Covariant framework to parametrize realistic deuteron wave functions

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January 19, 2024
Introduction

- **Original idea**: a covariant and separable but non-local model of nucleon-nucleon interactions.
  - Solve for deuteron from Bethe-Salpeter equation.
  - Calculate deuteron observables in manifestly covariant way.
  - Get generalized parton distributions that obey polynomiality.

- **Modified idea**: the formalism of the original idea can encode approximate parametrization of realistic wave functions.
  - Get *manifest covariance* (and GPD polynomiality) with existing, precision wave functions!
  - I use Argonne V18 as an example.

- I’ll explain the original idea first, and then how I adapted the framework to get covariant results from AV18.
Generalized parton distributions exhibit **polynomiality**.

\[
\int dx \, x H_1(x, \xi, t) = G_1(t) + \xi^2 G_3(t) \quad \text{etc.}
\]

- Required for unambiguous extraction of energy-momentum tensor from GPDs.

- Polynomiality requires covariance.

- Finite Fock expansion (standard method) violates covariance.
Adapted from **non-local NJL model.**

- Modified to be a nucleon-nucleon interaction.

**V** and **T** currents in *isosinglet* channel:

\[
B^\mu_V(x) = \frac{1}{2} \int \text{d}^4 z \, f(z) \psi^\dagger \left( z + \frac{z}{2} \right) C^{-1} \tau_2 \gamma^\mu \psi \left( z - \frac{z}{2} \right)
\]

\[
B^\mu_\nu_T(x) = \frac{1}{2} \int \text{d}^4 z \, f(z) \psi^\dagger \left( z + \frac{z}{2} \right) C^{-1} \tau_2 i \sigma^\mu_\nu \psi \left( z - \frac{z}{2} \right)
\]

- \( f(z) \) a spacetime form-factor; regulates UV divergences.
- \( C \) is charge conjugation matrix.
- \( \tau_2 \) isospin matrix.

**Interaction Lagrangian:**

\[
\mathcal{L}_I = g_V B^\mu_V (B_{V\mu})^* + \frac{1}{2} g_T B^\mu_\nu_T (B_{T\mu\nu})^*
\]
Momentum-space Feynman rule for interactions:

\[
\begin{align*}
    k_1 & \quad k_3 \\
    k_2 & \quad k_4
\end{align*}
\]

\[
\begin{align*}
    \{ g_V \gamma^\mu C \otimes C^{-1} \gamma_\mu + \frac{g_T}{2} \sigma^{\mu\nu} C \otimes C^{-1} \sigma_{\mu\nu} \} \tilde{f}(k_1 - k_2) \tilde{f}(k_3 - k_4)
\end{align*}
\]

- **Separable interaction**: initial & final momentum dependence factorize.
- (isospin dependence suppressed to compactify formula)

\[
\tilde{f}(k) \text{ is Fourier transform of } f(z), \text{ chosen:}
\]

\[
\tilde{f}(k) = \frac{\Lambda}{k^2 - \Lambda^2 + i0}
\]

- \(\Lambda\) is the regulator scale (non-locality scale).
Kernel encodes channels with multiple quantum numbers:

\[
\gamma^\mu C \otimes C^{-1} \gamma_\mu = \left( \gamma^\mu - \frac{\psi p^\mu}{p^2} \right) C \otimes C^{-1} \left( \gamma_\mu - \frac{\psi p_\mu}{p^2} \right) + \frac{1}{p^2} \psi C \otimes C^{-1} \psi
\]

\[
\sigma^{\mu\nu} C \otimes C^{-1} \sigma_{\mu\nu} = \frac{1}{p^2} \sigma^{\mu\nu} C \otimes C^{-1} \sigma_{\mu\nu} + \left( \sigma^{\mu\nu} - \frac{\sigma^{\mu\nu} p^\nu - \sigma^{\nu\mu} p^\mu}{p^2} \right) C \otimes C^{-1} \left( \sigma^{\mu\nu} - \frac{\sigma^{\mu\nu} p^\nu - \sigma^{\nu\mu} p^\mu}{p^2} \right)
\]

\( p \) is center-of-mass momentum (deuteron momentum)

