PHENOMENOLOGY OF $3 \to 3$ SCATTERING

ANDREW JACKURA

INDIANA UNIVERSITY
JOINT PHYSICS ANALYSIS CENTER (JPAC)

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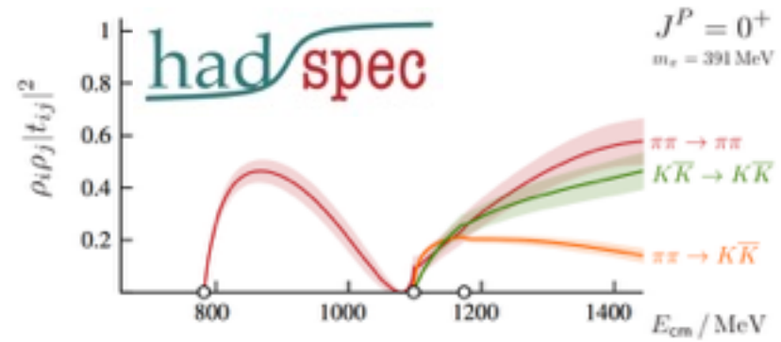
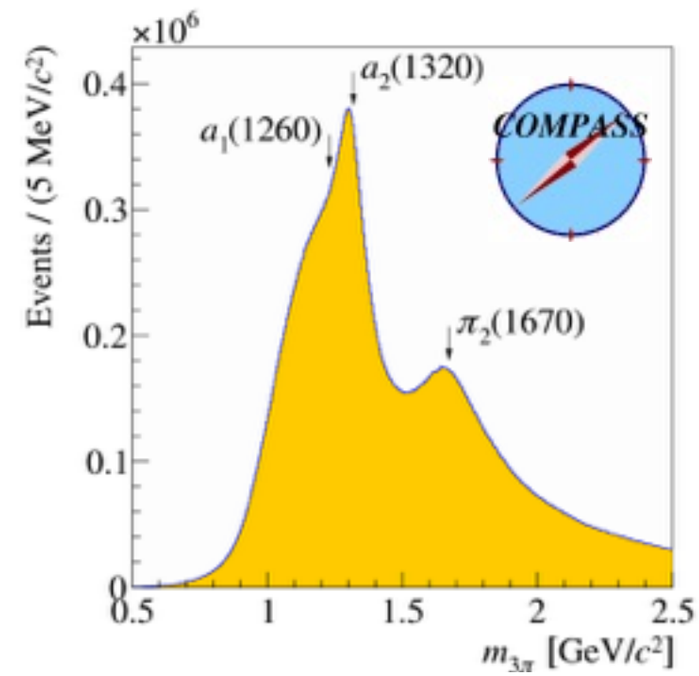
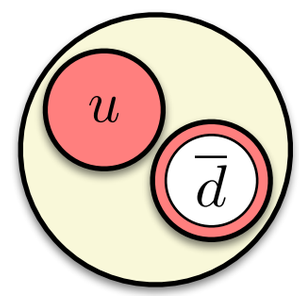
Hadrons and QCD

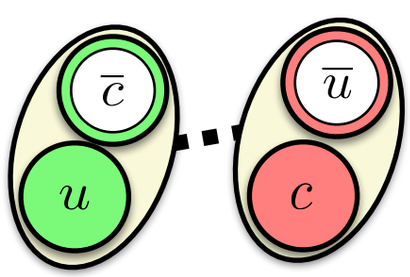
- Signatures of hadrons in data
  - Peaks and dips on **Real** energy axis
  - Patterns in mass/ spin
- Theoretical description
  - **Complex** energy plane singularities
- Interpretation
  - Constituent quarks, exotica



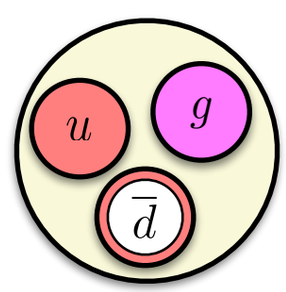




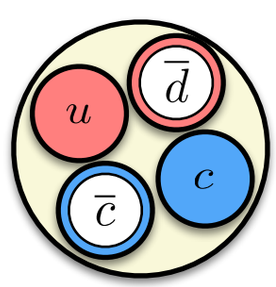
Mesons



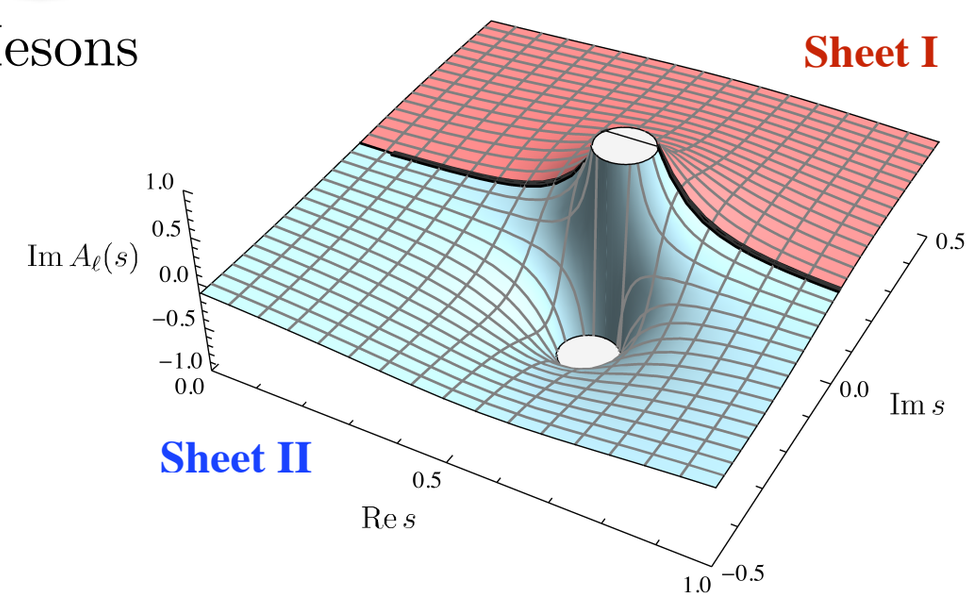
Mesonic-Molecules



Hybrids

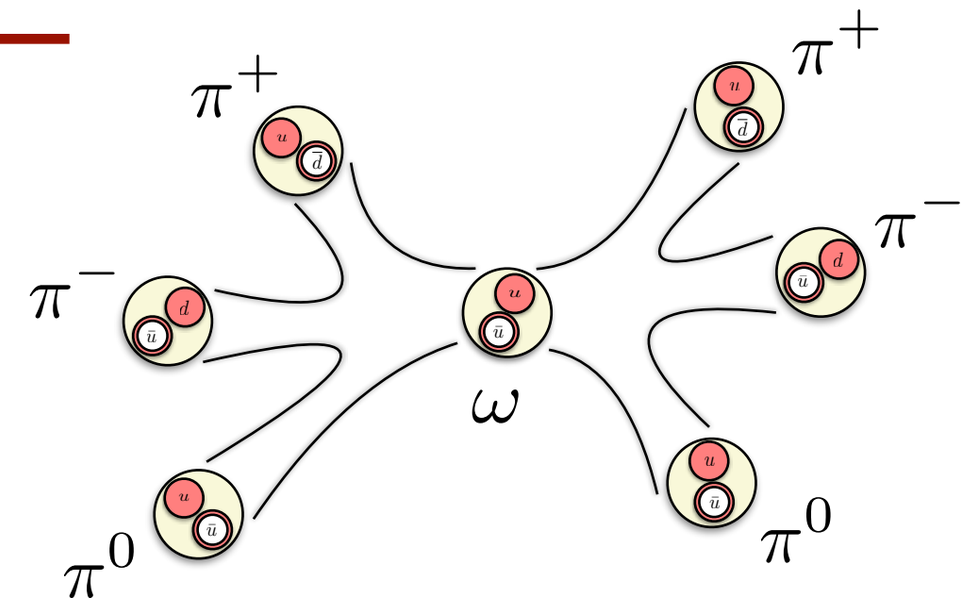


Tetraquarks

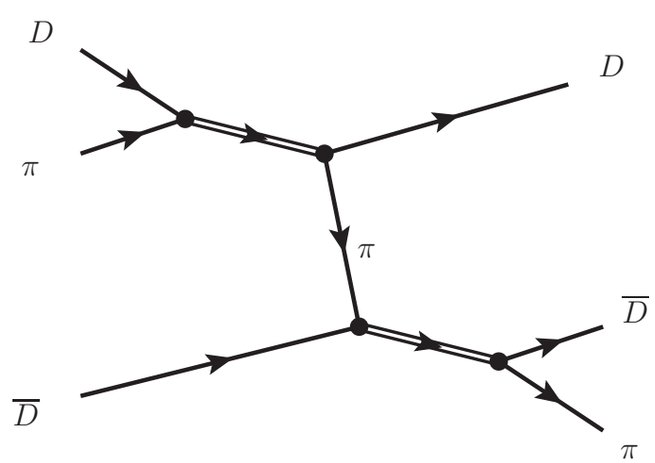


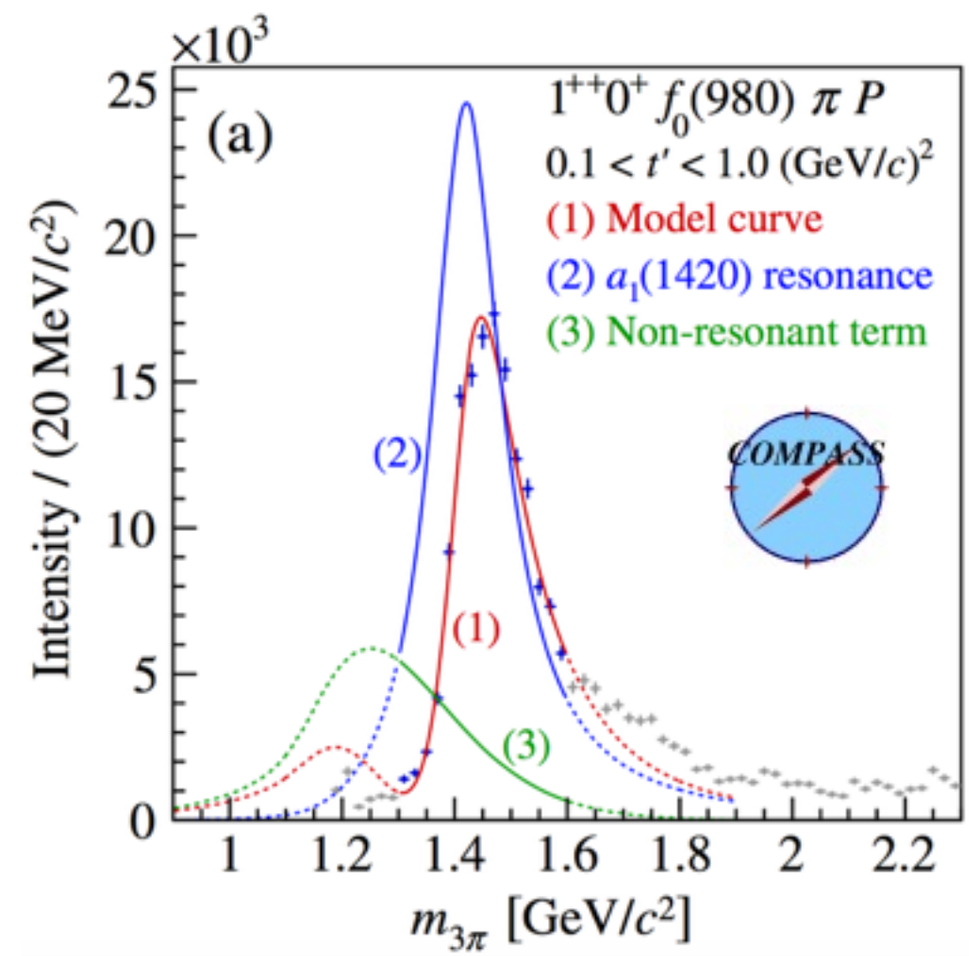
3→3 Phenomenology

- Many Resonances decay to 3-particles
  - New high-statistics experimental results
  - Advancements in Lattice Formalism
- Signatures of new resonances
  - High-Precision PWA of 3π yields $a_1(1420)$, Incompatible with quark model/ Regge expectations
- *XYZ* States
  - Lie near 3-body thresholds



$X(3872) \to D\overline{D}\pi$



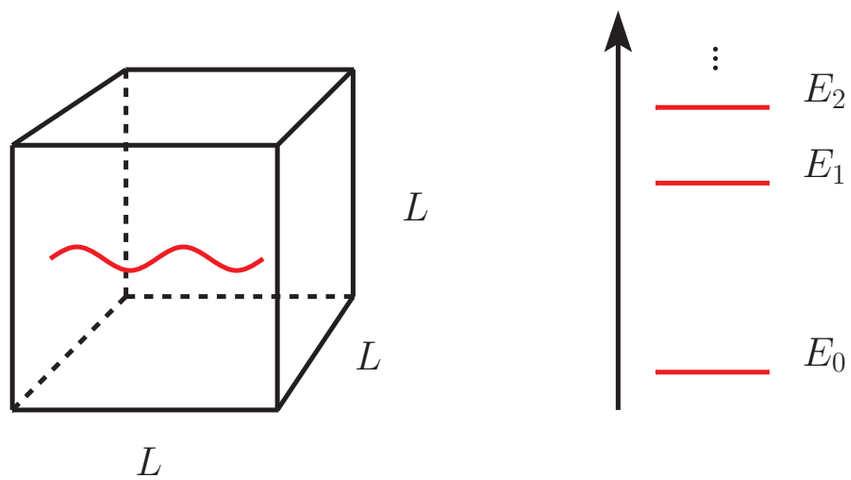


Connecting to LQCD

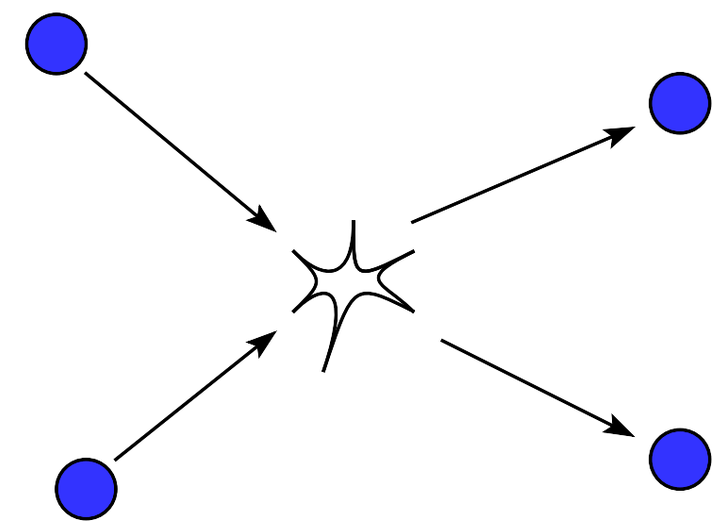
- Quantization condition relates FV spectra and IV amplitudes

