Energy and momentum densities in hadrons

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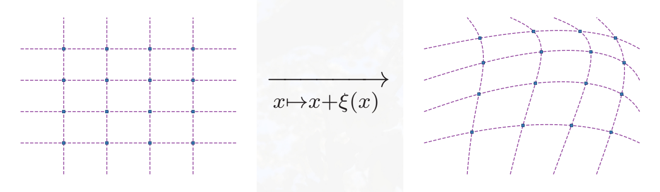
Introduction

- The **energy-momentum tensor** (EMT) is an operator characterizing the distribution and flow of energy and momentum.
  - It's related to space & time symmetry via Noether's theorems.
  - It's the source of gravitation in general relativity.
- Matrix elements between hadronic states characterize coveted properties of hadrons:
  - The distribution & decomposition of energy.
  - The distribution & decomposition of momentum.
  - The distribution & decomposition of internal stresses

$$\hat{T}^{\mu\nu}_{\text{QCD}}(x) = \sum_q \left\{ \frac{1}{2}\bar{q}(x) i\gamma^{\{\mu} \overleftrightarrow{D}^{\nu\}} q(x) \right\} - 2\text{Tr}\left[G^{\mu\lambda}G^{\nu}{}_{\lambda}\right] + \frac{1}{2}g^{\mu\nu}\text{Tr}\left[G^{\lambda\sigma}G_{\lambda\sigma}\right]$$

- Conserved current from *local* spacetime translations (**Noether's second theorem**):
  - $x^\mu \mapsto x^\mu + \xi^\mu(x)$
  - $S_{\text{QCD}} \mapsto S_{\text{QCD}}$
  - $\partial_\mu T^{\mu\nu}_{\text{QCD}} = 0.$
  - $T^{\mu\nu}_{\text{QCD}} = T^{\nu\mu}_{\text{QCD}}$
  - **Gauge-invariant**



AF, Phys. Rev. D 106 (2022) 125012

- Can be alternatively be derived from action variations:

$$\hat{T}^{\mu\nu}_{\text{Hil}}(x) = -\frac{2}{\sqrt{-g}} \frac{\delta S_{\text{QCD}}}{\delta g_{\mu\nu}(x)}$$

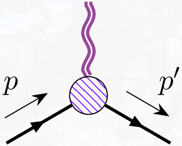
Form factors of the EMT

- EMT matrix elements give **gravitational form factors** (GFFs).
  - It's just a name.
  - EMT is the source of gravitation: $G^{\mu\nu} + \Lambda g^{\mu\nu} = 8\pi T^{\mu\nu}$
  - But we don't really use gravitation to measure them.
- Analogy to **electromagnetic form factors**.
- Spin-zero example:

$$\langle p' | \hat{J}^{\mu}(0) | p \rangle = 2P^{\mu} F(t)$$

$$\langle p' | \hat{T}^{\mu\nu}(0) | p \rangle = 2P^{\mu} P^{\nu} A(t) + \frac{1}{2}(\Delta^{\mu}\Delta^{\nu} - \Delta^2 g^{\mu\nu}) D(t)$$

- $A(t)$ encodes momentum density
- $D(t)$ encodes stress distributions (anisotropic pressures)
- Mix of both encodes **energy density**

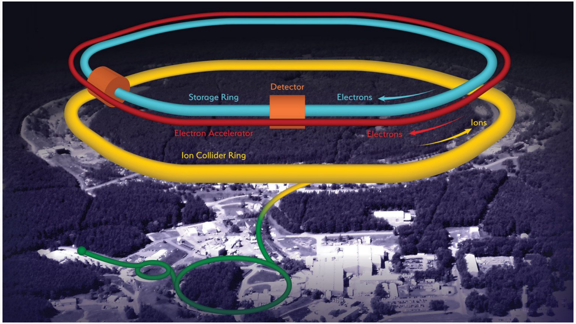


$$P = \frac{1}{2}(p + p')$$

$$\Delta = (p' - p)$$

$$t = \Delta^2$$

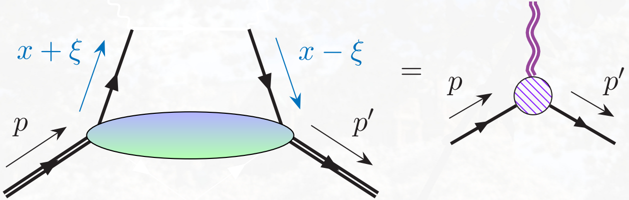
- **Hard exclusive reactions** are used to measure GFFs—not gravity experiments.
  - Deeply virtual Compton scattering (DVCS) to probe quark structure.
  - Deeply virtual meson production (DVMP), *e.g.*, J/ψ or Υ to probe gluon structure.
  - …and more!
- Measured at **Jefferson Lab** and the upcoming **Electron Ion Collider**.



