Covariant framework to parametrize realistic deuteron wave functions

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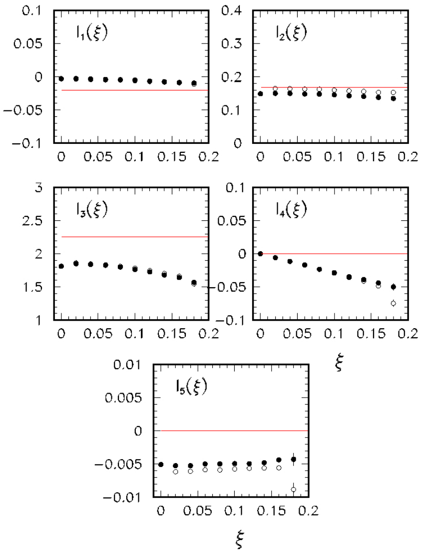
Introduction

- **Original idea**: a covariant and separable but non-local model of nucleon-nucleon interactions.
  - Solve for deuteron from Bethe-Salpeter equation.
  - Calculate deuteron observables in manifestly covariant way.
  - Get generalized parton distributions that obey polynomiality.
- **Modified idea**: the formalism of the original idea can encode approximate parametrization of **realistic wave functions**.
  - Get **manifest covariance** (and GPD polynomiality) with existing, precision wave functions!
  - I use Argonne V18 as an example.
- I'll explain the original idea first, and then how I adapted the framework to get covariant results from AV18.

- Generalized parton distributions exhibit **polynomiality**.

$$\int \mathrm{d}x\, x H_1(x,\xi,t) = \mathcal{G}_1(t) + \xi^2 \mathcal{G}_3(t) \qquad \text{etc.}$$

  - Required for unambiguous extraction of energy-momentum tensor from GPDs.
- Polynomiality requires covariance.
  - X. Ji, J. Phys. G24 (1998) 1181
- Finite Fock expansion (standard method) violates covariance.
  - Example: landmark calculation of Cano and Pire EPJA 19 (2004) 423



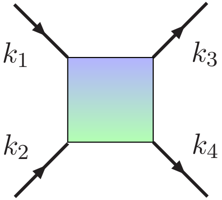
- Adapted from **non-local NJL model**.
  - Bowler & Birse, Nucl. Phys. A582 (1995) 655
  - Modified to be a nucleon-nucleon interaction.
- V and T currents in *isosinglet* channel:

$$B_V^{\mu}(x) = \frac{1}{2}\int \mathrm{d}^4 z\, F(z)\psi^{\mathsf{T}}\left(z+\frac{z}{2}\right)C^{-1}\tau_2\gamma^{\mu}\psi\left(z-\frac{z}{2}\right)$$
$$B_T^{\mu\nu}(x) = \frac{1}{2}\int \mathrm{d}^4 z\, F(z)\psi^{\mathsf{T}}\left(z+\frac{z}{2}\right)C^{-1}\tau_2\, \mathrm{i}\sigma^{\mu\nu}\psi\left(z-\frac{z}{2}\right)$$

  - $F(z)$ a spacetime form-factor; regulates UV divergences.
  - C is charge conjugation matrix.
  - τ_2 isospin matrix.
- Interaction Lagrangian:

$$\mathscr{L}_I = g_V B_V^{\mu}(B_{V\mu})^* + \frac{1}{2} g_T B_T^{\mu\nu}(B_{T\mu\nu})^*$$

- Momentum-space Feynman rule for interactions:



$$= \left\{ g_V \gamma^\mu C \otimes C^{-1}\gamma_\mu + \frac{g_T}{2}\sigma^{\mu\nu} C \otimes C^{-1}\sigma_{\mu\nu} \right\} \tilde{F}(k_1 - k_2)\tilde{F}(k_3 - k_4)$$

  - **Separable interaction**: initial & final momentum dependence factorize.
  - (isospin dependence suppressed to compactify formula)
- $\tilde{F}(k)$ is Fourier transform of $F(z)$, chosen:

$$\tilde{F}(k) \equiv \frac{\Lambda}{k^2 - \Lambda^2 + \mathrm{i}0}$$

  - Λ is the regulator scale (non-locality scale).

