mathbf{k}_f^\star) \, D(m, \mathbf{k}) K^\omega_{\mu_1\cdots\mu_n}(m, \mathbf{k}) \, \mathcal{Y}^*_{\ell_i m_i}(\mathbf{k}_i^\star)$$

Stable particle



Finite Volume

-



Infinite Volume

$$\sim e^{-mL}$$