Infinite volume amplitude **FV functions**

$$\det\left[\mathcal{A} + F\right] = 0$$



$\Longrightarrow$

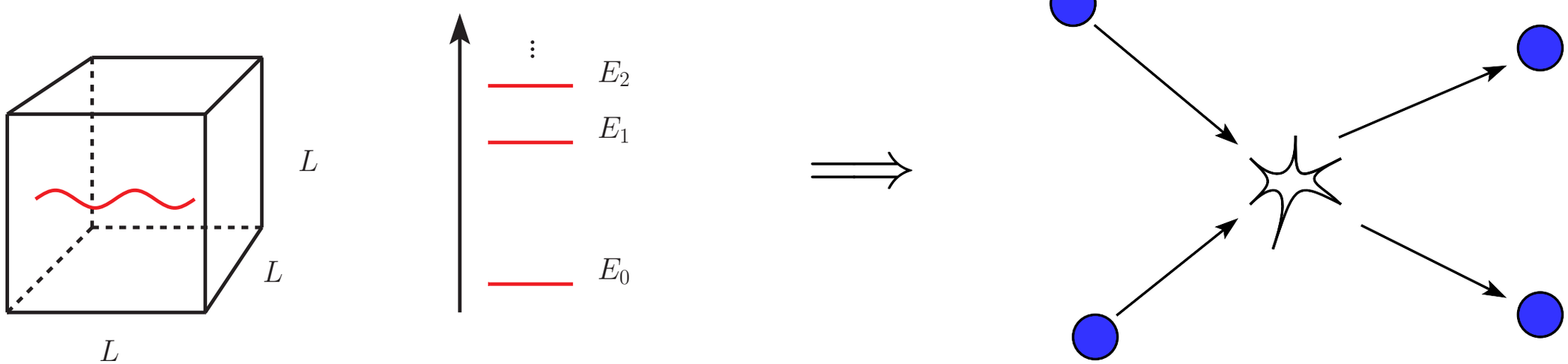


- Different approaches to the problem:
  - All orders QFT
  - The Isobar Approximation
  - EFT approach
  - …

Need infinite volume amplitudes for analytic continuation

Connecting to LQCD

$$\det\left[\mathcal{A} + F\right] = 0$$



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3→3 Phenomenology

- Our approach — S-matrix constraints within Isobar Approximation
  - Use information for on-shell amplitudes (particle production)
  - Introduce concept of isobars — fix cross-channel physics
  - Propose parameterization
  - Investigate analytic properties

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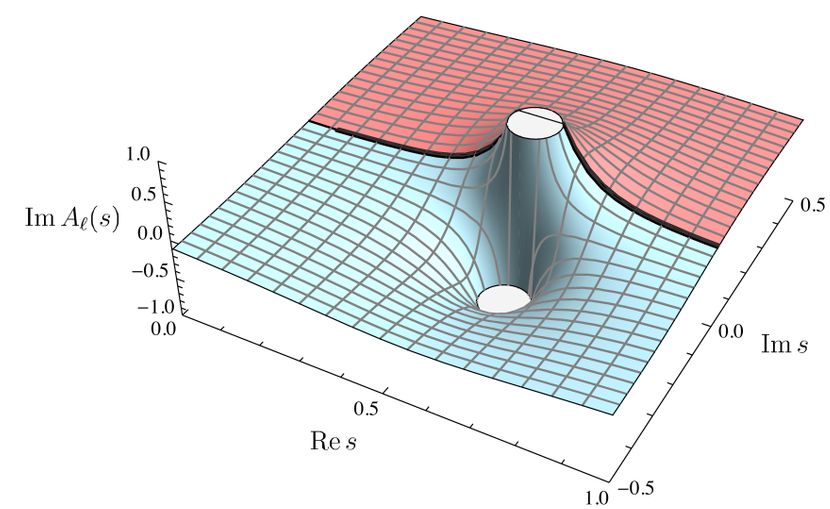
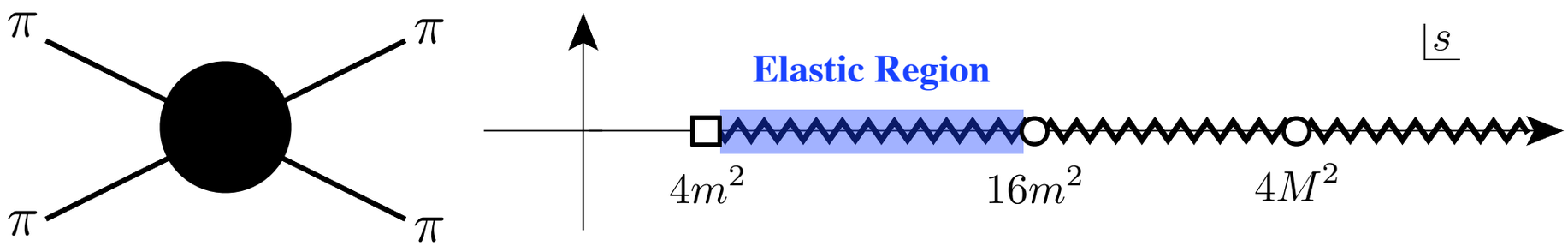
This work

- Ultimate Goals:
  - Analytically continuation
  - Investigate resonances and exchanges
  - Extension into the finite volume



2→2 Phenomenology

- To illustrate our approach, consider **2→2** elastic scattering, e.g. $\pi\pi\to\pi\pi$



- Constrain amplitude with unitarity

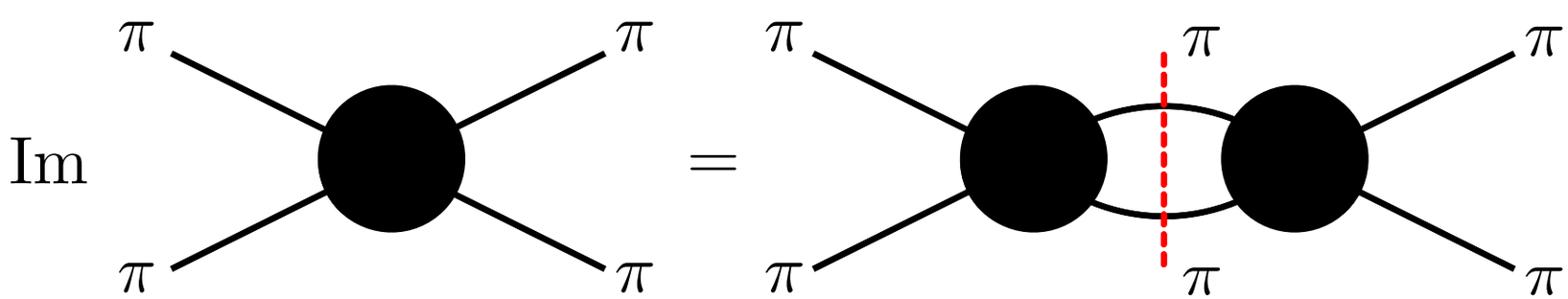
On-shell amplitudes only

$$\mathrm{Im}\ \mathcal{F}(s,\widehat{\mathbf{q}}'\cdot\widehat{\mathbf{q}}) = \rho(s)\int \mathrm{d}^2\widehat{\mathbf{k}}\,\mathcal{F}^*(s,\widehat{\mathbf{k}}\cdot\widehat{\mathbf{q}}')\mathcal{F}(s,\widehat{\mathbf{k}}\cdot\widehat{\mathbf{q}})\,\Theta(s-4m^2)$$

Imaginary part fixed

2-body phase space
from on-shell particles

$$\rho(s) \propto \frac{2q^\star}{\sqrt{s}}$$



2→2 Phenomenology

- Introduce partial wave amplitudes — simplifies unitarity relation

$$\mathcal{F}(s, \widehat{\mathbf{q}}' \cdot \widehat{\mathbf{q}}) = \sum_{\ell=0}^{\infty} \left(\frac{2\ell+1}{4\pi} \right) f_\ell(s) P_\ell(\widehat{\mathbf{q}}' \cdot \widehat{\mathbf{q}})$$

Cross channel singularities included in infinite sum

Partial wave amplitudes contains *s*-channel singularities

Rotational dependence

- Algebraic unitarity relation ⟹ Construct parameterization

$$\mathrm{Im}\ f_\ell(s) = \rho(s) |f_\ell(s)|^2\, \Theta(s - 4m^2)$$

$$\Longrightarrow \quad f_\ell^{-1}(s) = K_\ell^{-1}(s) - i\rho(s)$$

Dynamics from underlying theory

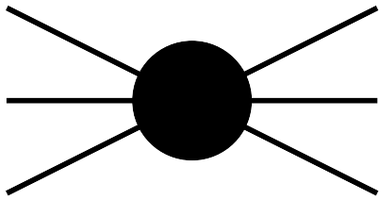
Kinematics from unitarity

- Can now analytically continue ⇒ Hunt for resonance poles

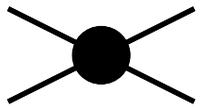
3→3 Phenomenology

- Use same methodology for elastic **3→3** scattering

Distinguishable particles



$= \mathcal{A}(\mathbf{p}';\mathbf{p})$

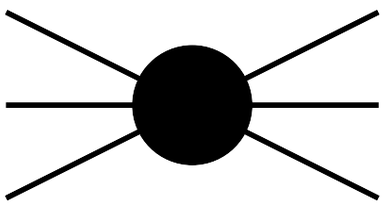


$= \mathcal{F}_j(\mathbf{p}';\mathbf{p})$

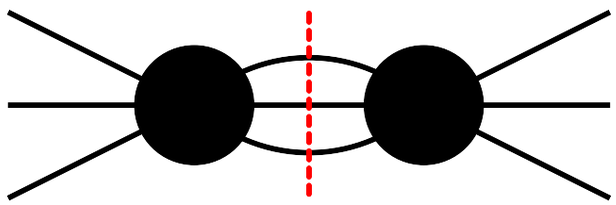
- Unitarity Relations

Intermediate state particles put on-shell

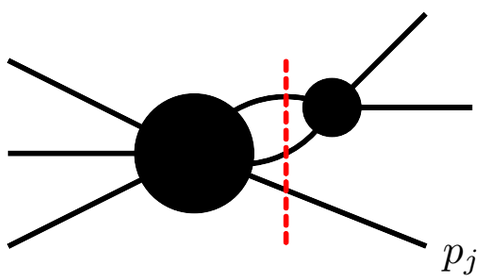
$2\,\mathrm{Im}$



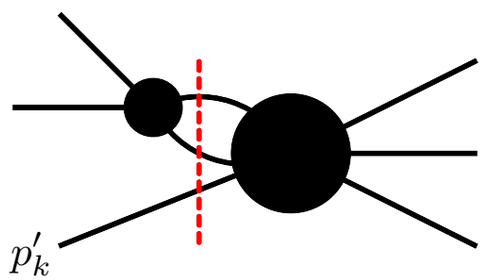
$=$



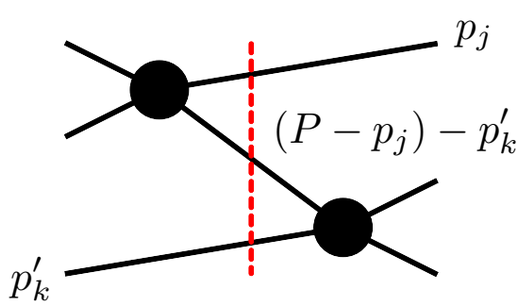
$+\sum_j$



$+\sum_k$



$+\sum_{\substack{j,k \\ j\neq k}}$



Relation between on-shell amplitudes

Kinematic singularity from real particle exchange

3→3 Phenomenology

- Partial wave expansion

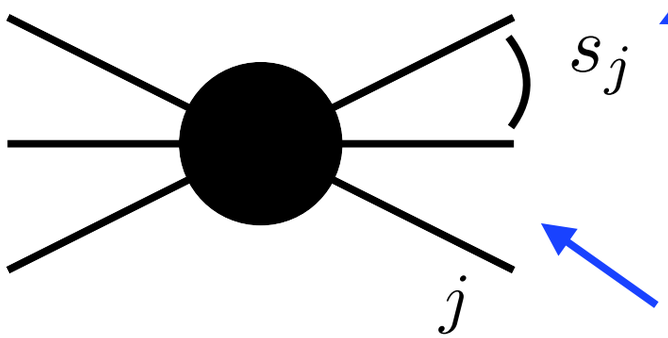
$$\mathcal{A}(\mathbf{p}';\mathbf{p}) = \sum_{J} \sum_{\ell'_k, s'_k} \sum_{\ell_j, s_j} \mathcal{M}^{J}_{\ell'_k s'_k; \ell_j s_j}(\sigma'_k, s, \sigma_j) \sum_{M} Z^{JM*}_{\ell'_k s'_k}\left(\widehat{\mathbf{P}}'^{\star}_k, \widehat{\mathbf{q}}'_k\right) Z^{JM}_{\ell_j s_j}\left(\widehat{\mathbf{P}}^{\star}_j, \widehat{\mathbf{q}}_j\right)$$

Infinite sums

Partial wave amplitude

Rotational functions

- Notice: Need complete infinite sum for cross-channel resonance



Sub-channel resonance has spin s_j

Sub-channel resonances need infinite sum of partial waves

- Point: For effective description of **3→3** scattering, need infinite sum!

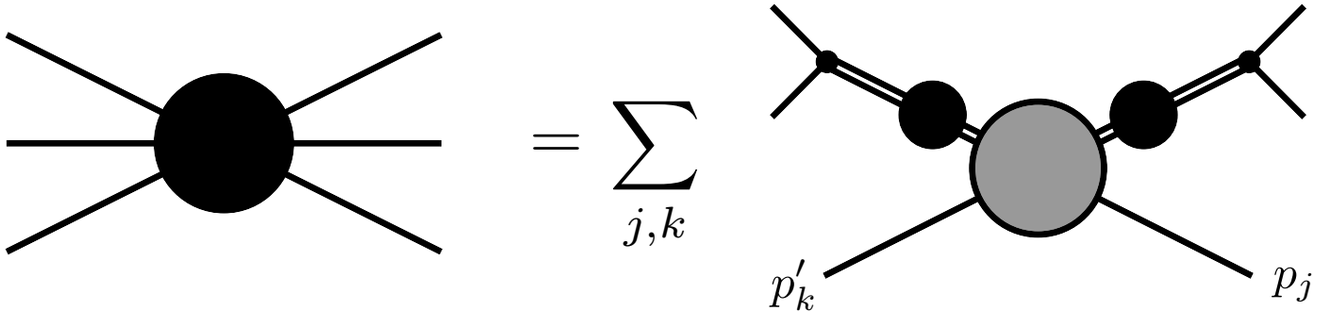
3→3 Phenomenology

- To work with truncated waves, introduce **Isobar Expansion**
  - Represent **3→3** amplitude as sum over different spectator-pairs:

$$\mathcal{A}(\mathbf{p}';\mathbf{p}) = \sum_{j,k} \mathcal{A}_{kj}(\mathbf{p}';\mathbf{p})$$

Complete 3→3 amplitude

Sum over all spectator-pair combinations
i.e. 123 = (12)3 + (23)1 + (31)2



Each pair = Isobar

- Partial wave expand each isobar-spectator amplitude
  - Effectively tripled amplitude, with PWE in each channel

3→3 Phenomenology

- Can now work with truncated partial waves — **Isobar approximation**

$$\mathcal{A}_{kj}\left(\mathbf{p}';\mathbf{p}\right)=\sum_{J}\sum_{\ell'_k,s'_k}^{\max'}\sum_{\ell_j,s_j}^{\max}\mathcal{A}^{J}_{\ell'_k s'_k;\ell_j s_j}\left(\sigma'_k,s,\sigma_j\right)\sum_{M}Z^{JM*}_{\ell'_k s'_k}\left(\widehat{\mathbf{P}}'^{\star}_k,\widehat{\mathbf{q}}'_k\right)Z^{JM}_{\ell_j s_j}\left(\widehat{\mathbf{P}}^{\star}_j,\widehat{\mathbf{q}}_j\right)$$

- Relation between full **3→3** amplitude and isobar-spectator amplitudes

$$\mathcal{M}^{J}_{\ell'_k s'_k;\ell_j s_j}\left(\sigma'_k,s,\sigma_j\right)=\mathcal{A}^{J}_{\ell'_k s'_k;\ell_j s_j}\left(\sigma'_k,s,\sigma_j\right)+X^{J}_{\ell'_k s'_k;\ell_j s_j}\left(\sigma'_k,s,\sigma_j\right)$$