Quantum numbers in kernel

- Kernel encodes channels with multiple quantum numbers:

$$\gamma^\mu C \otimes C^{-1}\gamma_\mu = \left(\gamma^\mu - \frac{\not p p^\mu}{p^2}\right) C \otimes C^{-1} \left(\gamma_\mu - \frac{\not p p_\mu}{p^2}\right) + \frac{1}{p^2}\not p C \otimes C^{-1} \not p$$

spin-one   spin-zero

$$\sigma^{\mu\nu} C \otimes C^{-1}\sigma_{\mu\nu} = \frac{1}{p^2}\sigma^{\mu p} C \otimes C^{-1}\sigma_{\mu p} + \left(\sigma^{\mu\nu} - \frac{\sigma^{\mu p}p^\nu - \sigma^{\nu p}p^\mu}{p^2}\right) C \otimes C^{-1} \left(\sigma^{\mu\nu} - \frac{\sigma^{\mu p}p^\nu - \sigma^{\nu p}p^\mu}{p^2}\right)$$

even parity   odd parity

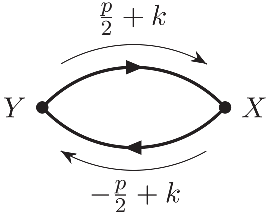
  - p is center-of-mass momentum (deuteron momentum)
- Need only structures with deuteron quantum numbers:

$$\gamma_V^\mu \equiv \gamma^\mu - \frac{\not p p^\mu}{p^2} \qquad\qquad \gamma_T^\mu \equiv \frac{\mathrm{i}\sigma_{\mu p}}{\sqrt{p^2}}$$

  - Other structures fully decouple in the T-matrix equation!

- Bubble diagrams defined via:

$$-\,\mathrm{i}\left(g^{\mu\nu}-\frac{p^{\mu}p^{\nu}}{p^{2}}\right)\Pi_{XY}(p^{2})=$$

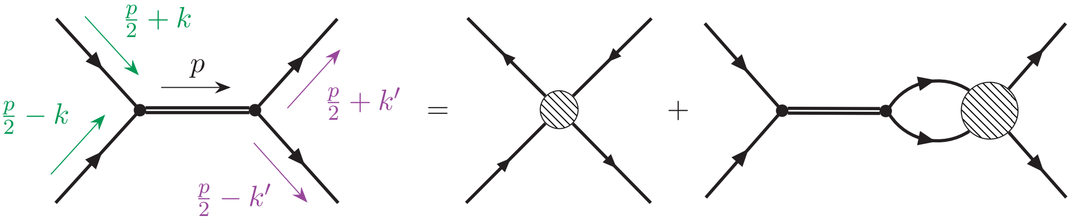


  - Either γ_V^{μ} or γ_T^{μ} can be on either end.
  - The regulator $\tilde{F}(k)$ appears twice inside loop integral—makes it UV finite.
- Can define bubble matrix:

$$\Pi=\begin{bmatrix}\Pi_{VV}(p^{2}) & \Pi_{VT}(p^{2})\\ \Pi_{TV}(p^{2}) & \Pi_{TT}(p^{2})\end{bmatrix}$$

  - Essential ingredient in calculations to follow.

- Bethe-Salpeter equation (BSE) for T-matrix given by:



- **Separability** of interaction permits a simple matrix form:

$$T = G - G\Pi T$$

  - Actual T-matrix related to simplified matrix via:

$$\mathcal{T}(p,k,k') = \frac{\Lambda}{k^2-\Lambda^2}\frac{\Lambda}{k'^2-\Lambda^2}\left\{T_{11}\gamma_V\otimes\gamma_V + T_{12}\gamma_V\otimes\gamma_T + T_{21}\gamma_T\otimes\gamma_V + T_{22}\gamma_T\otimes\gamma_T\right\}$$

  - Simplified kernel matrix:

$$G = \begin{bmatrix} g_V & 0 \\ 0 & g_T \end{bmatrix}$$

Deuteron bound state pole

- T-matrix solution given by:

$$T = (1 + G\Pi)^{-1} G$$

- Deuteron bound state pole exists where:

$$\det(1 + G\Pi) = 0$$

  - Use physical deuteron mass to fix g_V in terms of Λ and g_T.
- Residues at this pole give reduced form of deuteron vertex:

$$T(p^2 \approx M_{\mathrm{D}}^2) \approx -\frac{1}{p^2 - M_{\mathrm{D}}^2} \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix}$$

  - α and β are coefficients in deuteron Bethe-Salpeter vertex.
  - The k and k' dependence is fixed and **separable**.