Need only structures with deuteron quantum numbers:

\[
\gamma_V^\mu \equiv \gamma^\mu - \frac{\psi p^\mu}{p^2}
\]

\[
\gamma_T^\mu \equiv \frac{i\sigma_{\mu\nu}}{\sqrt{p^2}}
\]

Other structures fully decouple in the T-matrix equation!
Bubble diagrams defined via:

\[ -i \left( g^{\mu \nu} - \frac{p^\mu p^\nu}{p^2} \right) \Pi_{XY}(p^2) = Y \rightarrow X \]

- Either $\gamma_\nu^\mu$ or $\gamma_T^\mu$ can be on either end.
- The regulator $f(k)$ appears twice inside loop integral—makes it UV finite.

Can define bubble matrix:

\[
\Pi = \begin{bmatrix}
\Pi_{VV}(p^2) & \Pi_{VT}(p^2) \\
\Pi_{TV}(p^2) & \Pi_{TT}(p^2)
\end{bmatrix}
\]

- Essential ingredient in calculations to follow.
Bethe-Salpeter equation (BSE) for T-matrix given by:

\[
p \frac{p^2}{2} + k = \frac{p^2}{2} - k' = \frac{p^2}{2} + k' = +
\]

Separability of interaction permits a simple matrix form:

\[
T = G - G\Pi T
\]

Actual T-matrix related to simplified matrix via:

\[
\mathcal{T}(p,k,k') = \frac{\Lambda}{k^2 - \Lambda^2} \frac{\Lambda}{k'^2 - \Lambda^2} \left\{ T_{11} \gamma_V \otimes \gamma_V + T_{12} \gamma_V \otimes \gamma_T + T_{21} \gamma_T \otimes \gamma_V + T_{22} \gamma_T \otimes \gamma_T \right\}
\]

Simplified kernel matrix:

\[
G = \begin{bmatrix} g_V & 0 \\ 0 & g_T \end{bmatrix}
\]
T-matrix solution given by:

\[ T = (1 + G\Pi)^{-1}G \]

Deuteron bound state pole exists where:

\[ \det(1 + G\Pi) = 0 \]

Use physical deuteron mass to fix \( g_V \) in terms of \( \Lambda \) and \( g_T \).

Residues at this pole give reduced form of deuteron vertex:

\[ T(p^2 \approx M_D^2) \approx -\frac{1}{p^2 - M_D^2} \begin{bmatrix} \alpha^2 & \alpha\beta \\ \alpha\beta & \beta^2 \end{bmatrix} \]

\( \alpha \) and \( \beta \) are coefficients in deuteron Bethe-Salpeter vertex.

The \( k \) and \( k' \) dependence is fixed and separable.
The result of all this is a deuteron Bethe-Salpeter vertex:

\[ \Gamma_D^{\mu}(p, k) = \frac{\Lambda}{k^2 - \Lambda^2 + i0} \left\{ \alpha \gamma^\mu_V + \beta \gamma^\mu_T \right\} C\tau_2 \]

- Simple \( k \) dependence fixed by separable interaction.
- Can be used to covariantly calculate all sorts of observables.

Relationship to fundamental model parameters:

\[ (\Lambda, g_V, g_T) \rightarrow (M_D, \alpha, \beta) \]

- Eliminate one model parameter by fixing \( M_D \) to empirical value.
- Could fix other parameters via observables, e.g., charge radius & quadrupole moment.

A curious thing happens if we look at the non-relativistic limit …
Non-relativistic reduction

- Non-relativistic, momentum-space wave function:

\[ \psi_{\text{NR}}(k, \lambda) \sim \frac{-1}{\sqrt{8M_D}} \frac{\bar{u}(k, s_1)(\Gamma_D \cdot \varepsilon_\lambda)\bar{u}^T(-k, s_2)}{k^2 + m\epsilon_D} \]

- Working out the Dirac matrix algebra and using the limit \( k^2 \ll m^2 \) will give:

\[ \psi_{\text{NR}}(k, \lambda) = 4\pi \left\{ u(k)Y^\lambda_{101}(\hat{k}) + w(k)Y^\lambda_{121}(\hat{k}) \right\} \]

\[ u(k) = \sum_{j=0}^{1} \frac{C_j}{k^2 + B_j^2} \]

\[ w(k) = \sum_{j=0}^{1} \frac{D_j}{k^2 + B_j^2} \]

\[ B_0 = \sqrt{m\epsilon_D} \]

\[ B_1 = \Lambda \]