Full 3→3 PW Amplitude

PWIS Amplitude

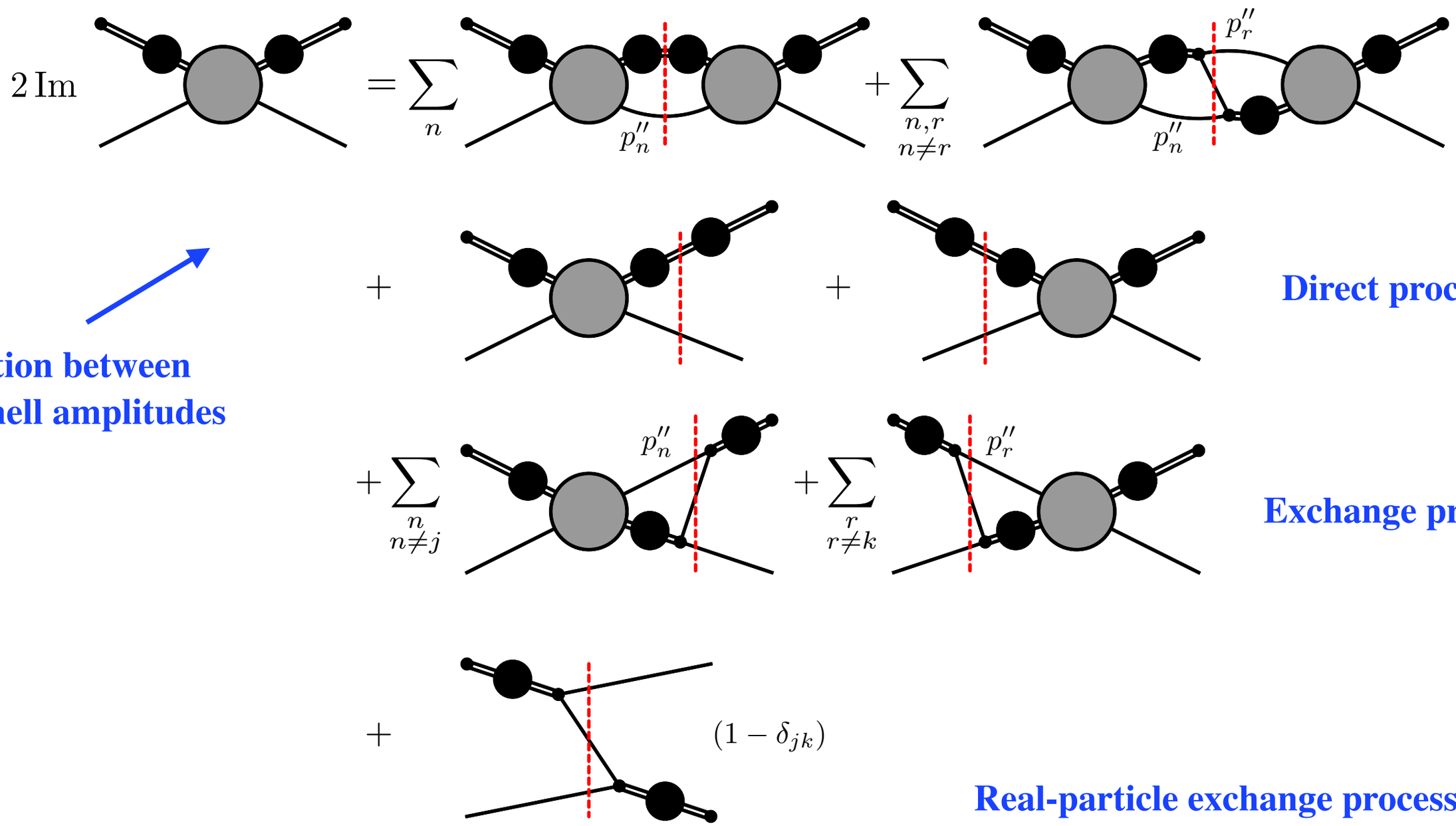
Recoupling PWIS Amplitudes
i.e. project cross-channels to direct channel

- Have now a description of **3→3** scattering
  - Truncated PW expansion
  - All sub-channel resonances easily described through cross-channels

3→3 Phenomenology

- Now, insert isobar approximation into unitarity relation

Direct process **Exchange process**



Relation between on-shell amplitudes

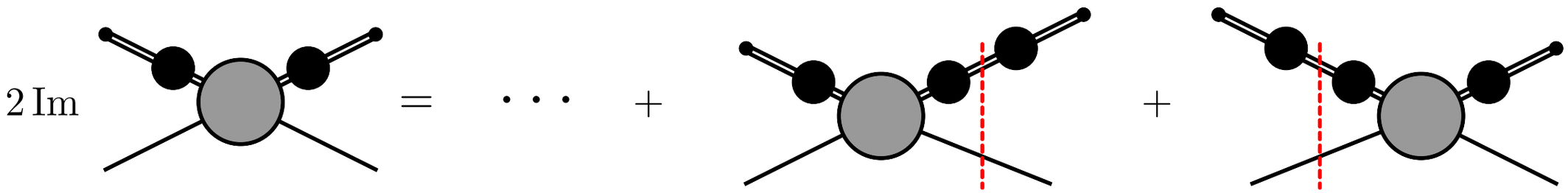
Direct processes

Exchange processes

Real-particle exchange process

3→3 Phenomenology

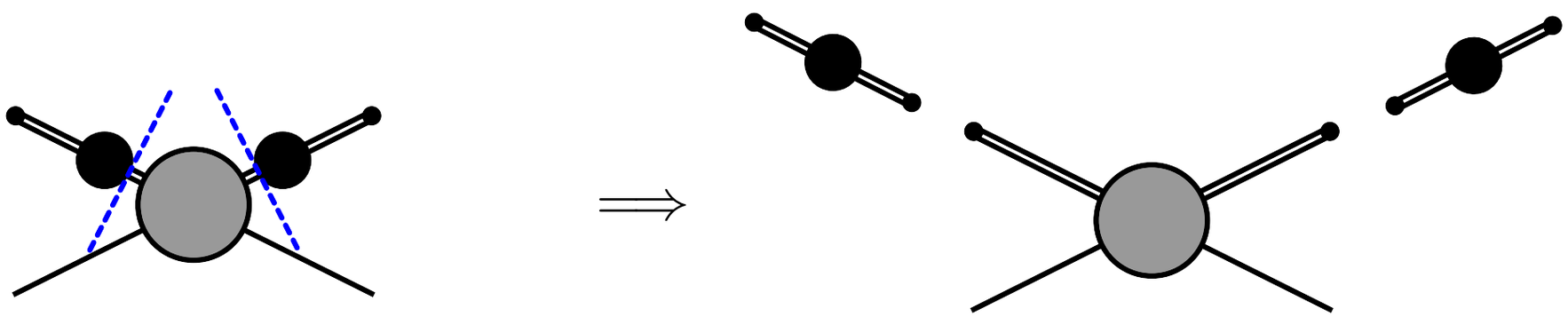
- Direct sub-channel terms can be removed via sub-channel unitarity



- Introduce amputated amplitude

Amputated amplitude

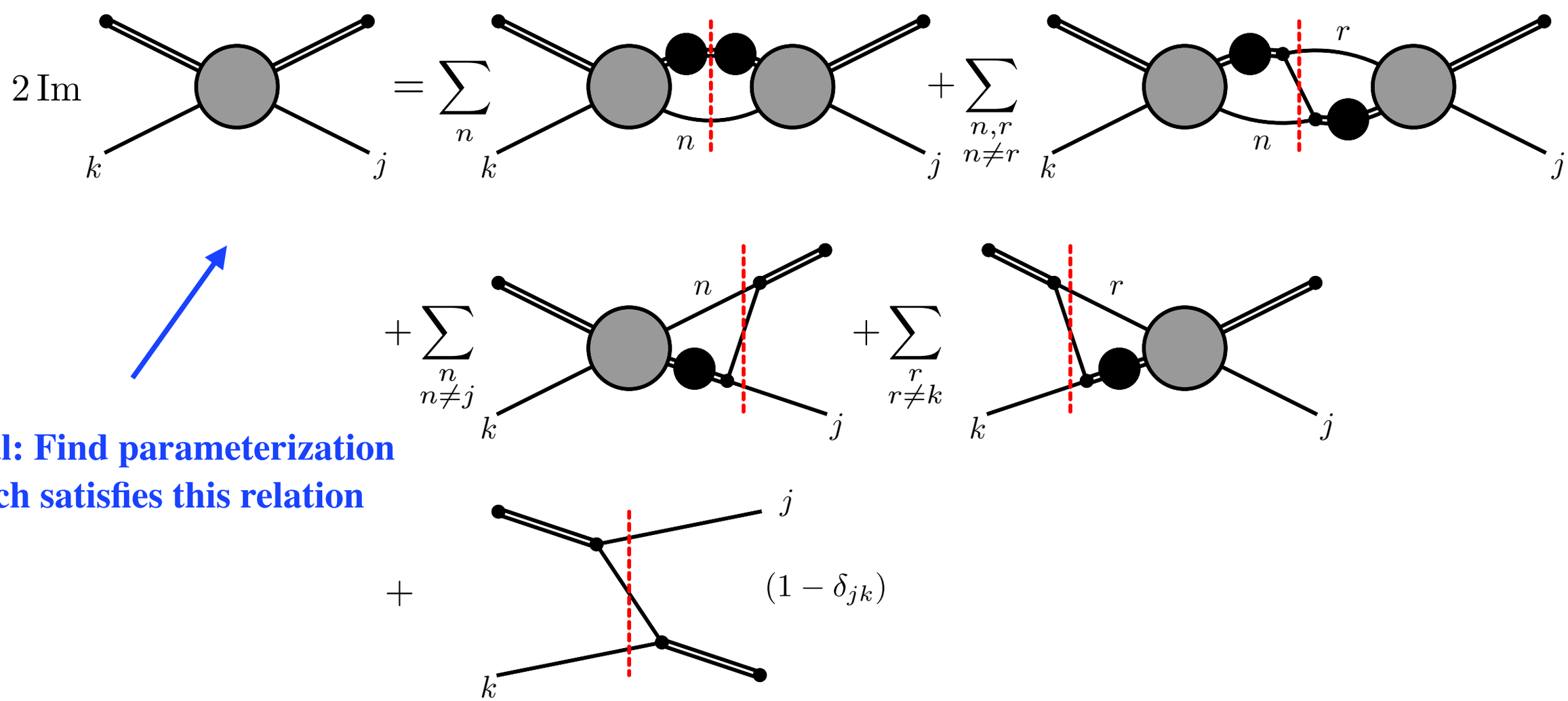
$$\mathcal{A}^{J}_{\ell'_k s'_k;\ell_j s_j} = f_{s'_k}\left(\sigma'_k\right) \widetilde{\mathcal{A}}^{J}_{\ell'_k s'_k;\ell_j s_j}\left(\sigma'_k, s, \sigma_j\right) f_{s_j}\left(\sigma_j\right)$$



Sub-channel unitarity → $\mathrm{Im}\, f_{s_j}\left(\sigma_j\right) = \rho_2\left(\sigma_j\right) \left|f_{s_j}\left(\sigma_j\right)\right|^2 \Theta\left(\sigma_j - \sigma_j^{(\mathrm{th})}\right)$

3→3 Phenomenology

- Now, have constraint for on-shell amplitudes
  - Next, construct representation for isobar-spectator amplitudes



Goal: Find parameterization which satisfies this relation

3→3 Phenomenology

$$
\begin{aligned}
&\mathrm{Im}\,\widetilde{\mathcal{A}}^{J}_{\ell'_k s'_k;\ell_j s_j}(\sigma'_k, s, \sigma_j) \\
&= \frac{1}{\pi(32\pi^2)^2}\sum_{n}\sum_{\ell''_n, s''_n}\int_{\sigma_n^{(\mathrm{th})}}^{(\sqrt{s}-m_n)^2} d\sigma''_n\,\frac{|\mathbf{q}''_n||\mathbf{p}''^{\star}_n|}{\sqrt{\sigma''_n}\sqrt{s}}|f_{s''_n}(\sigma''_n)|^2\,\widetilde{\mathcal{A}}^{J\,*}_{\ell''_n s''_n;\ell'_k s'_k}(\sigma''_n, s, \sigma'_k)\widetilde{\mathcal{A}}^{J}_{\ell''_n s''_n;\ell_j s_j}(\sigma''_n, s, \sigma_j)\Theta(s - s_{\mathrm{th}}) \\
&+ \frac{1}{2\pi s(32\pi^2)^2}\sum_{\substack{n,r\\ n\neq r}}\sum_{\ell''_n, s''_n}\sum_{\ell''_r, s''_r}\int_{\sigma_n^{(\mathrm{th})}}^{(\sqrt{s}-m_n)^2} d\sigma''_n\int_{\sigma_r^{(-)}(\sigma''_n)}^{\sigma_r^{(+)}(\sigma''_n)} d\sigma''_r\, f^*_{s''_r}(\sigma''_r) f_{s''_n}(\sigma''_n) \\
&\qquad\times \mathcal{C}^{J}_{\ell''_n s''_n;\ell''_r s''_r}(\sigma''_n, s, \sigma''_r)\,\widetilde{\mathcal{A}}^{J\,*}_{\ell''_r s''_r;\ell'_k s'_k}(\sigma''_r, s, \sigma'_k)\widetilde{\mathcal{A}}^{J}_{\ell''_n s''_n;\ell_j s_j}(\sigma''_n, s, \sigma_j)\Theta(s - s_{\mathrm{th}}) \\
&+ \frac{1}{64\pi^2\sqrt{s}}\frac{1}{|\mathbf{p}'^{\star}_k|}\sum_{r\neq k}\sum_{\ell''_r, s''_r}\int_{\sigma_r^{(\mathrm{th})}}^{(\sqrt{s}-m_r)^2} d\sigma''_r\,\mathcal{C}^{J}_{\ell'_k s'_k;\ell''_r s''_r}(\sigma'_k, s, \sigma''_r)\, f_{s''_r}(\sigma''_r)\,\widetilde{\mathcal{A}}^{J}_{\ell''_r s''_r;\ell_j s_j}(\sigma''_r, s, \sigma_j)\Theta(\sigma'_k - \sigma_k^{(\mathrm{th})}) \\
&+ \frac{1}{64\pi^2\sqrt{s}}\frac{1}{|\mathbf{p}^{\star}_j|}\sum_{n\neq j}\sum_{\ell''_n, s''_n}\int_{\sigma_n^{(\mathrm{th})}}^{(\sqrt{s}-m_n)^2} d\sigma''_n\,\mathcal{C}^{J}_{\ell''_n s''_n;\ell_j s_j}(\sigma''_n, s, \sigma_j)\, f^*_{s''_n}(\sigma''_n)\,\widetilde{\mathcal{A}}^{J\,*}_{\ell'_k s'_k;\ell''_n s''_n}(\sigma'_k, s, \sigma''_n)\Theta(\sigma_j - \sigma_j^{(\mathrm{th})}) \\
&+ \frac{\pi}{2|\mathbf{p}^{\star}_j||\mathbf{p}'^{\star}_k|}\,\mathcal{C}^{J}_{\ell'_k s'_k;\ell_j s_j}(\sigma'_k, s, \sigma_j)(1-\delta_{jk})\Theta(1-\cos^2\theta^{\star}_{kj})
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{C}^{J}_{\ell_k s_k;\ell_j s_j}(\sigma_k, s, \sigma_j) &= 2\pi\,\mathcal{N}_{s_j}\mathcal{N}_{s_k}\mathcal{N}_{\ell_j}\mathcal{N}_{\ell_k}\mathcal{N}_J^{-2}\sum_{\lambda_k,\lambda_k}\langle J\lambda_k|\ell_k 0 s_k\lambda_k\rangle\,\langle J\lambda_j|\ell_j 0 s_j\lambda_j\rangle \\
&\times d^{(s_k)}_{\lambda_k 0}(\cos\chi_k)\, d^{(J)}_{\lambda_k\lambda_j}(\cos\theta^{\star}_{kj})\, d^{(s_j)}_{\lambda_j 0}(\cos\chi_j)
\end{aligned}
$$