Deuteron vertex

- The result of all this is a deuteron Bethe-Salpeter vertex:

$$\Gamma^{\mu}_{\rm D}(p,k) = \frac{\Lambda}{k^2 - \Lambda^2 + {\rm i}0}\Big\{\alpha\gamma^{\mu}_{V} + \beta\gamma^{\mu}_{T}\Big\}C\tau_2$$

  - Simple k dependence fixed by separable interaction.
  - Can be used to **covariantly** calculate all sorts of observables.
- Relationship to fundamental model parameters:

$$(\Lambda, g_V, g_T) \to (M_{\rm D}, \alpha, \beta)$$

  - Eliminate one model parameter by fixing $M_{\rm D}$ to empirical value.
  - Could fix other parameters via observables, e.g., charge radius & quadrupole moment.
- A curious thing happens if we look at the non-relativistic limit …

Non-relativistic reduction

- Non-relativistic, momentum-space wave function:

$$\psi_{\mathrm{NR}}(\boldsymbol{k},\lambda) \sim \frac{-1}{\sqrt{8M_{\mathrm{D}}}} \frac{\bar{u}(\boldsymbol{k},s_1)(\Gamma_{\mathrm{D}}\cdot\varepsilon_\lambda)\bar{u}^{\mathsf{T}}(-\boldsymbol{k},s_2)}{\boldsymbol{k}^2+m\epsilon_{\mathrm{D}}}$$

- Working out the Dirac matrix algebra and using the limit $\boldsymbol{k}^2 \ll m^2$ will give:

$$\psi_{\mathrm{NR}}(\boldsymbol{k},\lambda) = 4\pi\left\{u(k)Y^{\lambda}_{101}(\hat{k}) + w(k)Y^{\lambda}_{121}(\hat{k})\right\}$$

$$u(k) = \sum_{j=0}^{1}\frac{C_j}{\boldsymbol{k}^2+B_j^2}$$

$$w(k) = \sum_{j=0}^{1}\frac{D_j}{\boldsymbol{k}^2+B_j^2}$$

$$B_0 = \sqrt{m\epsilon_{\mathrm{D}}}$$

$$B_1 = \Lambda$$

$$C_0 = \frac{m}{\sqrt{4\pi M_{\mathrm{D}}}}\frac{\Lambda}{\Lambda^2-\epsilon_{\mathrm{D}}m}\left(\alpha+\beta-\frac{(\alpha-\beta)\epsilon_{\mathrm{D}}m}{12m^2}\right)$$

$$C_1 = -\frac{m}{\sqrt{4\pi M_{\mathrm{D}}}}\frac{\Lambda}{\Lambda^2-\epsilon_{\mathrm{D}}m}\left(\alpha+\beta-\frac{(\alpha-\beta)\Lambda^2}{12m^2}\right)$$

$$D_0 = -\frac{m}{\sqrt{4\pi M_{\mathrm{D}}}}\frac{\Lambda}{\Lambda^2-\epsilon_{\mathrm{D}}m}\frac{\sqrt{2}(\alpha-\beta)\epsilon_{\mathrm{D}}m}{6m^2}$$

$$D_1 = \frac{m}{\sqrt{4\pi M_{\mathrm{D}}}}\frac{\Lambda}{\Lambda^2-\epsilon_{\mathrm{D}}m}\frac{\sqrt{2}(\alpha-\beta)\Lambda^2}{6m^2}$$

  - $u(k)$ is S-wave, $w(k)$ is D-wave.

Approximating non-relativistic wave functions

- The curious thing is that this is a standard parametrization for deuteron wave functions!

$$u(k) = \sum_{j=0}^{N} \frac{C_j}{\boldsymbol{k}^2 + B_j^2} \qquad w(k) = \sum_{j=0}^{N} \frac{D_j}{\boldsymbol{k}^2 + B_j^2}$$

  - First used by Paris group, Lacombe *et al.*, PLB 101 (1981) 139
  - Typically $N > 1$ of course.
- One requires $B_0 = \sqrt{\epsilon_{\mathrm{D}} m}$ to get the right asymptotic behavior. (**Check!**)
- One also requires the following sum rules for correct behavior at the origin:

$$\sum_{j=0}^{N} C_j = \sum_{j=0}^{N} D_j = \sum_{j=0}^{N} D_j B_j^{-2} = \sum_{j=0}^{N} D_j B_j^2 = 0$$

  - Model as given **fails** unless $\alpha = \beta$, meaning no D wave.
- But we can fix this by having N copies of the separable kernel!