- **Hard exclusive reactions** are used to measure GFFs—not gravitational experiments.
  - Deeply virtual Compton scattering (DVCS) to probe quark structure.
  - Deeply virtual meson production (DVMP), *e.g.*, J/ψ or Υ to probe gluon structure.
  - …and more!
- GFFs are related to **generalized parton distributions** (GPDs) through Mellin moments—spin-zero example:

$$\int_{-1}^{1} \mathrm{d}x\, x H_a(x,\xi,t) = A_a(t) + \xi^2 D_a(t)$$

$$\int_{-1}^{1} \mathrm{d}x\, x$$



Components of the EMT

Energy density

Momentum densities

$$T^{\mu\nu}(x) = \begin{bmatrix} T^{00}(x) & T^{01}(x) & T^{02}(x) & T^{03}(x) \\ T^{10}(x) & T^{11}(x) & T^{12}(x) & T^{13}(x) \\ T^{20}(x) & T^{21}(x) & T^{22}(x) & T^{23}(x) \\ T^{30}(x) & T^{31}(x) & T^{32}(x) & T^{33}(x) \end{bmatrix}$$

Energy fluxes

Stress tensor

- **Angular momentum densities** accessible too:

$$J_i(x) = \epsilon_{ijk}\left(x^j T^{0k}(x) - x^k T^{0j}(x)\right)$$

…basically, from $\mathbf{x} \times \mathbf{p}$

- For physical states, mixture of internal structure & wave packet dependence.
  - Removing wave packet dependence is tricky.
  - Several schemes for dealing with this exist.

Breit frame densities

- **Breit frame** densities most common approach.

$$T^{\mu\nu}_{\mathrm{BF}}(\boldsymbol{x}) \equiv \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{\langle \boldsymbol{q}/2|\hat{T}^{\mu\nu}(\boldsymbol{x})| - \boldsymbol{q}/2\rangle}{2\sqrt{m^2+\boldsymbol{q}^2/4}}$$

  - Original derivation by Sachs erroneous (see Miller, PRC99 (2019) 035202)
  - More recent justification by Lorcé et al., EPJC 79 (2019) 89
- Example: spin-zero energy density and stress tensor

$$\mathcal{E}(\boldsymbol{x}) = m \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{1+\boldsymbol{q}^2/4m^2}} \left\{ A(-\boldsymbol{q}^2) + \frac{\boldsymbol{q}^2}{4m^2}\Big(A(-\boldsymbol{q}^2)+D(-\boldsymbol{q}^2)\Big)\right\} e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$$

$$T^{ij}(\boldsymbol{x}) = m \int \frac{\mathrm{d}^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{1+\boldsymbol{q}^2/4m^2}} \left(\frac{q^i q^j - \boldsymbol{q}^2\delta^{ij}}{2}\right) D(-\boldsymbol{q}^2) e^{-i\boldsymbol{q}\cdot\boldsymbol{x}}$$

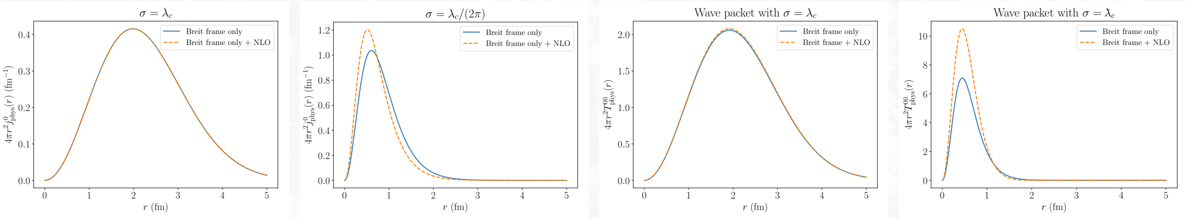
See Polyakov & Schweitzer, Int. J. Mod. Phys. A 33 (2018) for a great review