- \( u(k) \) is S-wave, \( w(k) \) is D-wave.

\[ C_0 = \frac{m}{\sqrt{4\pi M_D}} \frac{\Lambda}{\Lambda^2 - \epsilon_D m} \left( \alpha + \beta - \frac{(\alpha - \beta)\epsilon_D m}{12m^2} \right) \]

\[ C_1 = -\frac{m}{\sqrt{4\pi M_D}} \frac{\Lambda}{\Lambda^2 - \epsilon_D m} \left( \alpha + \beta - \frac{(\alpha - \beta)\Lambda^2}{12m^2} \right) \]

\[ D_0 = -\frac{m}{\sqrt{4\pi M_D}} \frac{\Lambda}{\Lambda^2 - \epsilon_D m} \frac{\sqrt{2}(\alpha - \beta)\epsilon_D m}{6m^2} \]

\[ D_1 = \frac{m}{\sqrt{4\pi M_D}} \frac{\Lambda}{\Lambda^2 - \epsilon_D m} \frac{\sqrt{2}(\alpha - \beta)\Lambda^2}{6m^2} \]
Approximating non-relativistic wave functions

The curious thing is that this is a standard parametrization for deuteron wave functions!

\[ u(k) = \sum_{j=0}^{N} \frac{C_j}{k^2 + B_j^2} \quad \text{and} \quad w(k) = \sum_{j=0}^{N} \frac{D_j}{k^2 + B_j^2} \]

- First used by Paris group, Lacombe et al., PLB 101 (1981) 139
- Typically \( N > 1 \) of course.
- One requires \( B_0 = \sqrt{\varepsilon_D m} \) to get the right asymptotic behavior. (Check!)
- One also requires the following sum rules for correct behavior at the origin:

\[ \sum_{j=0}^{N} C_j = \sum_{j=0}^{N} D_j = \sum_{j=0}^{N} D_j B_j^{-2} = \sum_{j=0}^{N} D_j B_j^2 = 0 \]

- Model as given fails unless \( \alpha = \beta \), meaning no D wave.
- But we can fix this by having \( N \) copies of the separable kernel!
Making $N$ copies of the separable kernel

- Just have $N$ copies of the original separable interaction with different $\Lambda_n$:

$$\mathcal{K}(k, k') = \sum_{n=1}^{N} \frac{\Lambda_n}{k^2 - \Lambda_n^2 + i0} \frac{\Lambda_n}{k'^2 - \Lambda_n^2 + i0} \left\{ g_{Vn} \gamma^\mu C \otimes C^{-1} \gamma_\mu + \frac{g_{Tn}}{2} \sigma^{\mu\nu} C \otimes C^{-1} \sigma_{\mu\nu} \right\}$$

- Kernel now has $3N$ parameters: $\{ \Lambda_n, g_{Vn}, g_{Tn} | n \in \{1, \ldots, N\} \}$.

- Simplified forms of kernel, T-matrix, and bubble are now all $2N \times 2N$ matrices.
  - The different $\Lambda_n$ mix, but the T-matrix equation is still separable and can be solved algebraically.

- Deuteron vertex is now:

$$\Gamma_D^\mu(p, k) = \sum_{n=1}^{N} \frac{\Lambda_n}{k^2 - \Lambda_n^2 + i0} \left\{ \alpha_n \gamma^\mu_V + \beta_n \gamma^\mu_T \right\} C\tau_2$$

- Non-relativistic reduction now has $N + 1$ terms in S and D waves!
Using the separable kernel as a parametrization

- The popular non-relativistic parametrization is:

\[
 u(k) = \sum_{j=0}^{N} \frac{C_j}{k^2 + B_j^2} \quad w(k) = \sum_{j=0}^{N} \frac{D_j}{k^2 + B_j^2}
\]

- Here \( B_0 = \sqrt{\epsilon_D m} \) and \( B_n = \Lambda_n \) (for \( n > 0 \)).