Making N copies of the separable kernel

- Just have N copies of the original separable interaction with different Λ_n:

$$\mathcal{K}(k,k') = \sum_{n=1}^{N} \frac{\Lambda_n}{k^2 - {\Lambda_n}^2 + \mathrm{i}0} \frac{\Lambda_n}{k'^2 - {\Lambda_n}^2 + \mathrm{i}0} \left\{ g_{Vn} \gamma^\mu C \otimes C^{-1} \gamma_\mu + \frac{g_{Tn}}{2} \sigma^{\mu\nu} C \otimes C^{-1} \sigma_{\mu\nu} \right\}$$

  - Kernel now has $3N$ parameters: $\{\Lambda_n, g_{Vn}, g_{Tn} | n \in \{1,\ldots,N\}\}$.
- Simplified forms of kernel, T-matrix, and bubble are now all $2N \times 2N$ matrices.
  - The different Λ_n mix, but the T-matrix equation is still separable and can be solved algebraically.
- Deuteron vertex is now:

$$\Gamma^\mu_{\mathrm{D}}(p,k) = \sum_{n=1}^{N} \frac{\Lambda_n}{k^2 - {\Lambda_n}^2 + \mathrm{i}0} \left\{ \alpha_n \gamma^\mu_V + \beta_n \gamma^\mu_T \right\} C\tau_2$$

  - Non-relativistic reduction now has $N+1$ terms in S and D waves!

Using the separable kernel as a parametrization

- The popular non-relativistic parametrization is:

$$u(k) = \sum_{j=0}^{N} \frac{C_j}{\boldsymbol{k}^2 + B_j^2} \qquad w(k) = \sum_{j=0}^{N} \frac{D_j}{\boldsymbol{k}^2 + B_j^2}$$

  - Here $B_0 = \sqrt{\epsilon_\mathrm{D} m}$ and $B_n = \Lambda_n$ (for $n > 0$).
- From the kernel coupling strengths:

$$\{g_{Vn}\}, \{g_{Tn}\} \to \{\alpha_n\}, \{\beta_n\} \to \{C_j\}, \{D_j\}$$

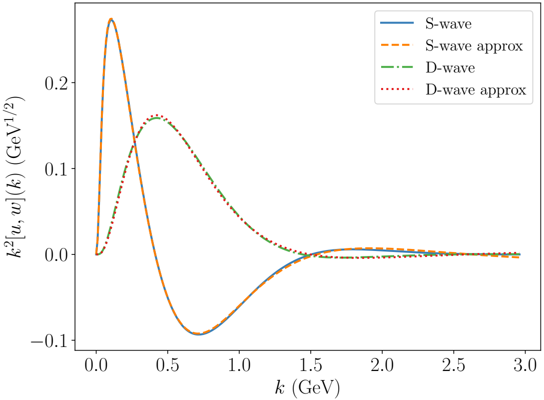
  - Won't fill up a slide with all the formulas (see preprint when it comes out).
- The formulas are linear & invertible!

$$\{C_j\}, \{D_j\} \to \{\alpha_n\}, \{\beta_n\} \to \{g_{Vn}\}, \{g_{Tn}\}$$

- So why not **start with** the C_j, D_j and B_j from a **well-established wave function**?
  - Automatically get precision of established wave function.
  - Get correct constraints by starting with $\{B_j, C_j, D_j\}$ that obey them.
  - Guaranteed **Lorentz covariance** from using separable framework.

Approximating Argonne V18

- Example: Argonne V18 fit using $N = 7$.
  - Get $u(k)$ and $w(k)$ from ANL website*.
  - B_n $(n > 0)$ were allowed to float in fit.
  - $B_0 = \sqrt{\epsilon_{\text{D}} m}$.
  - $r \to 0$ constraints were enforced.
- From $\{B_j, C_j, D_j\}$ get $\{\Lambda_n, g_{Vn}, g_{T,n}\}$.
  - See future preprint for numerical values!