B-Matrix Parameterization

- Review **2→2** case:

$$f_{s_j}^{-1}(\sigma_j) = K_{s_j}^{-1}(\sigma_j) - i\rho_2(\sigma_j)$$

$$\Longrightarrow \quad f_{s_j}(\sigma_j) = K_{s_j}(\sigma_j) + i\rho_2(\sigma_j)K_{s_j}(\sigma_j)f_{s_j}(\sigma_j)$$

- Form similar to integral scattering equation, e.g. Lippman-Schwinger
- Postulate *B*-matrix parameterization

AJ et al. [*JPAC*], *ArXiV: 1809.10523*

$$\widetilde{\mathcal{A}}_{kj} = \widetilde{\mathcal{B}}_{kj} + \sum_n \int \widetilde{\mathcal{B}}_{kn}\tau_n\widetilde{\mathcal{A}}_{nj}$$

Isobar-spectator amplitudes

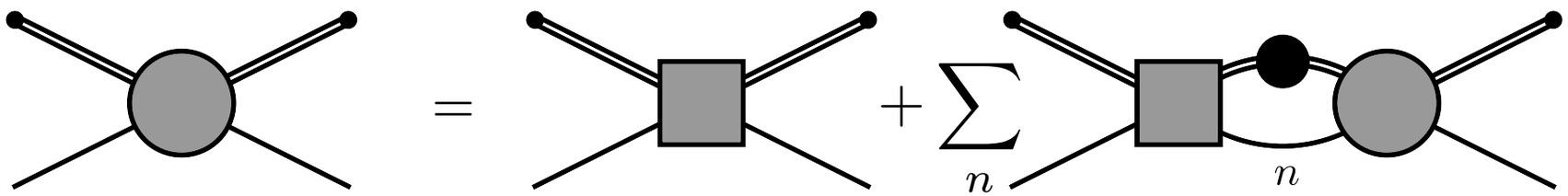
Driving term

Known function

B-Matrix Parameterization

- To ensure unitarity relation is satisfied, must have form

$$\widetilde{\mathcal{A}}_{kj} = \widetilde{\mathcal{B}}_{kj} + \sum_n \int \widetilde{\mathcal{B}}_{kn} \tau_n \widetilde{\mathcal{A}}_{nj}$$



$$\widetilde{\mathcal{B}}_{kj} = \widetilde{\mathcal{R}}_{kj} + \widetilde{\mathcal{E}}_{kj}$$

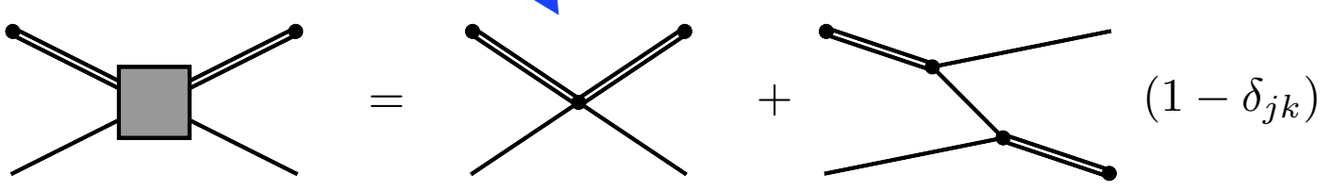
Real Function - Unconstrained

OPE (required by unitarity)

2→2 amplitude

$$\tau_n = \rho_3(s, \sigma_n) f_n(\sigma_n)$$

Phase space between isobar and spectator



B-Matrix Parameterization

- For simplicity, assume isobars in (12) and (23) only
  - Considering *B*-matrix as matrix equations, can invert

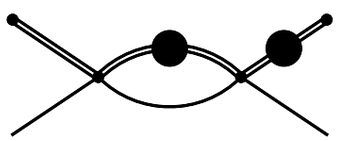
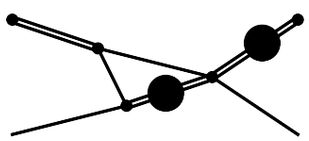
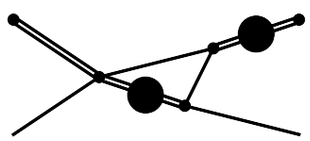
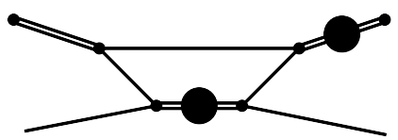
$$\widetilde{\mathcal{A}}_{13}(s) = [\mathbb{1} - \mathcal{K}_{11}(s)\tau_1(s)]^{-1}\,\widetilde{\mathcal{B}}_{13}(s)$$
$$\widetilde{\mathcal{A}}_{33}(s) = [\mathbb{1} - \mathcal{K}_{33}(s)\tau_3(s)]^{-1}\,\mathcal{K}_{33}(s)$$
$$\widetilde{\mathcal{A}}_{31}(s) = [\mathbb{1} - \mathcal{K}_{33}(s)\tau_3(s)]^{-1}\,\widetilde{\mathcal{B}}_{31}(s)$$
$$\widetilde{\mathcal{A}}_{11}(s) = [\mathbb{1} - \mathcal{K}_{11}(s)\tau_1(s)]^{-1}\,\mathcal{K}_{11}(s)$$

$$\mathcal{K}_{kj} = \sum_n \widetilde{\mathcal{B}}_{kn}\tau_n\widetilde{\mathcal{B}}_{nj}$$

Understand analytic properties of kernel $\Rightarrow$ Understand solution!

Kernel function — Contains rescattering physics

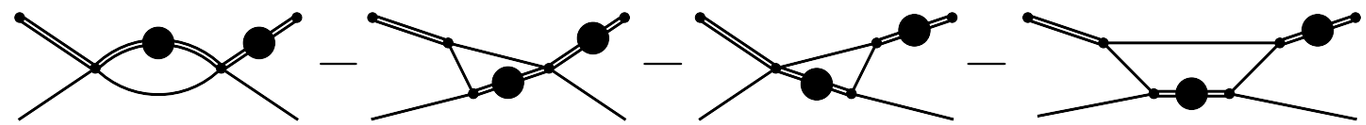
Bubble **Triangles** **Box**

$[\mathbb{1} - \mathcal{K}_{33}(s)\tau_3(s)]^{-1} = \Big[\ \mathbb{1} -$  $-$  $-$  $-$  $\Big]^{-1}$

B-Matrix Parameterization

- B-Matrix relation unitarizes kernel functions

$$[\mathbb{1} - \mathcal{K}_{33}(s)\tau_3(s)]^{-1} = \Big[\, \mathbb{1} -$$



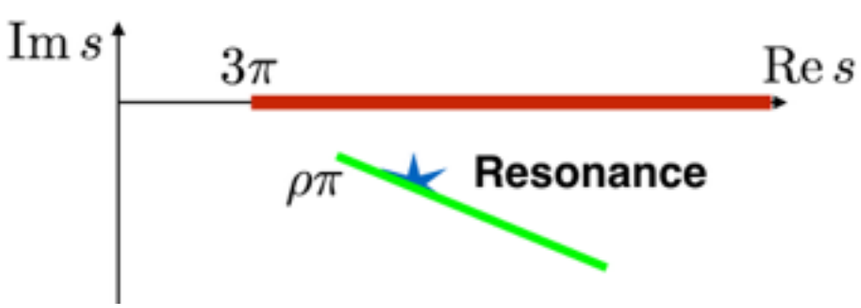
$$\Big]^{-1}$$

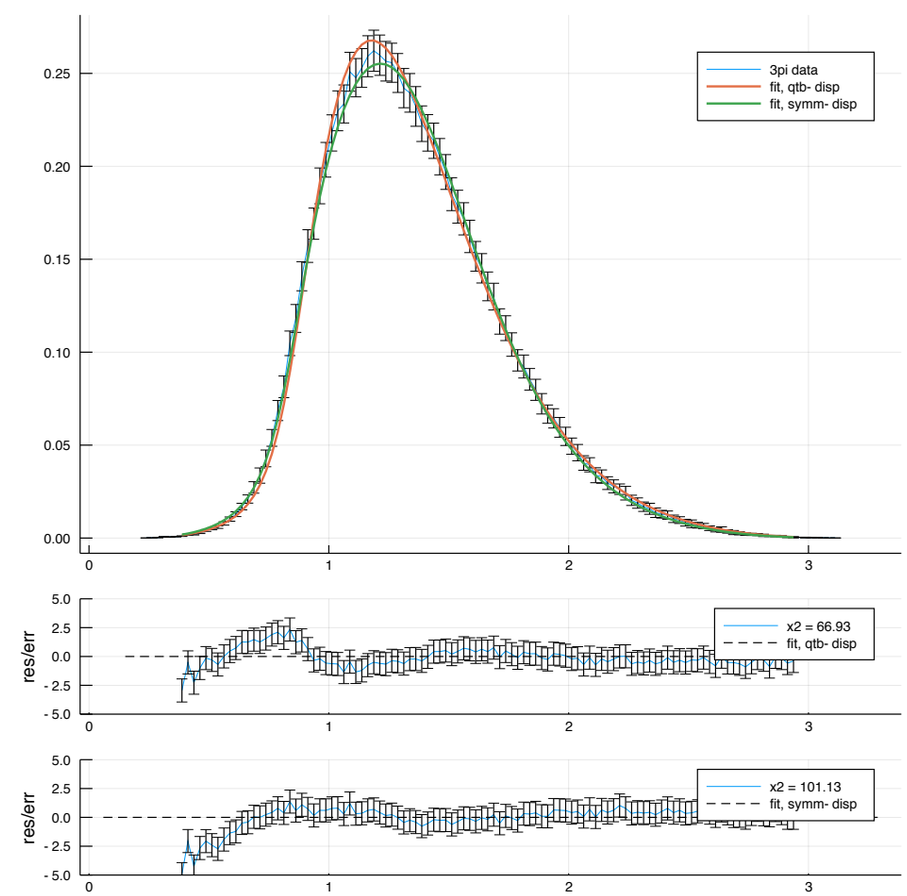
- Bubble unitarization gives quasi-two-body phase space
  - e.g. $a_1(1260) \to 3\pi$

M. Mikhasenko et al. [JPAC], arXiv:1810.00016v1 (2018)

$$\rho_{\mathrm{QTB}}(s) = \frac{1}{2\pi(8\pi)^2 s} \int_{4m_\pi^2}^{\left(\sqrt{s}-m_\pi\right)^2} f_\rho^*\left(\sigma_1\right) f_\rho\left(\sigma_1\right) \frac{\sqrt{\lambda_1 \lambda_{s1}}}{\sigma_1} \mathrm{d}\sigma_1$$

Two-body phase space weighted by isobar



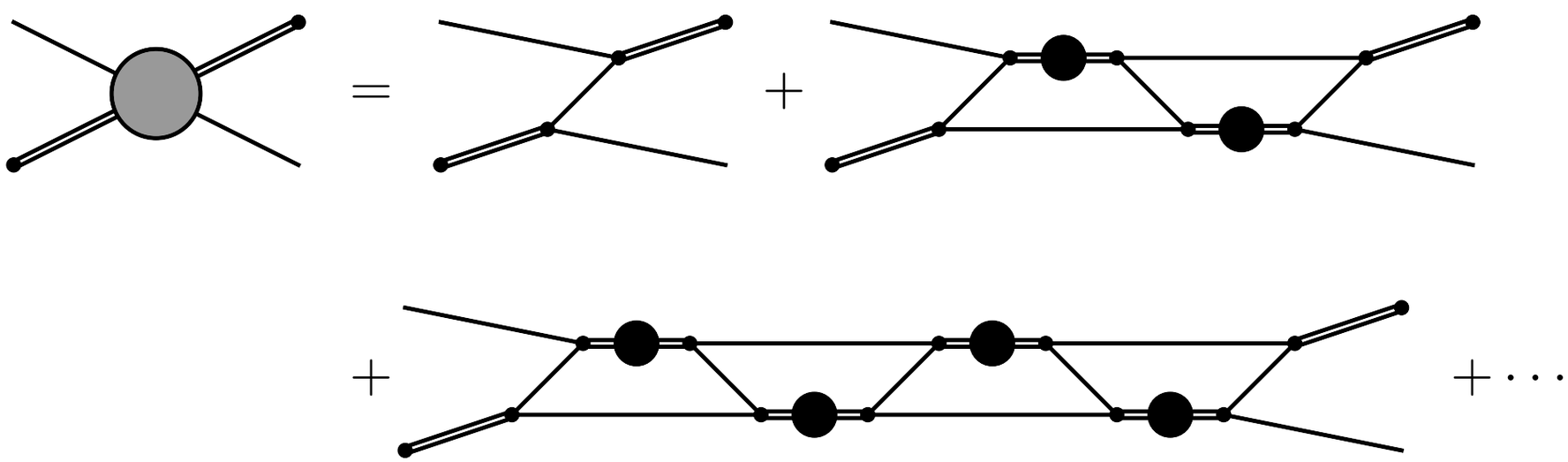


B-Matrix Parameterization

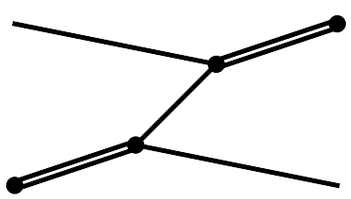
- Ladder summation of exchanges

e.g. OPE driving term only

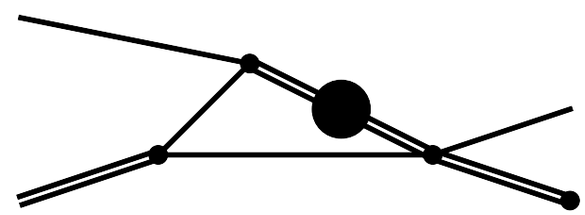
$$\widetilde{\mathcal{A}}_{13} = [\mathbb{1} - \mathcal{K}_{11} \cdot \tau_1]^{-1} \widetilde{\mathcal{E}}_{13}$$



- Focus on properties of OPE and triangle amplitudes



OPE



Triangle

One-Particle Exchange

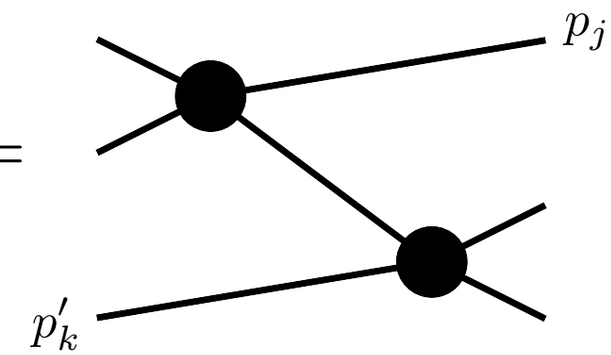
- Consider S-wave projection

$$\widetilde{\mathcal{E}}^{S}_{31}\left(\sigma'_k, s, \sigma_j\right) = \frac{1}{4\left|\mathbf{p}'^{\star}_3\right|\left|\mathbf{p}^{\star}_1\right|} \log\left(\frac{z_{13}-1}{z_{13}+1}\right)$$