- Consider hadron as a *medium* & wave function as an arbitrary test function
  - Newer idea due to Yang Li et al., PLB 838 (2023) 137676
  - Hadron has potential to contribute to monopole, dipole, etc. densities.
  - Each of these is an intrinsic property!
- Breit frame density emerges as *leading*, **monopole** term in infinite expansion:

$$T^{\mu\nu}(\boldsymbol{x},t) = \int \mathrm{d}^3\boldsymbol{R}\,\mathcal{P}(\boldsymbol{R},t)T^{\mu\nu}_{\mathrm{BF}}(\boldsymbol{x}-\boldsymbol{R}) + \dots$$

  - Higher-order (e.g. quadrupole) densities negligible if packet width $\gtrsim \lambda_C$.

see AF & Miller, 2210.03807

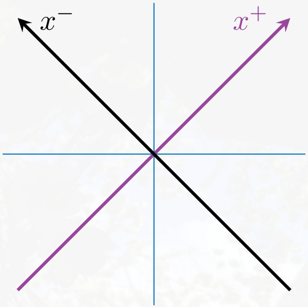


- Stark contrast to the non-relativistic case, where:

$$\rho_{\text{phys}}(\mathbf{x}, t) = \int \mathrm{d}^3\mathbf{R} \left|\Psi_{\text{bar}}(\mathbf{R}, t)\right|^2 \rho_{\text{internal}}(\mathbf{x} - \mathbf{R})$$

- The simplicity owes to the **Galilean symmetry** of non-relativistic physics.
- But the Poincaré group has a Galilean subgroup!
- **Light front coordinates** exploit this subgroup to simplify densities.

Light front coordinates

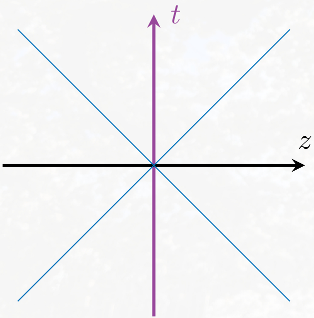


$$x^\pm = t \pm z \qquad \mathbf{x}_\perp = (x, y) \qquad \tau = x^+ = \text{time}$$

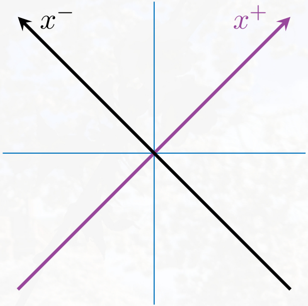
Light front coordinates

- **Light front coordinates** are a different foliation of spacetime.
- Entail a new **synchronization convention**.
- Entail a new spatial grid.

$$x^{\pm} = t \pm z \qquad \mathbf{x}_{\perp} = (x, y) \qquad x^{+} = t + z = \text{time}$$



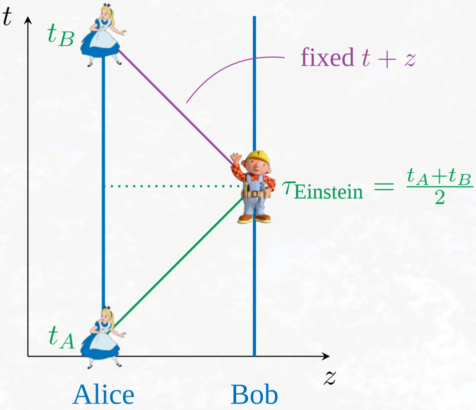
Minkowski coordinates



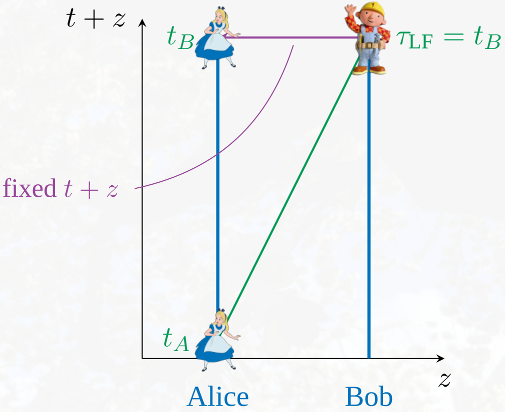
Light front coordinates

Synchronization conventions

Einstein synchronization



Light front synchronization



- **Einstein synchronization** defined to be isotropic.
- **Light front synchronization** defines hyperplanes with fixed $t+z$ to be "simultaneous."
  - Light travels instantaneously in $-z$ direction *by definition*.
  - We take what we see as literally happening now.