- From the kernel coupling strengths:

\[
 \{g_{Vn}\}, \{g_{Tn}\} \rightarrow \{\alpha_n\}, \{\beta_n\} \rightarrow \{C_j\}, \{D_j\}
\]

- Won’t fill up a slide with all the formulas (see preprint when it comes out).

- The formulas are linear & invertible!

\[
 \{C_j\}, \{D_j\} \rightarrow \{\alpha_n\}, \{\beta_n\} \rightarrow \{g_{Vn}\}, \{g_{Tn}\}
\]

- So why not **start with** the \( C_j, D_j \) and \( B_j \) from a well-established wave function?

  - Automatically get precision of established wave function.
  - Get correct constraints by starting with \( \{B_j, C_j, D_j\} \) that obey them.
  - Guaranteed **Lorentz covariance** from using separable framework.
Example: Argonne V18 fit using $N = 7$.

- Get $u(k)$ and $w(k)$ from ANL website\(^*\).
- $B_n \ (n > 0)$ were allowed to float in fit.
- $B_0 = \sqrt{\epsilon D m}$. 
- $r \to 0$ constraints were enforced.

From $\{B_j, C_j, D_j\}$ get $\{\Lambda_n, gV_n, gT,n\}$.

- See future preprint for numerical values!

Now have covariant Lagrangian that reproduces AV18 wave function in NR limit!

\(^*\): [https://www.phy.anl.gov/theory/research/av18/](https://www.phy.anl.gov/theory/research/av18/)
Things to do with this framework

▶ Electromagnetic form factors (obtained!)

▶ Gravitational form factors (in progress)
  ▶ Manifest covariance helpful here.
  ▶ Previous non-covariant work (AF & Cosyn, PRD) found inconsistencies in EMT components.

▶ Collinear parton distributions (obtained!)

▶ $b_1$ structure function (obtained!)

▶ Generalized parton distributions (in progress)
  ▶ GPDs are the main goal of this project.
  ▶ Existing deuteron GPDs violate polynomiality.
  ▶ Manifest covariance of this framework guarantees polynomiality.
Sum of triangle and bicycle diagrams

- Bicycle diagram comes from gauge invariance.
- Non-local interaction requires Wilson lines.

Diagrams can be evaluated *exactly* within the present framework!

- Symbolic algebra program needed though—results are *long* (hundreds of lines of generated Fortran code)
- Results are covariant too.
Coulomb form factor

- Vast improvement over non-relativistic impulse approximation (NRIA)!
  - Despite same NR limit.
  - Likely just form relativistic kinematics.

- Bicycle diagram makes negligible contribution.

- Reasonable description of data—sanity check passed for framework!
Good agreement with elastic structure functions

Static moments a bit large …

<table>
<thead>
<tr>
<th></th>
<th>Separable framework</th>
<th>Empirical value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Charge radius</td>
<td>2.156 fm</td>
<td>2.12799 fm</td>
</tr>
<tr>
<td>Magnetic moment</td>
<td>0.876 $\mu_N$</td>
<td>0.8574382284 $\mu_N$</td>
</tr>
<tr>
<td>Quadrupole moment</td>
<td>0.301 fm$^2$</td>
<td>0.2859 fm$^2$</td>
</tr>
</tbody>
</table>
Virtual Compton scattering amplitude

- Sum of **convolution** and **interaction** diagrams:

\[
\sum_{q} e_{q}^{2} \frac{1}{2} \int \frac{dz}{2\pi} e^{i x dp_{+}z} \frac{\gamma}{q} \left( -\frac{z^{-}}{2} \right) \gamma^{+} q \left( \frac{z^{-}}{2} \right)
\]