- Now have **covariant Lagrangian** that reproduces AV18 wave function in NR limit!

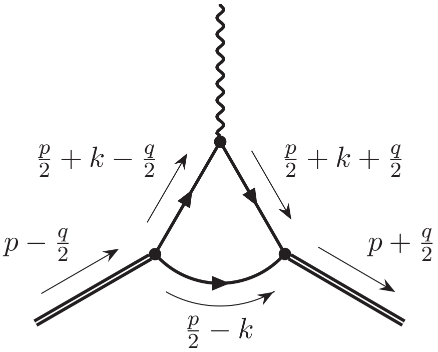
*: https://www.phy.anl.gov/theory/research/av18/

Things to do with this framework

- Electromagnetic form factors (**obtained!**)
  - Still need to check whether bicycle diagram contributes at $Q^2 > 0$.
- Gravitational form factors (in progress)
  - Manifest covariance helpful here.
  - Previous non-covariant work (AF & Cosyn, PRD) found inconsistencies in EMT components.
- Collinear parton distributions (**obtained!**)
- b_1 structure function (**obtained!**)
- Generalized parton distributions (in progress)
  - GPDs are the **main goal** of this project.
  - Existing deuteron GPDs violate polynomiality.
  - Manifest covariance of this framework *guarantees* polynomiality.

Electromagnetic form factors

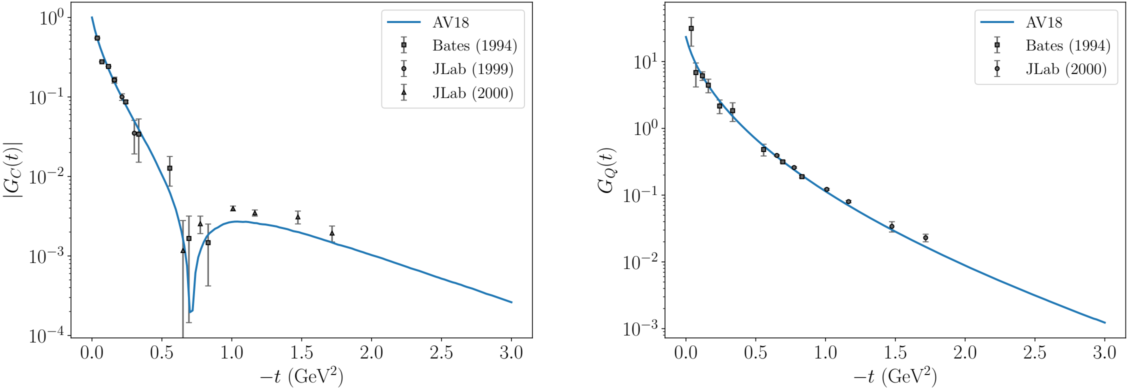
- Triangle diagrams for electromagnetic form factors:



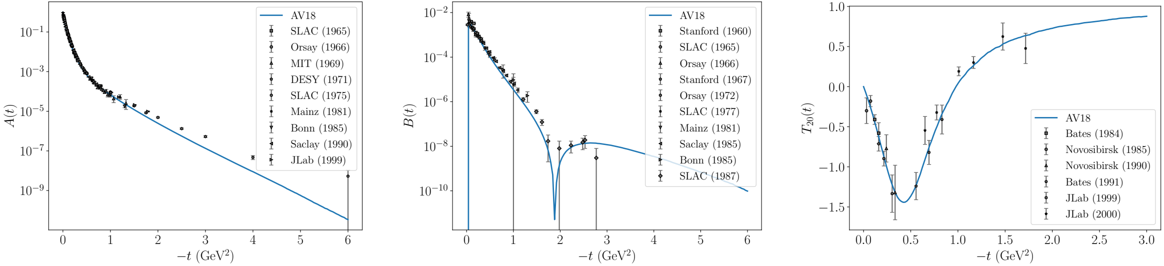
$$= \begin{array}{l} -2p^\mu(\varepsilon \cdot \varepsilon'^*)G_1(q^2) \\ + \left[\varepsilon'^{*\mu}(\varepsilon \cdot q) - \varepsilon^\mu(\varepsilon'^* \cdot q)\right] G_2(q^2) \\ + \frac{p^\mu}{M_{\mathrm{D}}^2}(\varepsilon \cdot q)(\varepsilon'^* \cdot q) G_3(q^2) \end{array}$$

- This diagram can be evaluated *exactly* within the present framework!
  - Symbolic algebra program needed though—results are *long* (hundreds of lines of generated Fortran code)
  - Results are covariant too.