- Lorentz-boosted objects *appear rotated*.
  - **Terrell rotation** (PR116, 1959)
  - Optical effect: contraction + delay

- **Light front transverse boost** *undoes* Terrell rotation:

$$B_x^{(\mathrm{LF})} = K_x - J_y$$

  - Standard boost + counter-rotation
  - Leaves x^+ (time) invariant
  - Part of the **Galilean subgroup**

Dice images by Ute Kraus,
`https://www.spacetimetravel.org/`

Galilean subgroup

- Poincaré group has a $(2+1)$D **Galilean subgroup**.
  - x^+ is time and $\mathbf{x}_\perp$ is space under this subgroup.
  - x^- can be integrated out.
  - $P^+ = \frac{1}{\sqrt{2}}(E_\mathbf{p} + p_z)$ is the central charge.
  - x^+ and P^+ are invariant under this subgroup!
- Basically, light front coordinates should give a **fully relativistic** 2D picture that looks like *non-relativistic* physics.
  - But with P^+ in place of M.

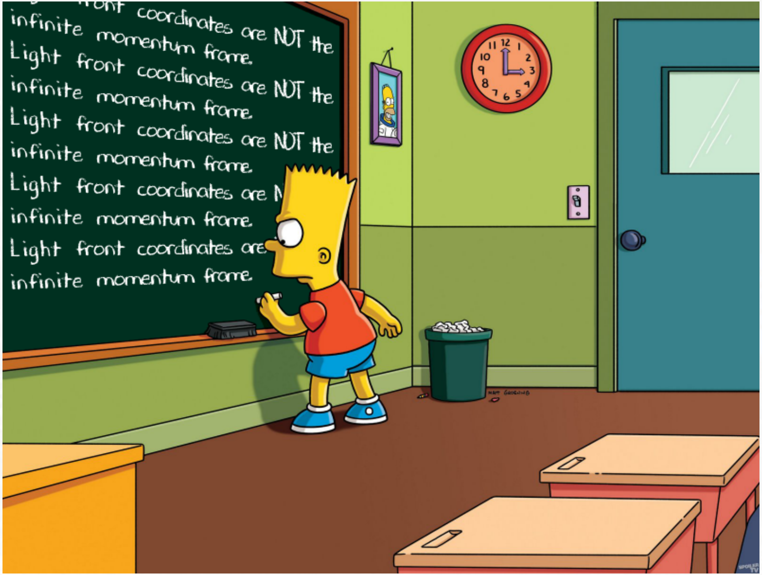
$$\frac{\mathrm{d}\mathbf{P}_\perp}{\mathrm{d}x^+} = P^+ \frac{\mathrm{d}^2\mathbf{x}_\perp}{\mathrm{d}x^{+2}}$$

$$H = P^- = H_{\text{rest}} + \frac{\mathbf{P}_\perp^2}{2P^+}$$

$$\mathbf{v}_\perp = \frac{\mathbf{P}_\perp}{P^+}$$

The cost: lose one spatial dimension (2D densities).

- All momenta can be finite.
- We didn't boost.
- LFCs are not the IMF.



Not the rest frame!

- LFCs are not the IMF.
- They're also not rest frames.
- They're not even Cartesian.
- The reason is x^-.
  - Fixed x^- is lightlike.

Tilted light front coordinates

Tilted coordinates

$$\tilde{\tau} = t + z$$
$$\tilde{x} = x$$
$$\tilde{y} = y$$
$$\tilde{z} = z$$

- Mind the strange metric…

$$\tilde{g}_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

- First defined by Blunden, Burkardt & Miller.
  - Phys. Rev. C61 (2000) 025206
- Use light front time.
  - Use light front synchronization!
  - Time invariant under **Galilean subgroup**.
- Use Cartesian spatial coordinates.
  - Can furnish a **rest frame**!

$$\mathrm{d}s^2 = \mathrm{d}\tilde{\tau}^2 - 2\,\mathrm{d}\tilde{\tau}\,\mathrm{d}\tilde{z} - \mathrm{d}\tilde{\boldsymbol{x}}_\perp^2$$
$$\partial^2 = -2\tilde{\partial}_z\tilde{\partial}_\tau - \tilde{\boldsymbol{\nabla}}^2$$

See AF & Miller, PRD107 (2023) 074036 for full description!