- Diagrams evaluated in the Bjorken limit:

\[
0 < x_{d} < 1, \text{ in contrast to usual normalization.}
\]

\[
0 < x_{\text{Bj}} = \frac{M_{d}}{m_{N}} x_{d} < \frac{M_{d}}{m_{N}} \approx 2 \text{ is the usual variable.}
\]

\[
x_{d} \text{ is easier to use in calculations.}
\]

\[
\text{Compare empirical data in terms of } x_{\text{Bj}}.
\]
Effective triangle diagram (Bjorken limit):

\[
H_i(x_d, \xi, t, Q^2; \lambda) = \sum_{N=p,n} \int_{-1}^{1} \frac{dy}{|y|} \left[ h_i(y, \xi, t; \lambda) H_N \left( \frac{x_d}{y}, \frac{\xi}{y}, t, Q^2 \right) 
+ e_i(y, \xi, t; \lambda) E_N \left( \frac{x_d}{y}, \frac{\xi}{y}, t, Q^2 \right) \right]
\]

Forward limit \((t \to 0)\) gives standard PDF convolution:

\[
q_d(x_d, Q^2; \lambda) = \sum_{N=p,n} \int_{x_d}^{1} \frac{dy}{y} f(y; \lambda) q_N \left( \frac{x_d}{y}, Q^2 \right)
\]
Light cone density: a PDF assuming pointlike nucleons.

Nucleon sum rule obeyed:
\[
\sum_{N=p,n} \int_0^1 dy \ f(y; \lambda) = 2
\]

Momentum sum rule violated!
\[
\sum_{N=p,n} \int_0^1 dy \ yf(y; \lambda) = \begin{cases} 
1.0036 & : \lambda = \pm 1 \\
0.9984 & : \lambda = 0 
\end{cases}
\]

Interaction carries momentum
Effective **bicycle diagram** (Bjorken limit):

\[ q \rightarrow Q^2 \rightarrow \infty \]

New Feynman rule for operator insertion:

\[
\begin{align*}
&\sum_{X} g_x S \left\{ \tilde{h}_X^+(k, 0) \left( \delta(xp^+ - k_1^+) + \delta(xp^+ - k_2^+) \right) \tilde{f}_X(k') \\
&- \tilde{f}_X(k) \tilde{h}_X^+(k', 0) \left( \delta(xp^+ - k_3^+) + \delta(xp^+ - k_4^+) \right) \right\} \left( \gamma^\nu \mathcal{C} \tau_2 \right)_{i'j'} \left( C^{-1} \tau_2 \gamma^\nu \right)_{ij}
\end{align*}
\]

...assuming pointlike nucleon.

Use convolution formula to fold in $NN$ vertex structure???
Mellin moment of non-local correlator:

\[
\int d^6x \, d^6x' \, \frac{1}{q} \left( -\frac{z^-}{2} \right) \frac{\gamma^+}{2} q \left( \frac{z^-}{2} \right) = \frac{1}{(2p^+)^s} q(0) \gamma^+ (i \overset{\rightarrow}{\partial}^+) \, \gamma^+ = (2p^+)^s q(0). 
\]

- Quark interaction with (entirely hypothetical) spin-s gauge field.
- Operator inserted on the \( NN \) kernel.

How does the non-local kernel interact with a spin-s gauge field?
- Got the photon insertion rule from a gauge link.
- Assert invariance under spin-s gauge transforms.
- Spin-s gauge link???
Higher-spin gauge fields transform like
\[ \delta h_{\mu_1 \mu_2 \ldots \mu_s}(x) = \partial_{\mu_1} \xi_{\mu_2 \ldots \mu_s}(x) + \partial_{\mu_2} \xi_{\mu_1 \ldots \mu_s}(x) + \ldots + \partial_{\mu_s} \xi_{\mu_1 \ldots \mu_{s-1}}(x) \]
with \( \xi_{\mu_1 \ldots}(x) \) totally symmetric.

- Fronsdal, PRD (1978); de Wit & Freedman, PRD (1980)
- Not compatible with Poincaré symmetry (Coleman & Mandula, Phys. Rev. (1967))

My guess for gauge link (would love a proper derivation!):
\[ \psi(x + y) \to \exp \left\{ i \int_x^{x+y} dz^{\mu_1} \ h_{\mu_1 \ldots \mu_s}(z) i\partial^{\mu_2} \ldots i\partial^{\mu_s} \right\} \psi(x + y) \]

- Gives EM case for \( s = 1 \).
- Gives linearized gravity case for \( s = 2 \) (Green, PRD (2008); S. Wikeley’s PhD thesis).