Coulomb and quadrupole form factors



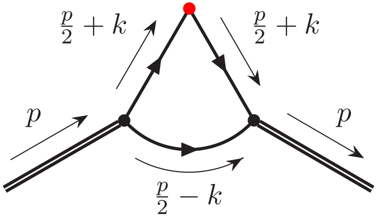
- Charge sum rule $G_C(0) = 1$ ensured by relating α_n & β_n to T-matrix residue.
- Charge radius: 2.156 fm (empirical: 2.12799 fm)
- Quadrupole moment: 0.259 fm^2 (empirical: 0.2859 fm^2)



- Magnetic moment: $0.874\ \mu_N$ (empirical: $0.8574382284\ \mu_N$)
- AV18 is already known to describe these well.
- This is basically a sanity check for the covariant framework.

Light cone density: triangle diagram

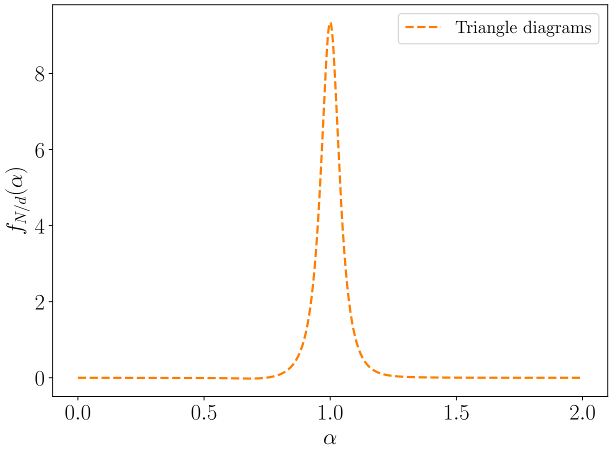
- Triangle diagrams for light cone density:

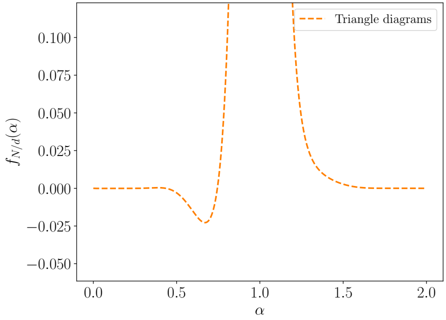


$$= -(\varepsilon \cdot \varepsilon'^*) f_1^{(\mathrm{tri})}(\alpha) + \left(M_{\mathrm{D}}^2 \frac{(\varepsilon \cdot n)(\varepsilon'^* \cdot n)}{(p \cdot n)^2} + \frac{1}{3}(\varepsilon \cdot \varepsilon'^*) \right) b_1^{(\mathrm{tri})}(\alpha)$$

  - $0 < \alpha < 2$ for A-scaled light cone fraction.
  - Diagram can be evaluated using residue theorem.
- $f_1(\alpha)$: unpolarized density; $b_1(\alpha)$: tensor-polarized density.

Triangle diagram versus momentum sum rule





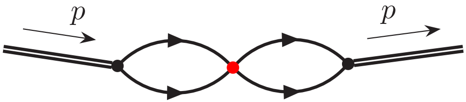
$$\sum_{N=p,n} \int_0^2 \mathrm{d}\alpha\, f_{N/D}^{(\mathrm{tri})}(\alpha) = 2$$

$$\sum_{N=p,n} \int_0^2 \mathrm{d}\alpha\, \alpha f_{N/D}^{(\mathrm{tri})}(\alpha) \approx 2.005$$

- Slight asymmetry in distribution causes violation.
- Problem is we've missed a diagram!