Momentum and velocity

- Energy & momentum are spacetime translation generators.

$$\mathrm{i}[\tilde{E}, \hat{M}] = \frac{\partial \hat{M}}{\partial \tilde{\tau}} \qquad\qquad -\mathrm{i}[\tilde{\boldsymbol{p}}, \hat{M}] = \tilde{\boldsymbol{\nabla}} \hat{M}$$

- On-shell dispersion relation:

$$\tilde{E} = \frac{m^2 + \tilde{\boldsymbol{p}}^2}{2\tilde{p}_z} = \frac{m^2 + \tilde{p}_z^2}{2\tilde{p}_z} + \frac{\tilde{\boldsymbol{p}}_\perp^2}{2\tilde{p}_z}$$

Energy-momentum

$$\tilde{E} = E$$
$$\tilde{p}_x = p_x$$
$$\tilde{p}_y = p_y$$
$$\tilde{p}_z = E + p_x = p^+$$

Velocity

$$\tilde{\boldsymbol{v}} = \boldsymbol{\nabla}_{\boldsymbol{p}} \tilde{E}$$
$$\tilde{v}_x = \tilde{p}_x / \tilde{p}_z$$
$$\tilde{v}_y = \tilde{p}_y / \tilde{p}_z$$
$$\tilde{v}_z = 1 - \tilde{E} / \tilde{p}_z$$

- **Rest** occurs when $\tilde{\boldsymbol{v}} = 0$.

Energy-momentum tensor in tilted coordinates

Energy density

Momentum densities

$$\tilde{T}^{\mu}{}_{\nu}(x) = \begin{bmatrix} \tilde{T}^{0}{}_{0}(x) & \tilde{T}^{0}{}_{1}(x) & \tilde{T}^{0}{}_{2}(x) & \tilde{T}^{0}{}_{3}(x) \\ \tilde{T}^{1}{}_{0}(x) & \tilde{T}^{1}{}_{1}(x) & \tilde{T}^{1}{}_{2}(x) & \tilde{T}^{1}{}_{3}(x) \\ \tilde{T}^{2}{}_{0}(x) & \tilde{T}^{2}{}_{1}(x) & \tilde{T}^{2}{}_{2}(x) & \tilde{T}^{2}{}_{3}(x) \\ \tilde{T}^{3}{}_{0}(x) & \tilde{T}^{3}{}_{1}(x) & \tilde{T}^{3}{}_{2}(x) & \tilde{T}^{3}{}_{3}(x) \end{bmatrix}$$

Energy fluxes

Stress tensor

- All 16 components of EMT have clear meaning in tilted coordinates.
- The **energy density** integrates to the usual “instant form” energy.

$$\tilde{E} = E$$

  - *Relativistically exact* energy density.
  - Will give standard mass decomposition.
  - Can describe system at rest.

Smearing functions

- Physical energy-momentum tensor:

$$\int \mathrm{d}\tilde{z}\, \langle \Psi | \hat{T}^{\mu}{}_{\nu}(x) | \Psi \rangle = \int \mathrm{d}^3 \tilde{\boldsymbol{R}}\, \mathscr{P}^{\mu}{}_{\nu\alpha}{}^{\beta}(\tilde{\boldsymbol{R}}, \tilde{\tau}, \Psi) [\tilde{T}^{\alpha}{}_{\beta}(\tilde{\boldsymbol{x}}_\perp - \tilde{\boldsymbol{R}}_\perp)]_{\text{internal}}$$

Internal density

Smearing function

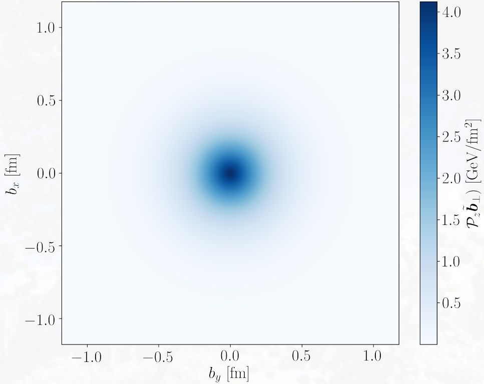
invariant under LF boosts

  - **Smearing function** contains all wave packet & velocity dependence.
  - Only **smearing function** modified by Lorentz boosts.
  - **Internal density** is boost-invariant. (due to Galilean subgroup)
  - **Internal density** is rest frame density!
- Galilean subgroup allows such a separation.
  - Multiple separations exist.
  - Will describe scheme for separation in upcoming work.
- I'm short on time; will just give results for **internal densities**.