S-wave

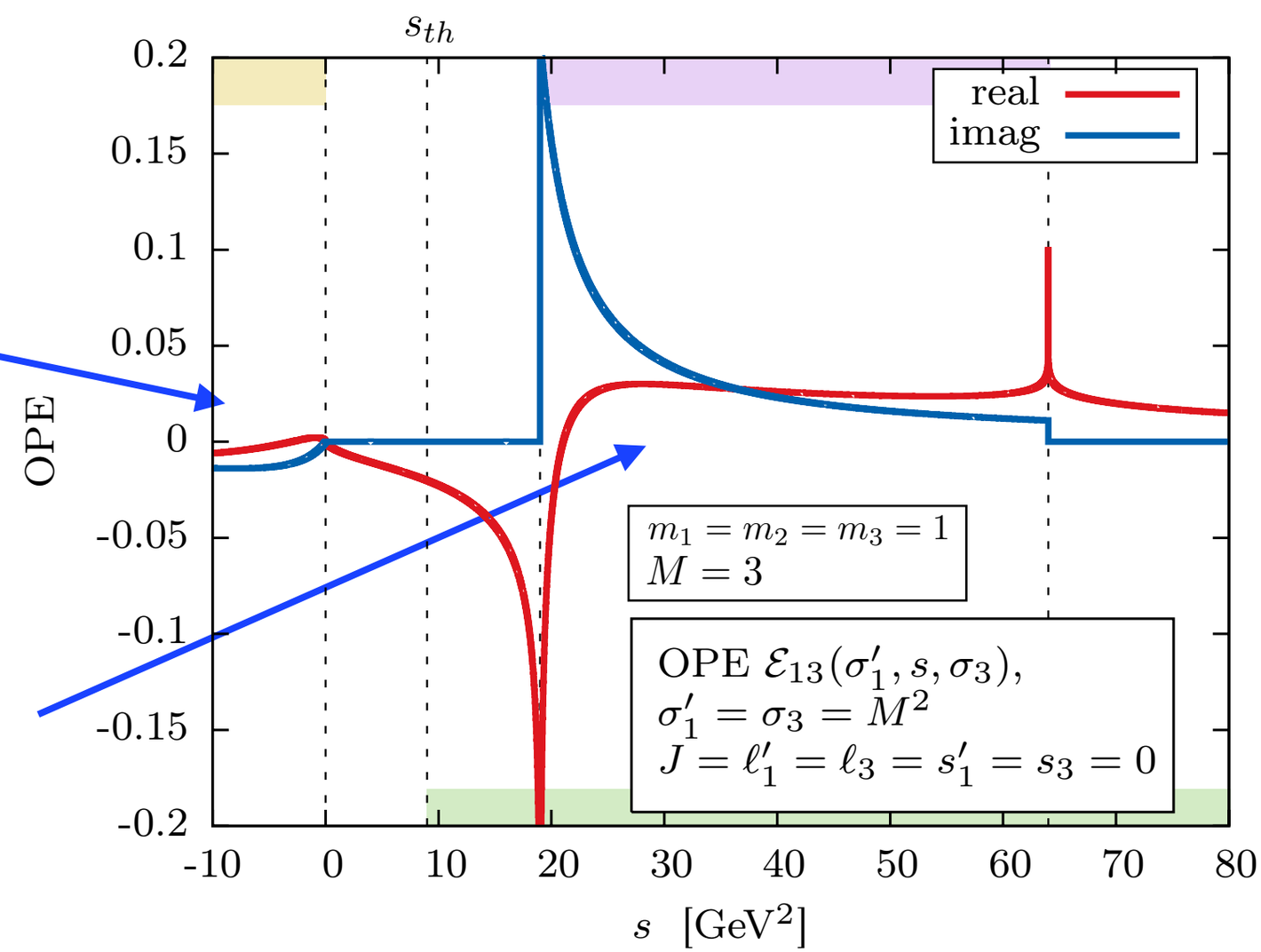
kinematics

$\mathcal{E}_{kj}(\mathbf{p}';\mathbf{p}) =$



Virtual Particle Exchange

Real Particle Exchange



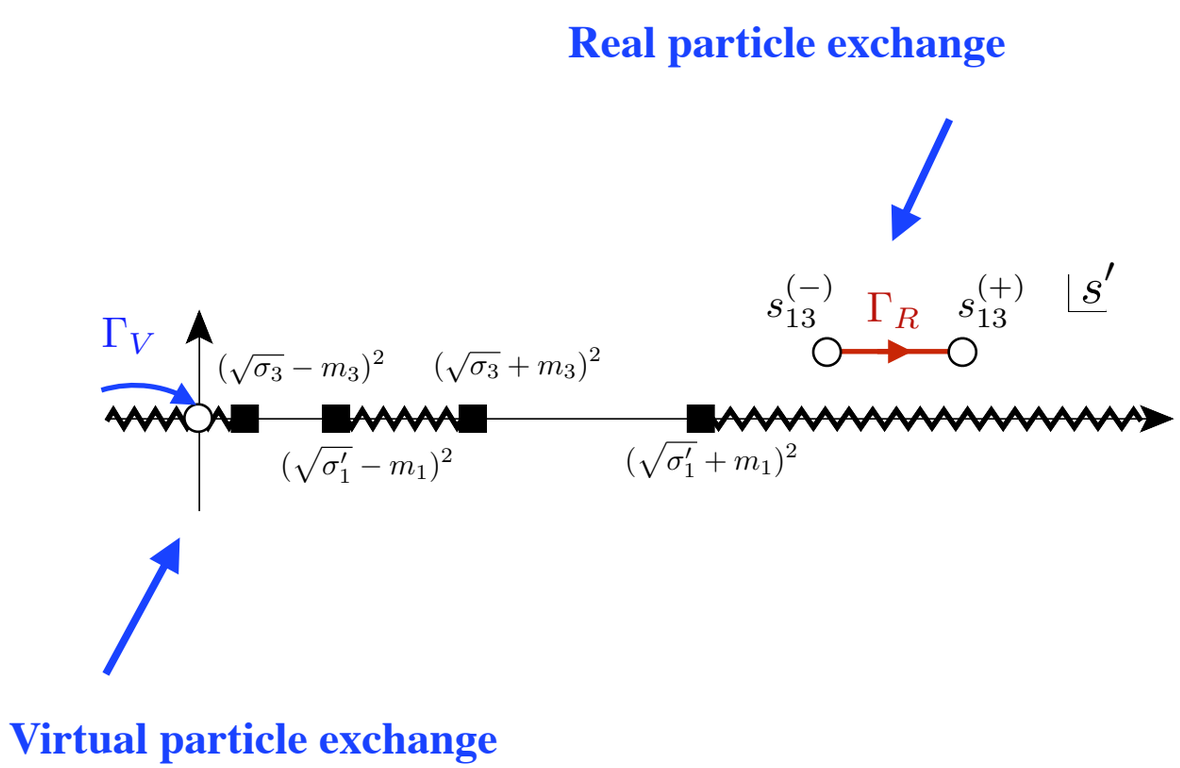
One-Particle Exchange

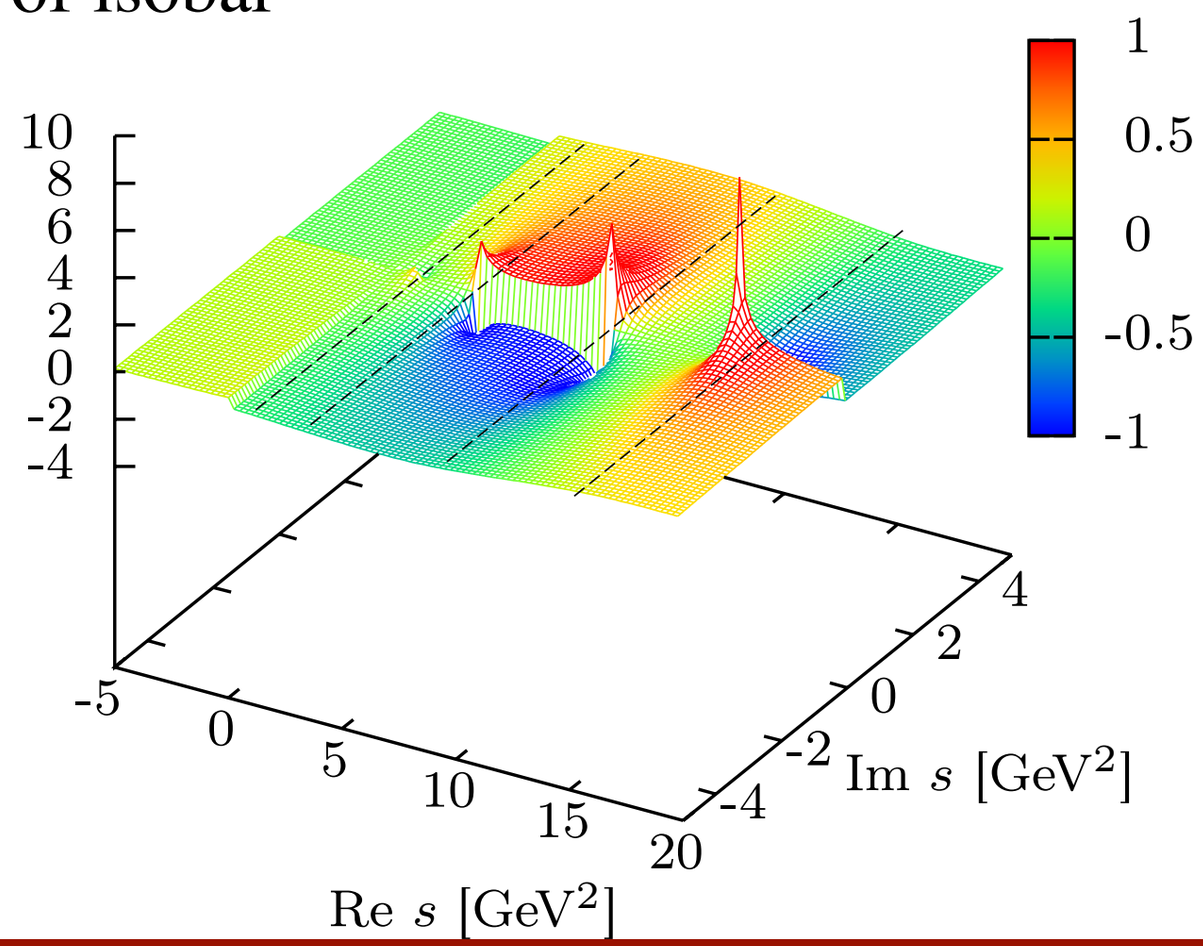
- Consider dispersive representation

$$\widetilde{\mathcal{E}}^S_{kj}(\sigma'_k, s, \sigma_j) = \int_{\Gamma_V} ds' \frac{1}{s' - s - i\epsilon} \frac{1}{4|\mathbf{p}'^\star_k||\mathbf{p}^\star_j|} + \int_{\Gamma_R} ds' \frac{1}{s' - s - i\epsilon} \frac{1}{4|\mathbf{p}^\star_k||\mathbf{p}^\star_j|}$$

$$\mathrm{Im}\,\widetilde{\mathcal{E}}^S_{kj}(\sigma'_k, s, \sigma_j) = \frac{\pi}{4\,|\mathbf{p}^\star_j|\,|\mathbf{p}'^\star_k|}$$

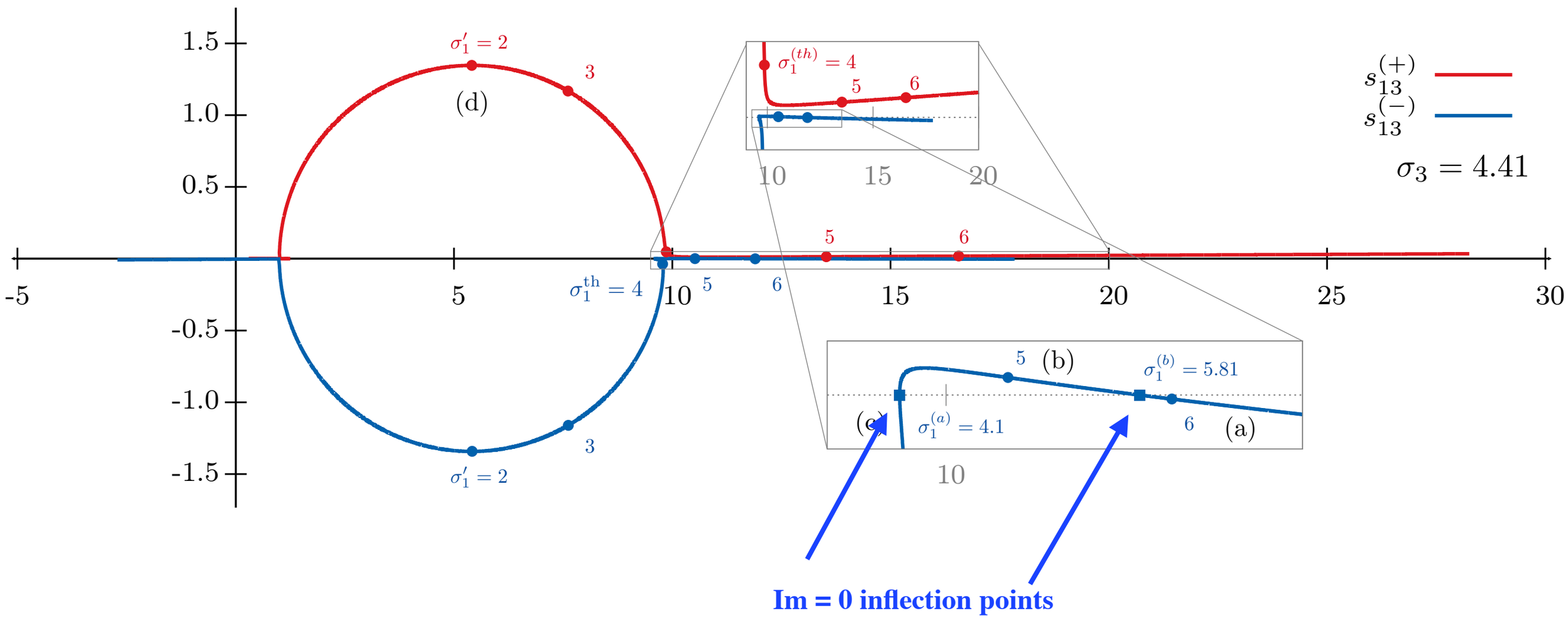
- OPE has 4 branch points: $0,\ \infty,\ s_{kj}^{(+)},\ s_{kj}^{(-)}$
  - $s_{kj}^{(+)},\ s_{kj}^{(-)}$ depend on kinematics of isobar





OPE Branch Points

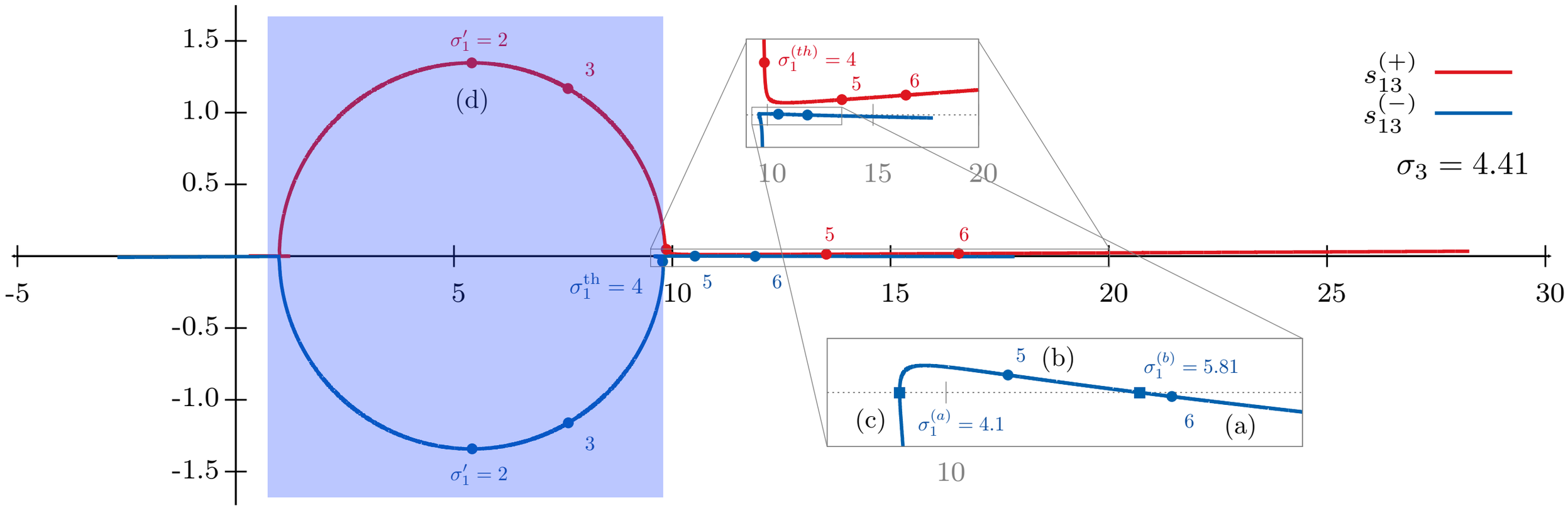
- $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ depend on kinematics of isobar



$$s_{kj}^{(\pm)} = \frac{1}{2\mu_{jk}^2}\left[\left(m_k^2-\sigma_j\right)\left(m_j^2-\sigma_k'\right)-\mu_{jk}^4+\mu_{jk}^2\left(m_k^2+m_j^2+\sigma_j+\sigma_k'\right)\pm\lambda^{1/2}\left(\mu_{jk}^2,m_k^2,\sigma_j\right)\lambda^{1/2}\left(\mu_{jk}^2,m_j^2,\sigma_k'\right)\right]$$