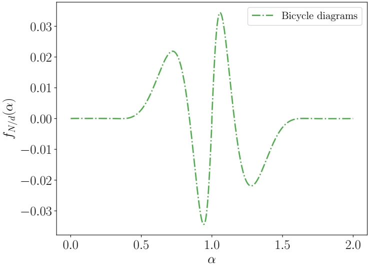
Light cone density: bicycle diagram

- Bicycle diagrams for light cone density:



$$= -(\varepsilon\cdot\varepsilon'^*)\,f_1^{(\mathrm{bi})}(\alpha) + \left(M_{\mathrm{D}}^2\frac{(\varepsilon\cdot n)(\varepsilon'^*\cdot n)}{(p\cdot n)^2} + \frac{1}{3}(\varepsilon\cdot\varepsilon'^*)\right) b_1^{(\mathrm{bi})}(\alpha)$$

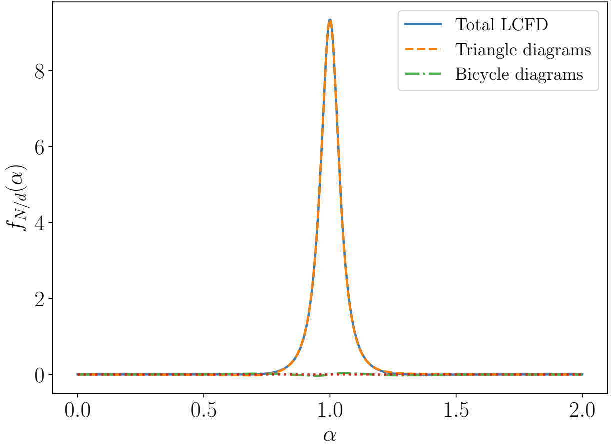
  - Vertex carries energy/momentum: **equivalence principle**.
  - Feynman rule derived using local spacetime translations, cf. AF, PRD106 (2022) 125012
  - **Having the Lagrangian** to which Γ^μ_{D} is a solution was necessary for derivation!
  - Diagram can be evaluated using residue theorem.

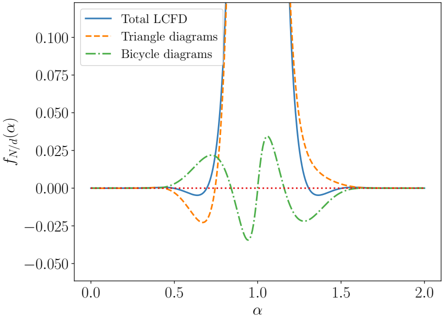


$$\sum_{N=p,n}\int_0^2 \mathrm{d}\alpha\, f_{N/D}^{(\mathrm{bi})}(\alpha) = 0$$

$$\sum_{N=p,n}\int_0^2 \mathrm{d}\alpha\, \alpha\, f_{N/D}^{(\mathrm{bi})}(\alpha) \approx -0.005$$

Light cone density: sum rules redeemed





$$\sum_{N=p,n} \int_0^2 \mathrm{d}\alpha \, f_{N/D}(\alpha) = 2$$

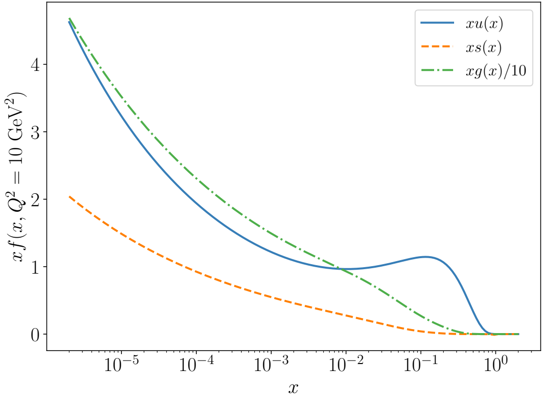
$$\sum_{N=p,n} \int_0^2 \mathrm{d}\alpha \, \alpha f_{N/D}(\alpha) = 2$$

- Slight negative support still.
  - Either a flaw with the framework …
  - …or a feature of renormalization?
    cf. Collins, Rogers & Sato, PRD105 (2022)