- $-\tilde{T}^0{}_3 = T^{++}$ gives $\tilde{P}_z = P^+$ density

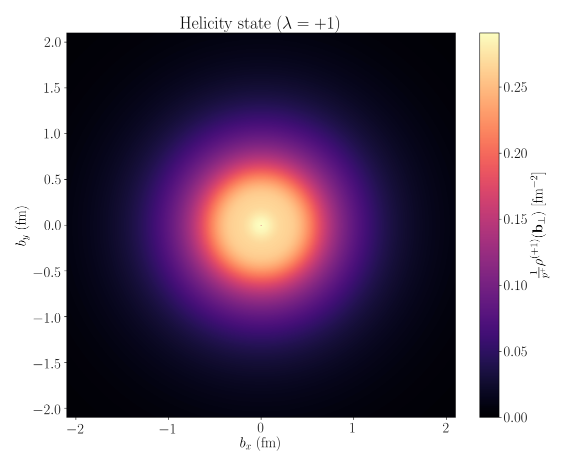
$$\rho^{(\mathrm{LF})}_{P^+}(\mathbf{b}_\perp, \mathbf{s}_\perp) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \frac{\langle p', \mathbf{s}_\perp | T^{++}(0) | p, \mathbf{s}_\perp \rangle}{2m} e^{-i\boldsymbol{\Delta}_\perp \cdot \mathbf{b}_\perp}$$

  - Set $\tilde{P}_z = P^+ = m$ for rest.
  - Works for any polarization state.
  - Structure *relative to* center-of-P^+.
  - Boost invariance: $\mathbf{P}_\perp$ independent!
- Proton dipole model on right.
  - $f_2(1270)$ pole
  - Agrees with Kharzeev's analysis in RPD104 (2021) 054015
  - 2D radius of 0.45 fm
- Get back standard P^+ density!

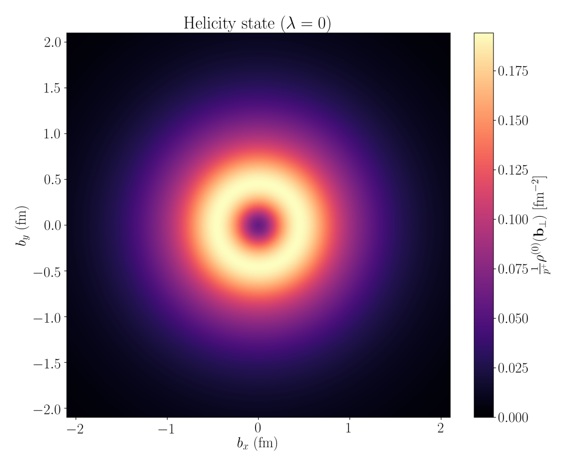


Spin-one targets

Helicity +1



Helicity 0

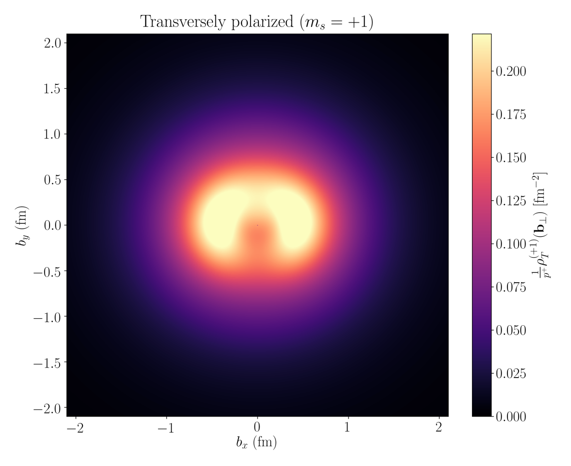


$\tilde{P}_z = P^+$ density depends on helicity for spin-one targets.