OPE Branch Points

- $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ depend on kinematics of isobar

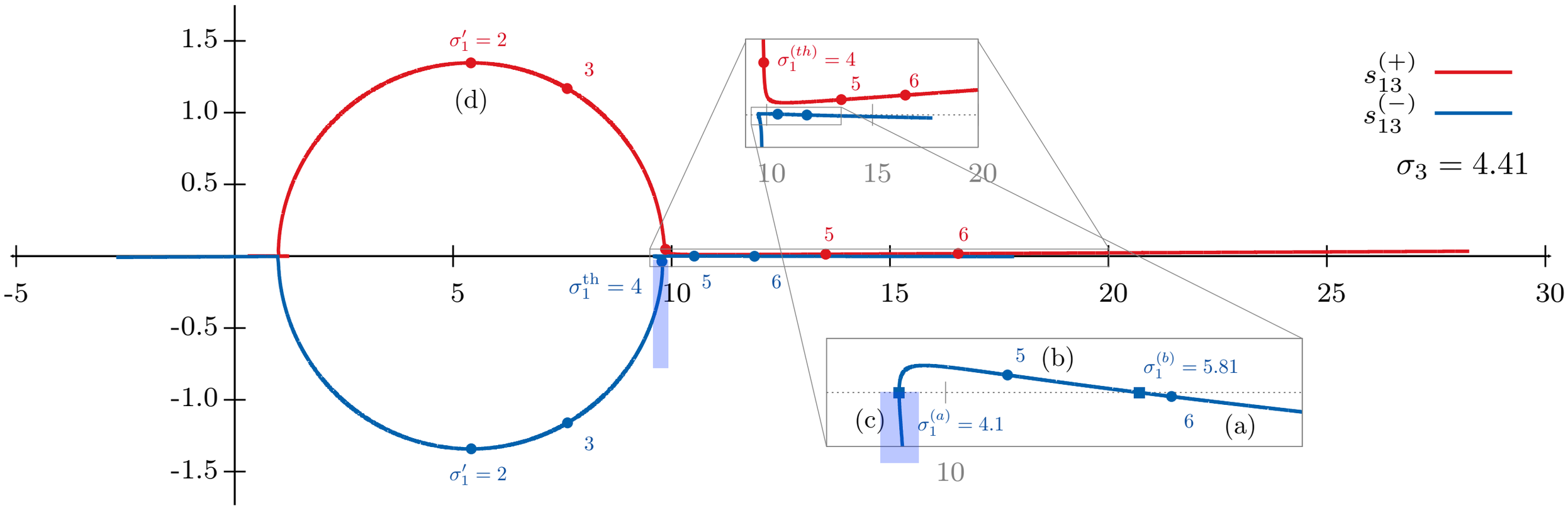


- Case: $\sigma_1' \leq \sigma_1^{(\mathrm{th})}$ — Isobar cannot form/decay

$$s_{kj}^{(\pm)} = \frac{1}{2\mu_{jk}^2}\left[\left(m_k^2 - \sigma_j\right)\left(m_j^2 - \sigma_k'\right) - \mu_{jk}^4 + \mu_{jk}^2\left(m_k^2 + m_j^2 + \sigma_j + \sigma_k'\right) \pm \lambda^{1/2}\left(\mu_{jk}^2, m_k^2, \sigma_j\right)\lambda^{1/2}\left(\mu_{jk}^2, m_j^2, \sigma_k'\right)\right]$$

OPE Branch Points

- $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ depend on kinematics of isobar

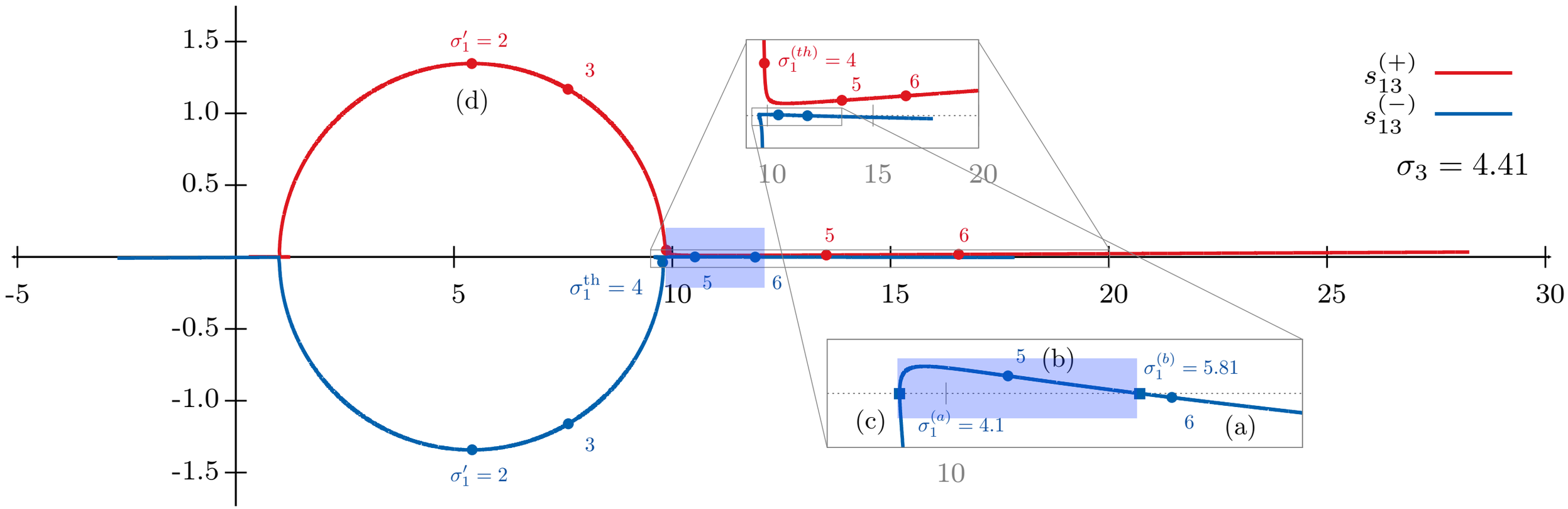


- Case: $\sigma_1^{(\mathrm{th})} \leq \sigma_1' \leq \sigma_1^{(a)}$

$$s_{kj}^{(\pm)} = \frac{1}{2\mu_{jk}^2}\left[\left(m_k^2 - \sigma_j\right)\left(m_j^2 - \sigma_k'\right) - \mu_{jk}^4 + \mu_{jk}^2\left(m_k^2 + m_j^2 + \sigma_j + \sigma_k'\right) \pm \lambda^{1/2}\left(\mu_{jk}^2, m_k^2, \sigma_j\right)\lambda^{1/2}\left(\mu_{jk}^2, m_j^2, \sigma_k'\right)\right]$$

OPE Branch Points

- $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ depend on kinematics of isobar

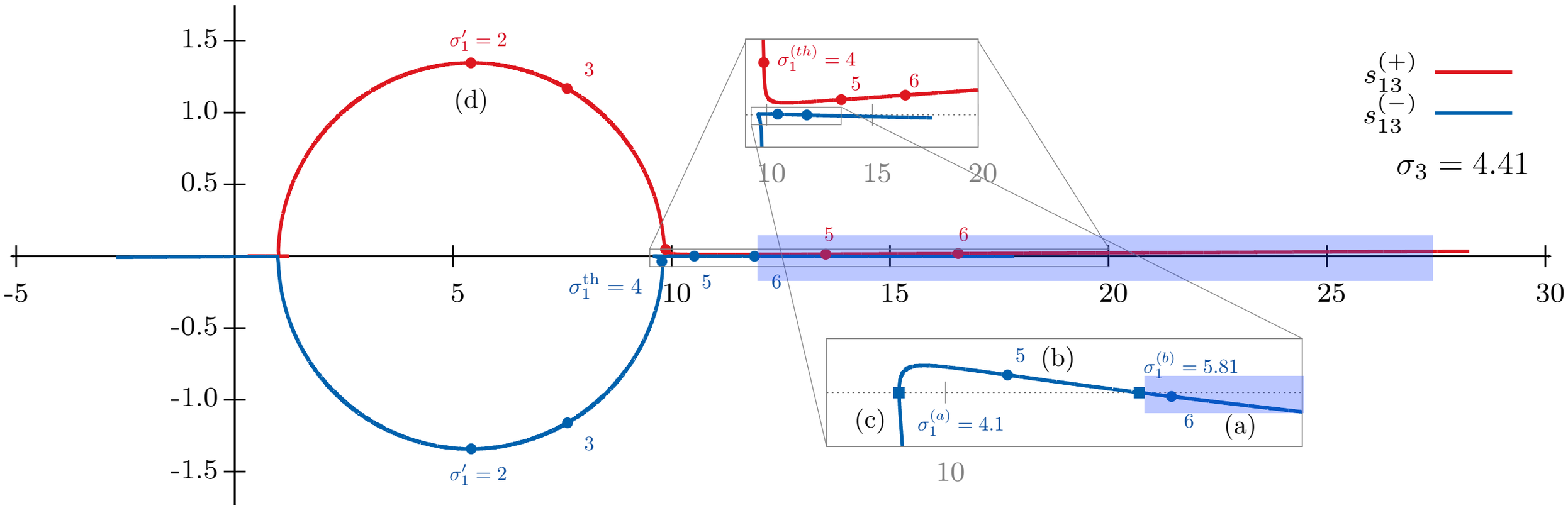


- Case: $\sigma_1^{(a)} \leq \sigma_1' \leq \sigma_1^{(b)}$ — Real Particle Exchange

$$s_{kj}^{(\pm)} = \frac{1}{2\mu_{jk}^2}\left[\left(m_k^2 - \sigma_j\right)\left(m_j^2 - \sigma_k'\right) - \mu_{jk}^4 + \mu_{jk}^2\left(m_k^2 + m_j^2 + \sigma_j + \sigma_k'\right) \pm \lambda^{1/2}\left(\mu_{jk}^2, m_k^2, \sigma_j\right)\lambda^{1/2}\left(\mu_{jk}^2, m_j^2, \sigma_k'\right)\right]$$

OPE Branch Points

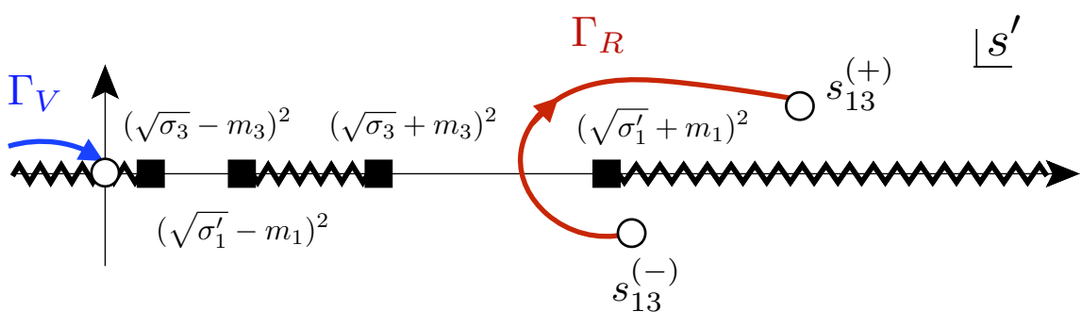
- $s_{kj}^{(+)}$, $s_{kj}^{(-)}$ depend on kinematics of isobar



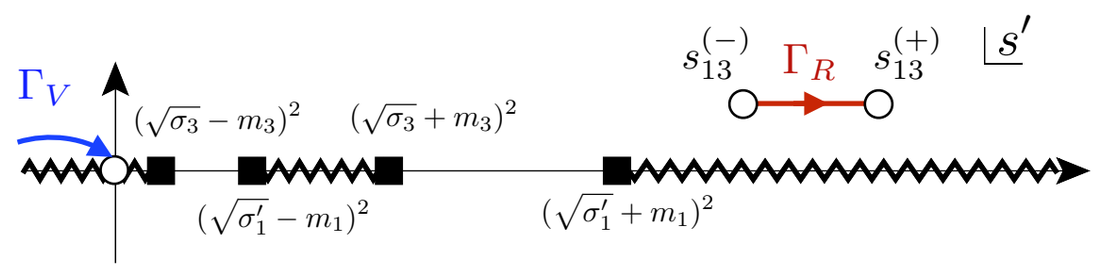
- Case: $\sigma_1' \geq \sigma_1^{(b)}$

$$s_{kj}^{(\pm)} = \frac{1}{2\mu_{jk}^2}\left[\left(m_k^2 - \sigma_j\right)\left(m_j^2 - \sigma_k'\right) - \mu_{jk}^4 + \mu_{jk}^2\left(m_k^2 + m_j^2 + \sigma_j + \sigma_k'\right) \pm \lambda^{1/2}\left(\mu_{jk}^2, m_k^2, \sigma_j\right)\lambda^{1/2}\left(\mu_{jk}^2, m_j^2, \sigma_k'\right)\right]$$