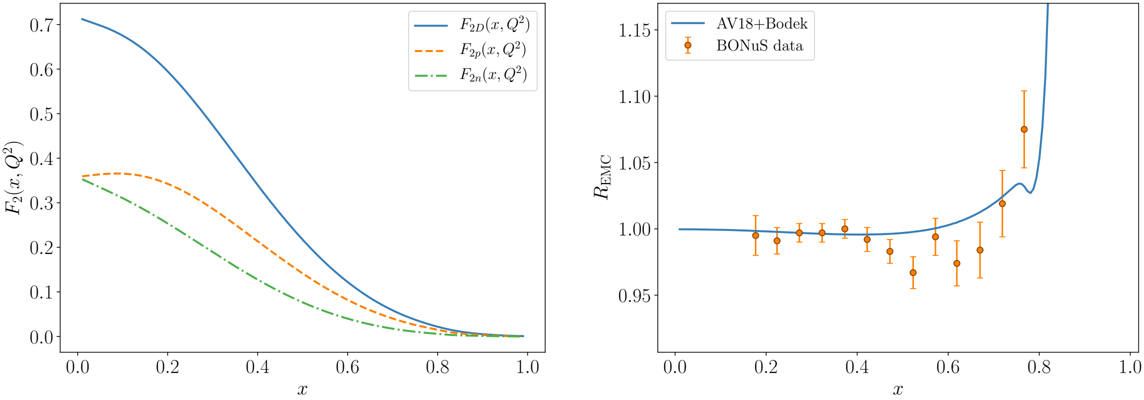
- Deuteron PDFs via convolution:

$$q_{\mathrm{D}}(x, Q^2) = \sum_{N=p,n} \int_x^2 \mathrm{d}\alpha\, q_N\left(\frac{x}{\alpha}\right) f_{N/\mathrm{D}}(\alpha)$$

- Use JAM PDFs for nucleon.
  - C. Cocuzza *et al.*, PRD106 (2022) L031502

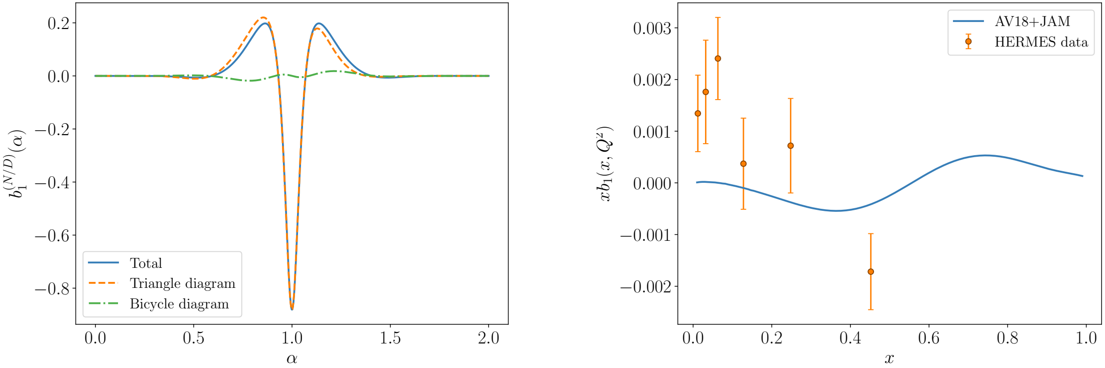


Structure function



- Bodek-Richtie parametrization for proton & neutron F_2, PRD23 (1981) 1070
- BONuS data: Griffioen *et al.*, PRC92 (2015) 015211
- Don't get an EMC effect with this framework.
  - Not surprising: known that modified nucleons are necessary.
  - See Miller & Smith, PRC65 (2002)

Tensor-polarized structure function



- HERMES data: PRL95 (2005) 242001
- Bicycle diagram **needed** for symmetry & sum rules.
  - $\int \mathrm{d}\alpha\, b_1(\alpha) = \int \mathrm{d}\alpha\, \alpha b_1(\alpha) = 0$
  - Sum rules entailed by Lorentz covariance!
- Cannot describe HERMES data.
  - Not surprising: this is a fancier convolution formalism.
  - See Cosyn *et al.*, PRD95 (2017) 074036

Outlook

- Presented a framework for **covariant calculations** using **realistic deuteron wave functions**.
  - Used Argonne's AV18 wave function as an example.
- Reproduced known deuteron properties in this framework
  - Not new or exciting, but a necessary sanity check.
  - Learned an important lesson: **bicycle diagrams** must be accounted for!
- Much more to be done:
  - Energy momentum tensor and gravitational form factors.
  - **Generalized parton distributions** (the main purpose of this project!)

Thank you for your time!