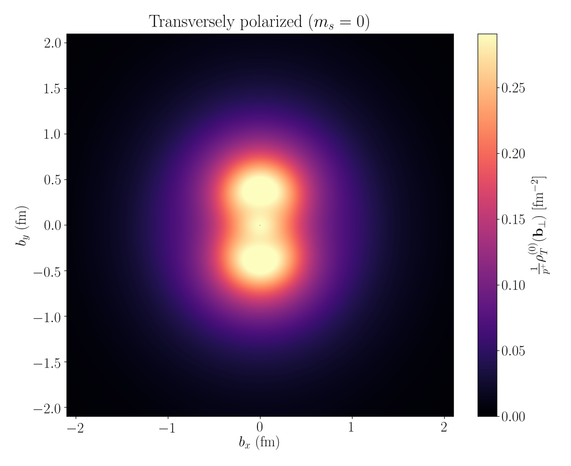
AF & Wim Cosyn, PRD**106** (2022) 114013

Transverse polarization

Transverse, $m_s = +1$



Transverse, $m_s = 0$



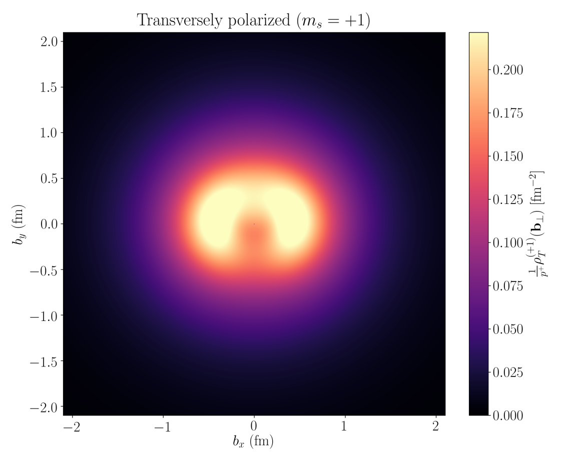
Transverse polarization contains helicity-flip contributions.

AF & Wim Cosyn, PRD**106** (2022) 114013

Why sin ϕ modulations?

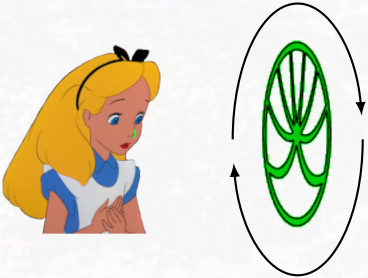
- P^+ density of transpol. deuteron.
  - Spin-up along x-axis
  - But left-handed coords.
- This is the P^+ density in every frame.
  - Includes the rest frame.
- Not an IMF artifact!
  - Never boosted to IMF.
- Effect of **synchronization scheme**.
  - Effect of taking what we see literally.
  - This is a known effect; relativistic wheel.
  - Explained by George Gamow in 1938, *Mr Tompkins in Wonderland*

Trans. pol. deuteron



Static wheel

Spinning wheel

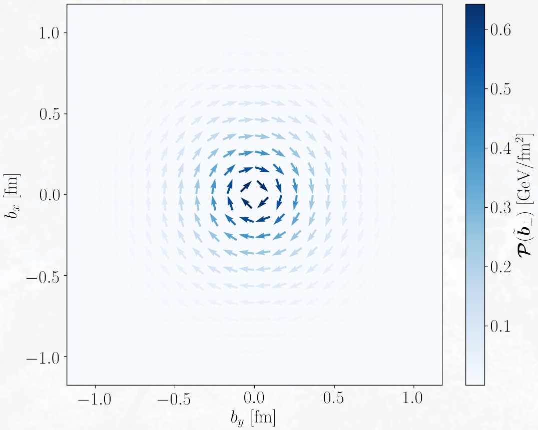


- **Static wheel** has regularly-placed spokes.
- Consider **spinning wheel**, axis oblique to observer.
  - *The wheel is considered at rest.*
- Spokes moving away are **redshifted**.
  - *Appear to* move slower.
  - Pile up; *appear to* become denser.
- Spokes moving towards are **blueshifted**.
  - *Appear to* move faster.
  - *Appear to* become rarer.
- These same distortions are present in nuclei!
  - **Light front densities bake in optical effects.**
- Also see videos at: `https://www.spacetimetravel.org/rad` (green wheel is relevant case)