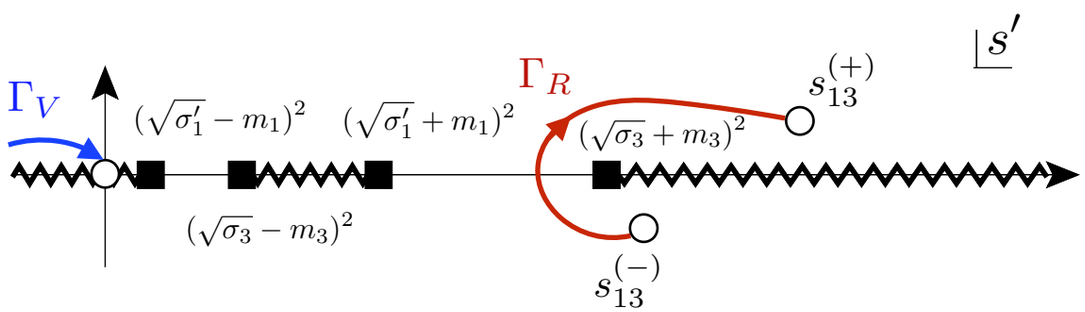
OPE Branch Points



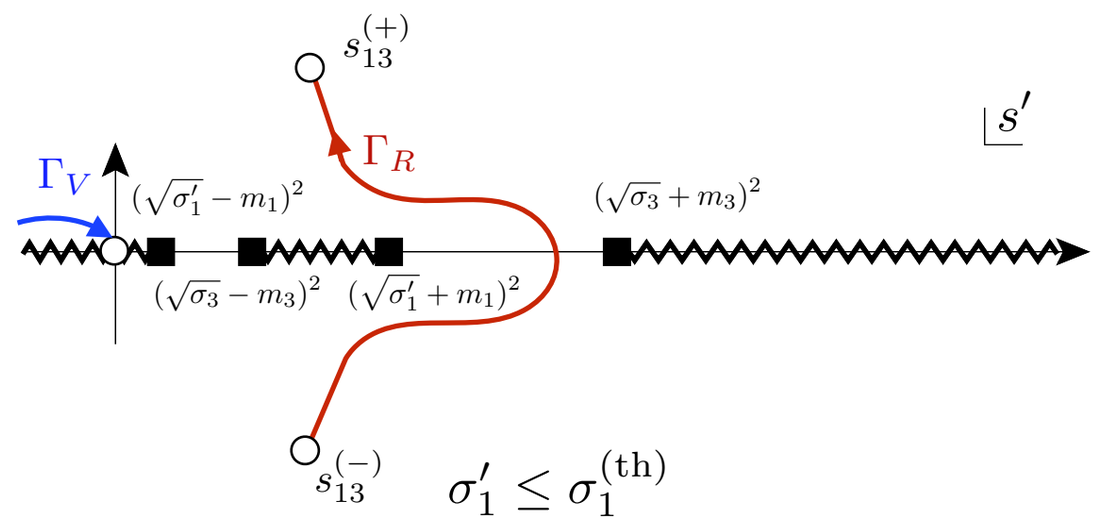
$\sigma_1' \geq \sigma_1^{(b)}$



$\sigma_1^{(a)} \leq \sigma_1' \leq \sigma_1^{(b)}$



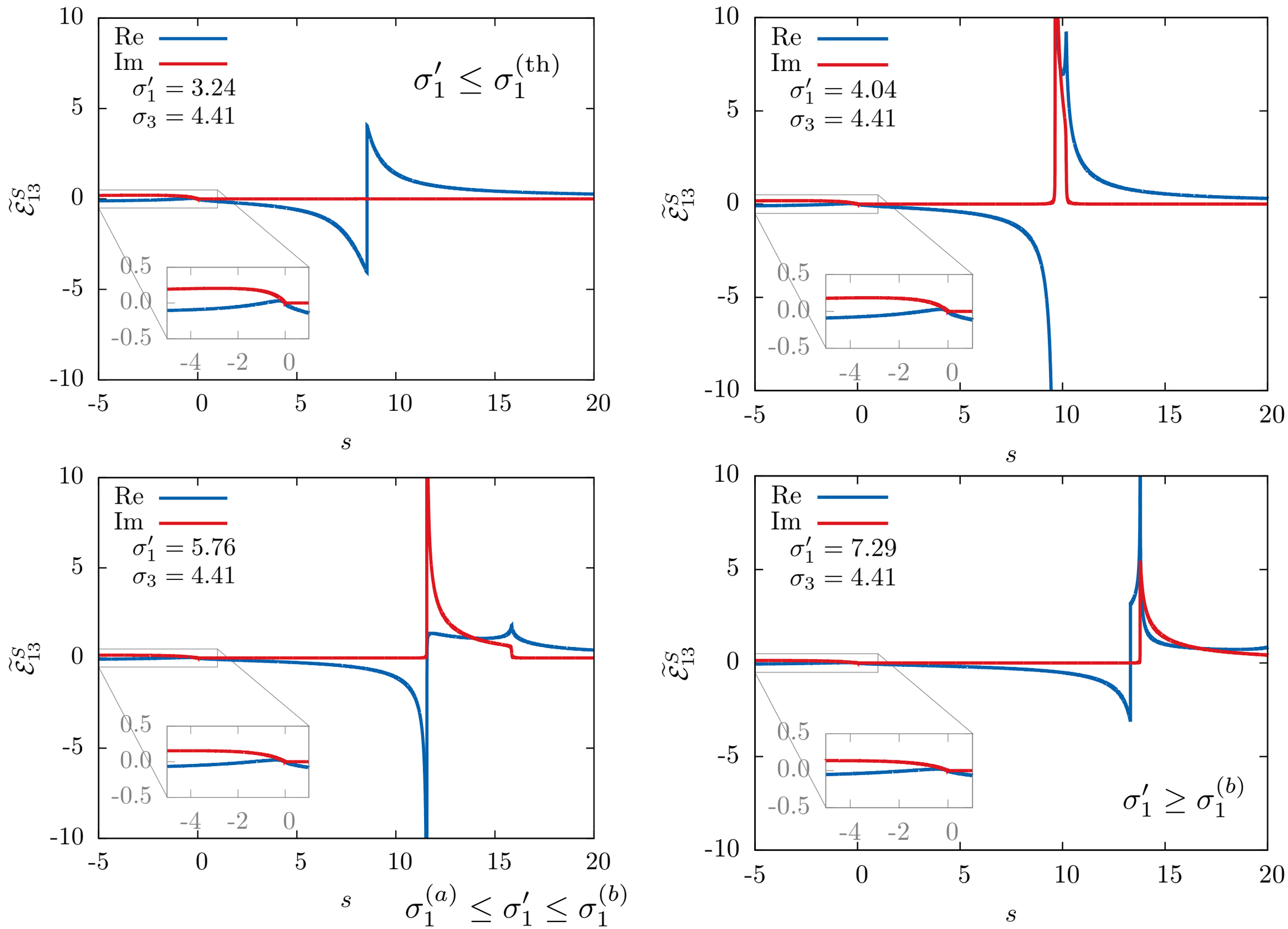
$\sigma_1^{(\mathrm{th})} \leq \sigma_1' \leq \sigma_1^{(a)}$



$\sigma_1' \leq \sigma_1^{(\mathrm{th})}$

OPE

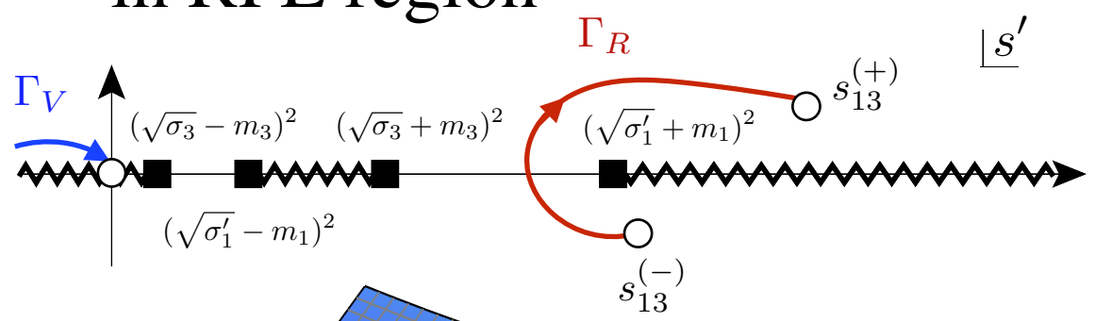
$\sigma_1^{(\mathrm{th})} \leq \sigma_1' \leq \sigma_1^{(a)}$

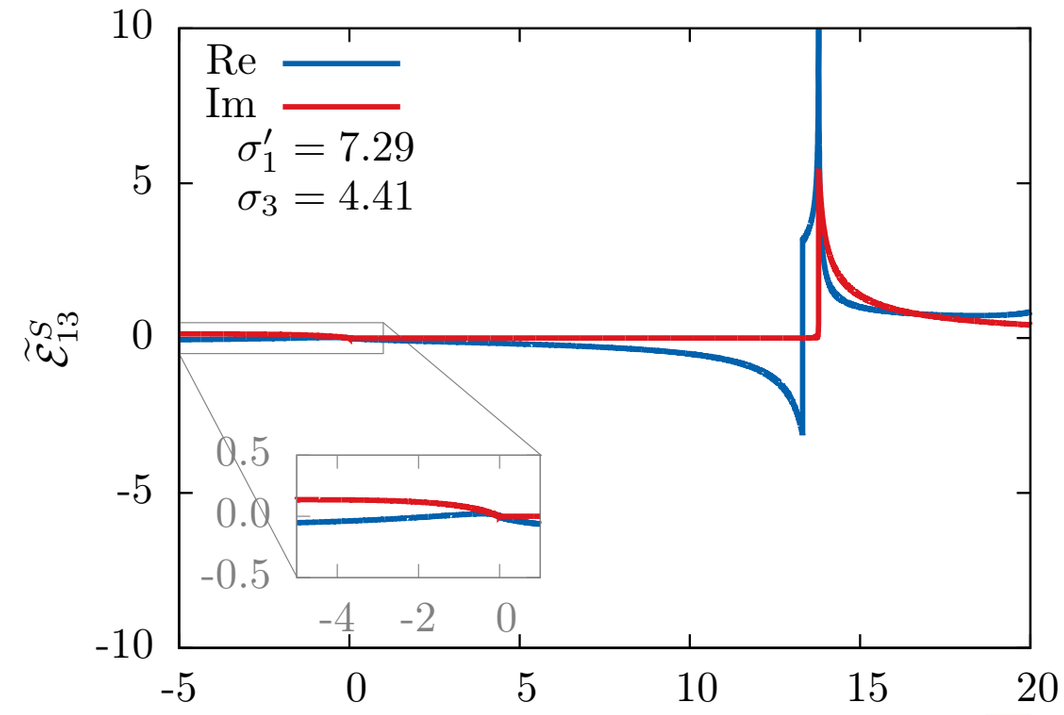


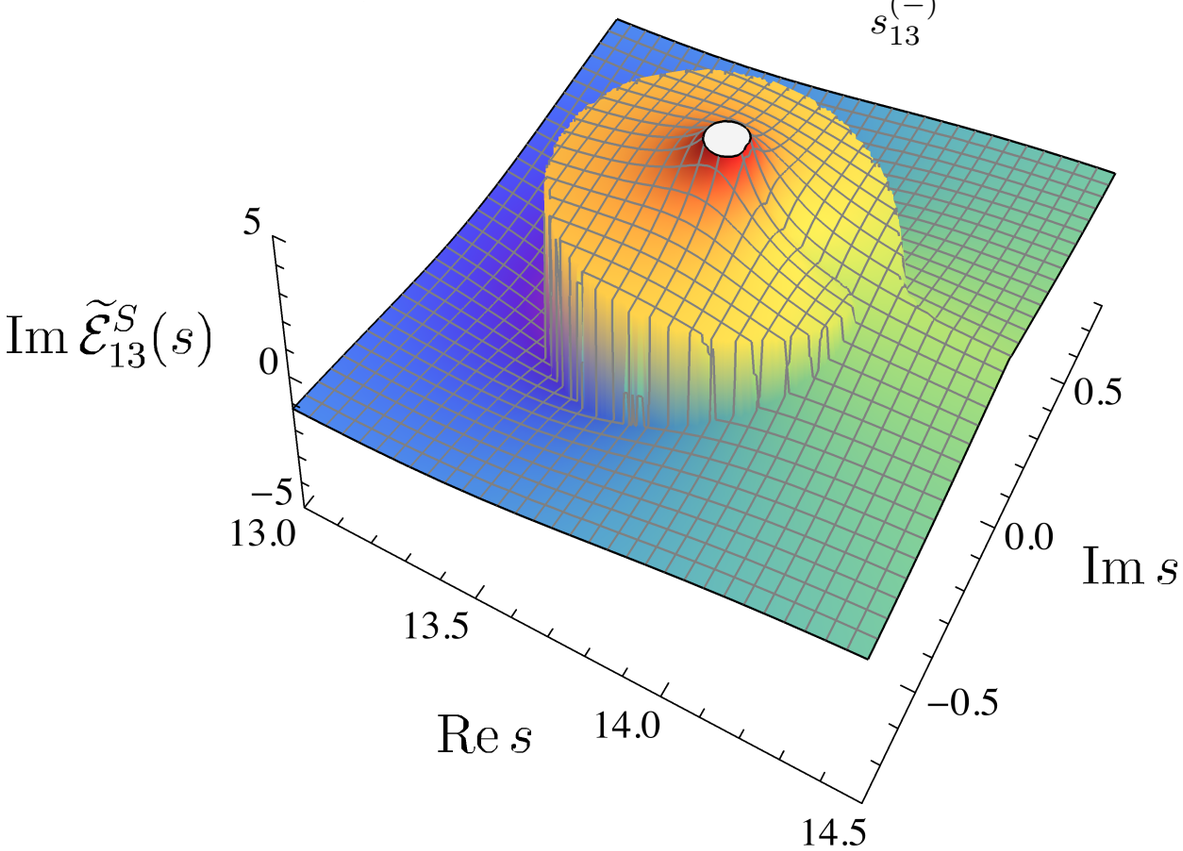
$\sigma_1^{(a)} \leq \sigma_1' \leq \sigma_1^{(b)}$

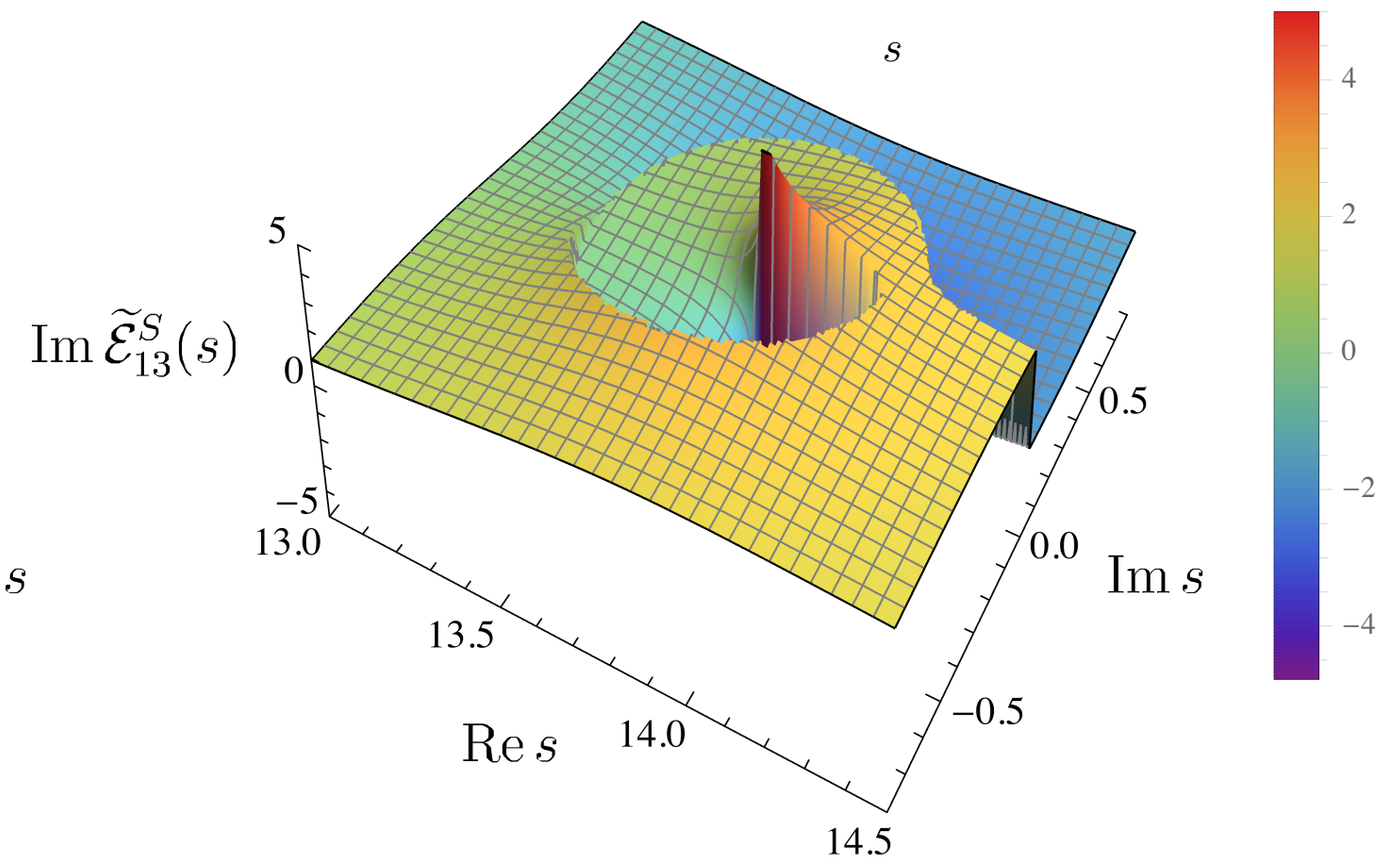
OPE

- In some cases with real particle exchange, OPE has circular cut
  - Triangle singularities occur in RPE region