Resulting Feynman rule:

\[ \Delta \]

\[ j, k_1 \]

\[ i', k_3 \]

\[ i, k_2 \]

\[ j', k_4 \]

\[ = \]

\[ -i \sum_X g_X S \left\{ \tilde{h}^{\mu_1 X}_X(k, \Delta)(k_1^{\mu_2} k_1^{\mu_3} \ldots k_1^{\mu_s} + k_2^{\mu_2} k_2^{\mu_3} \ldots k_2^{\mu_s}) \tilde{f}_X(k') \right\} \]

\[ - \tilde{f}_X(k) \tilde{h}^{\mu_1 X}_X(k', \Delta)(k_3^{\mu_2} k_3^{\mu_3} \ldots k_3^{\mu_s} + k_4^{\mu_2} k_4^{\mu_3} \ldots k_4^{\mu_s}) \right\} \left( \gamma_{\nu}^{\nu} C r_2 \right)_{ij', j'} \left( C^{-1} \tau_2 \tilde{\gamma}_{\nu} \right)_{ij} \]

Inverse Mellin transform of \( ++ \ldots \) component gives rule from two slides ago.
Momentum sum rule obeyed.
  - Saved by the bicycle diagram!
  - Also makes LCD symmetric.

Slight negative support.
  - Either a flaw with the framework …
  - …or a feature of renormalization?

cf. Collins, Rogers & Sato, PRD (2022)
Deuteron PDFs via convolution:

\[ q_d(x_d, Q^2; \lambda) = \sum_{N=p,n} \int_{x_d}^{1} \frac{dy}{y} q_N \left( \frac{x_d}{y}, Q^2 \right) f(y; \lambda) \]

Use JAM PDFs for nucleon.

C. Cocuzza et al., PRD106 (2022) L031502

Same PDF for triangle & bicycle diagrams.

Plot in terms of

\[ x = x_{\text{Bj}} \equiv \frac{Q^2}{2m_N \nu} = \frac{M_d}{m_N} x_d \approx 2x_d \]

This is the standard \( x \) variable.
Use free nucleon PDFs inside both triangle & bicycle diagrams.

- Use JAM22 PDFs.

**Unpolarized** $F_{2D}(x_{Bj}, Q^2)$ structure function.

- Looks reasonable.
- But no EMC effect when using free PDFs.

**Tensor-polarized** $b_{1d}(x_{Bj}, Q^2)$ structure function looks standard.

- Total (blue curve) looks similar to Cosyn &al., PRD (2017).
- Can’t explain HERMES $b_1$ data.
Regarding $b_1$ ...

- 2005 HERMES data at low $Q^2$.
  - From 0.51-4.69 GeV$^2$.
  - Lower $x$ points at lower $Q^2$.
  - PRL 95 (2005) 242001

- Partonic picture may not be valid for low-$x$ data here.

- So, maybe fold in Bodek parametrization of nucleon structure functions?
  - Includes resonances at low $Q^2$.
  - Bodek et al., PRD (1979)

- Something surprising happens when I do this…
Separable model with Bodek structure functions

- Nucleon $F_1$ at $Q^2 = 2.5$ GeV$^2$.
  - Bodek et al., PRD (1979)

- Model was not fit to these data.
  - Parameters fixed on slide 15.
  - I’m actually surprised by this result!

- Seems too good to be true?
  - Cosyn (PRD (2017)) don’t see this.
  - Model $f_T(y)$ responsible?
  - Maybe due to negative support?
  - Specifically needs Bodek $F_1$.
  - Using PDFs doesn’t work.
  - Found no code errors yet.

- Result is preliminary and subject to double-checking!
Presented a framework for **covariant calculations** using **realistic deuteron wave functions**.

- Used Argonne’s AV18 wave function as an example.
- Covariance means GPDs *will obey polynomiality*.

Reproduced known deuteron properties in this framework

- Necessary sanity check.
- Learned an important lesson: **bicycle diagrams** must be accounted for!

**Maybe** able to explain HERMES $b_1(x, Q^2)$ data.

- I’m double-checking the result though.

Much more to be done:

- Energy momentum tensor and gravitational form factors.
- **Generalized parton distributions** (the main purpose of this project!)

Thank you for your time!