Transverse momentum density

$$\mathcal{P}_\perp^{(\mathrm{LF})}(\mathbf{b}_\perp, \mathbf{s}_\perp) = \int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \frac{\langle p', \mathbf{s}_\perp | T^{+i}(0) | p, \mathbf{s}_\perp \rangle}{2m} e^{-i\boldsymbol{\Delta}_\perp \cdot \mathbf{b}_\perp} \Big|_{\boldsymbol{P}_\perp = 0, P^+ = m}$$

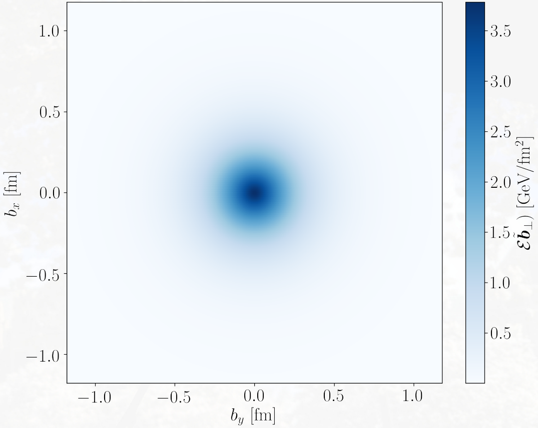
- $-\tilde{T}^0{}_i = T^{+i}$ gives $\tilde{\boldsymbol{P}}_\perp = \boldsymbol{P}_\perp$ density
  - Works for any polarization state.
  - Structure *relative to* center-of-P^+.
- Proton dipole model on right
  - $f_2(1270)$ pole.
  - Longitudinal polarization

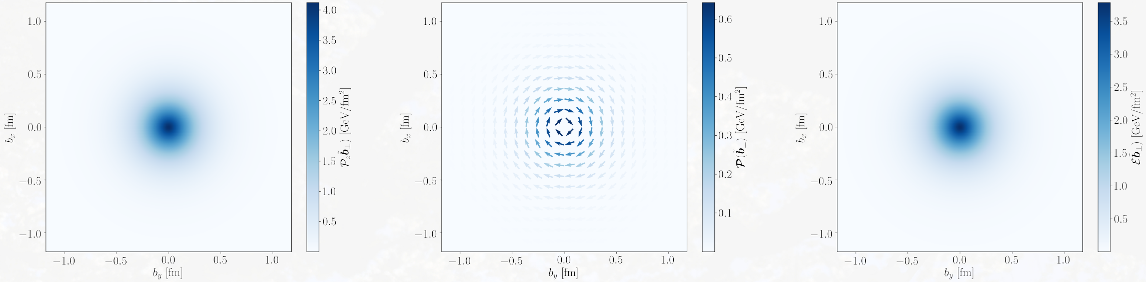


Energy density

$$\mathcal{E}_{\perp}^{(\mathrm{LF})}(\mathbf{b}_\perp, \mathbf{s}_\perp) = \frac{1}{2}\int \frac{\mathrm{d}^2\boldsymbol{\Delta}_\perp}{(2\pi)^2} \frac{\langle p', \mathbf{s}_\perp | T^{++}(0) + T^{+-}(0) | p, \mathbf{s}_\perp\rangle}{2m} e^{-i\boldsymbol{\Delta}_\perp\cdot\mathbf{b}_\perp}\Big|_{\boldsymbol{P}_\perp=0, P^+=m}$$

- $-\tilde{T}^0{}_i = T^{+i}$ gives $\tilde{\boldsymbol{P}}_\perp = \boldsymbol{P}_\perp$ density
  - Works for any polarization state.
  - Showing light front helicity state.
  - Structure *relative to* center-of-P^+.
- Proton dipole model on right
  - $f_2(1270)$ pole.
  - Longitudinal polarization
  - Use $D(0) = -2$ [from lattice]
    [Pefkou et al., PRD 105 (2022)]
  - $D(t)$ has extra σ pole
  - Radius 0.53 fm > 0.45 fm





- The **energy-momentum tensor** encodes interesting internal properties of hadrons.
  - Energy & momentum densities among them
  - Also stresses, but I didn't have time to cover these
- There are subtleties in how to identify "internal" properties here.
  - Tilted light front coordinates allow exact relativistic rest frame densities

…and, most importantly:

Thanks for your time and attention!