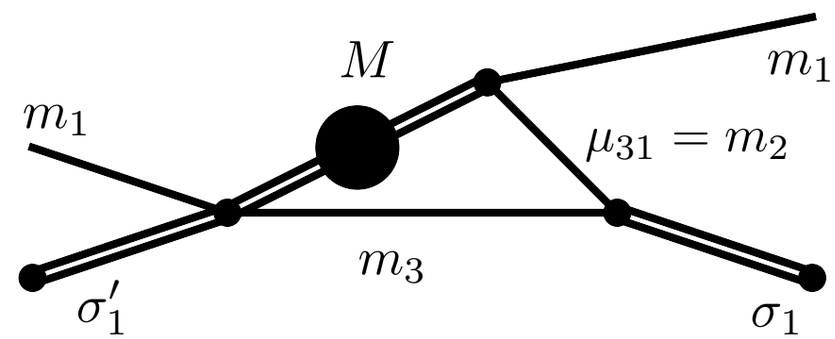






Triangle Amplitude

- Start with kernel - triangle amplitude
  - again, S-wave
  - narrow width approx for isobar



phase space

OPE

$$\mathcal{T}_B(s) = \int_{\sigma_3^{(\mathrm{th})}}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_3'',s,\sigma_1\right)}{M^2-\sigma_3''-i\epsilon}$$

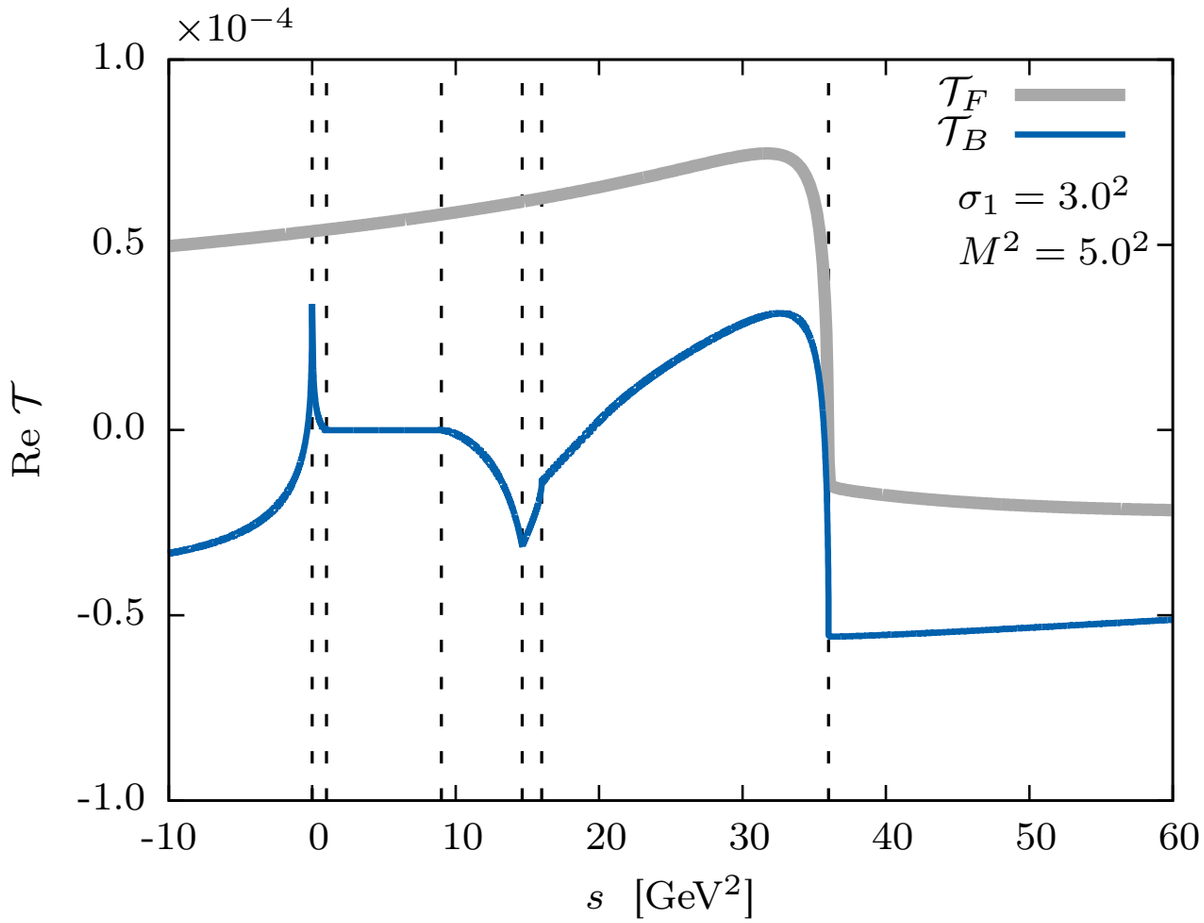
Integration over physical region

Isobar amplitude

- Compare analytic structure with Feynman triangle

$$\mathcal{T}_F(s) = \int_{\Gamma_T} ds' \frac{\rho_3\left(s',M^2\right)\widetilde{\mathcal{E}}_{13}^{S}\left(M^2,s',\sigma_1\right)}{s'-s-i\epsilon}$$

Triangle Amplitude

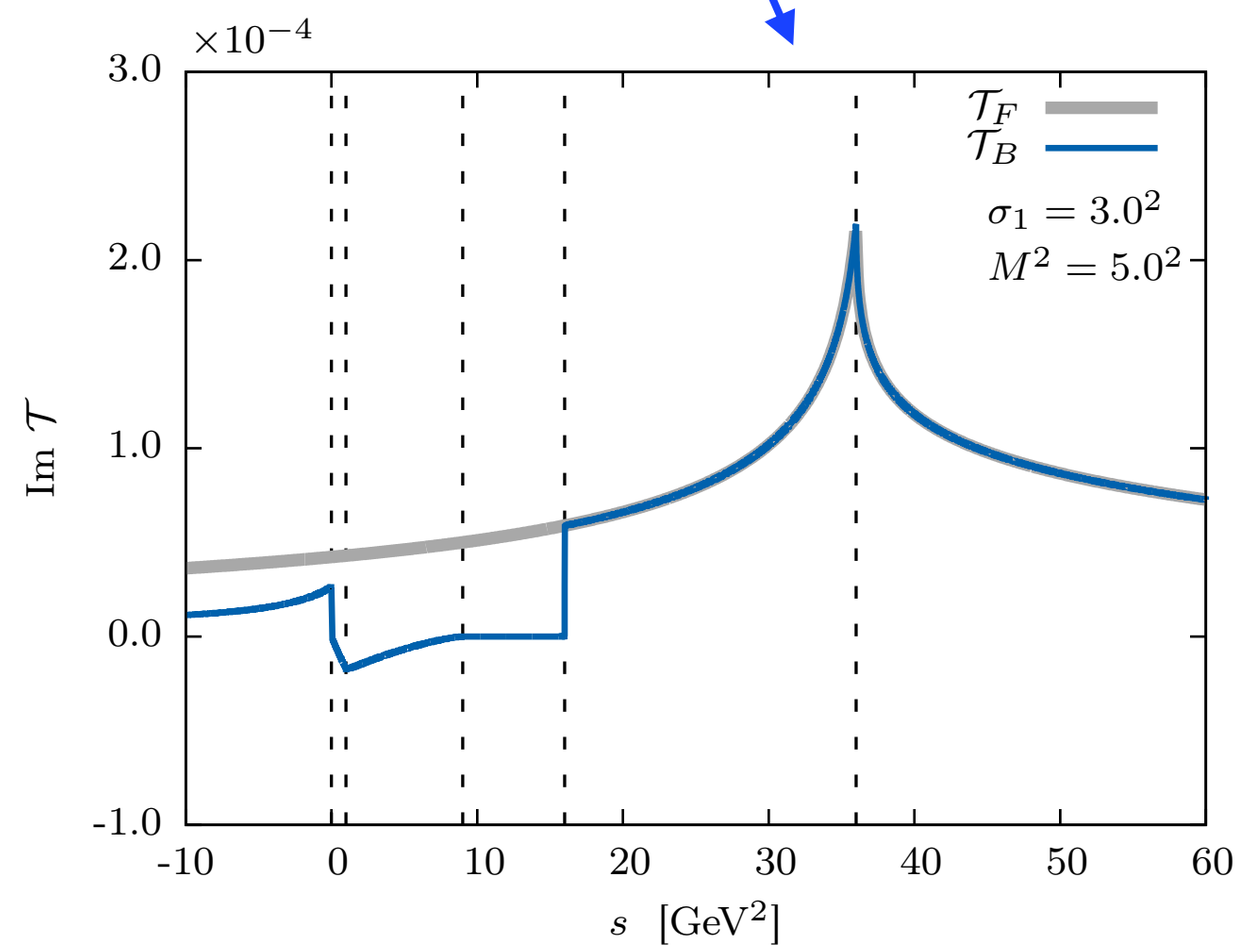


$$\mathcal{T}_B(s) = \int_{\sigma_3^{(\mathrm{th})}}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_3'',s,\sigma_1\right)}{M^2-\sigma_3''-i\epsilon}$$

Real parts differ everywhere

Imaginary parts identical above production threshold

Extra kinks due to Endpoint and Pinch singularities of B-matrix Triangle



Triangle Amplitude

- Can compare to other formalisms, e.g, Mai et al.

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak, Eur. Phys. J. A53, 177 (2017)

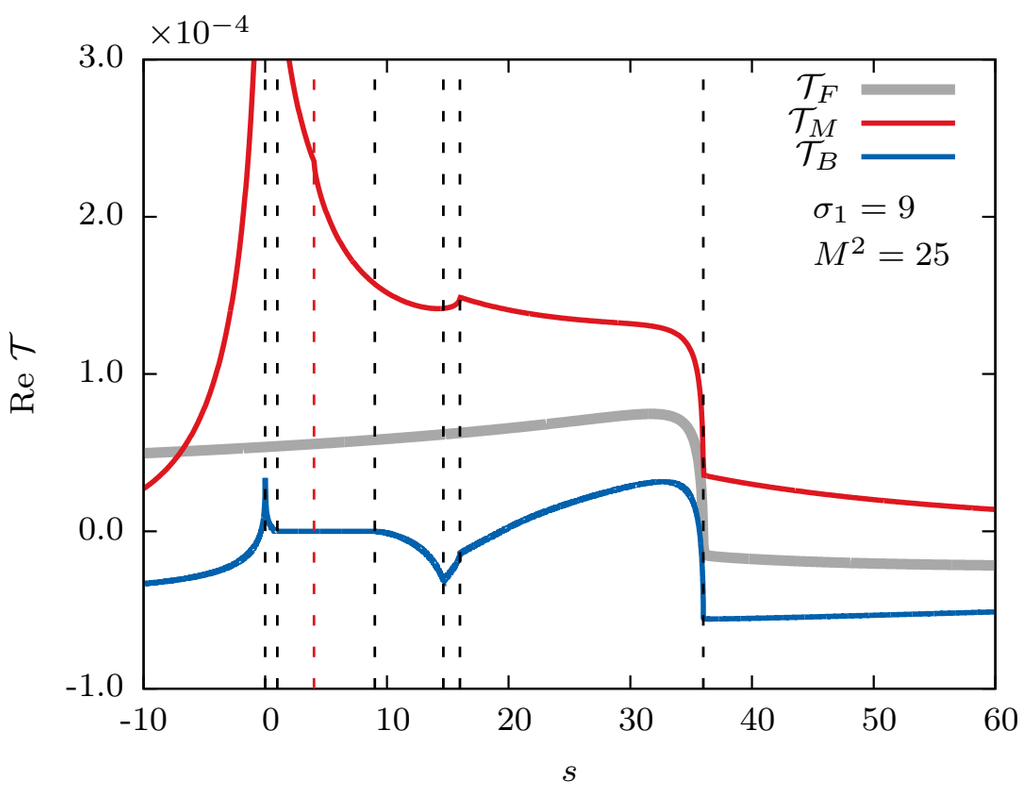
$$\mathcal{T}_M(s) = \int_{-\infty}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3'' \frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_3'',s,\sigma_1\right)}{M^2-\sigma_3''-i\epsilon}$$

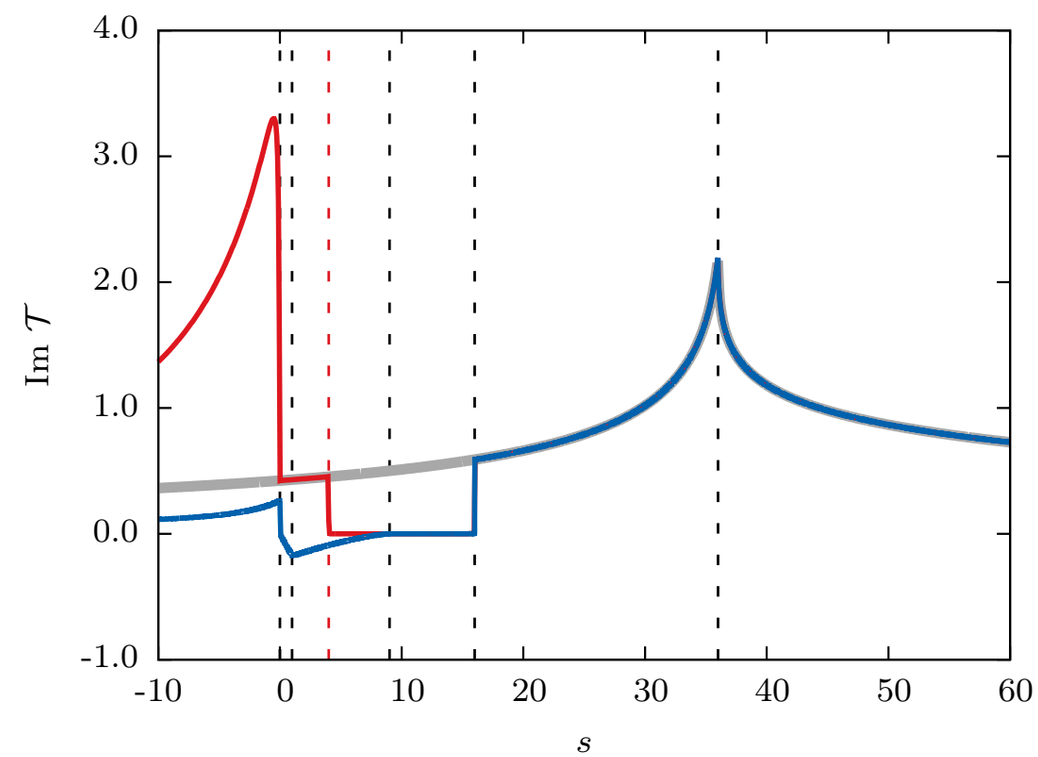
Includes off-shell physics

- Aitchison & Pasquier resolve the differences to Feynman

I. J. R. Aitchison and R. Pasquier, Phys. Rev. 152, 1274 (1966)

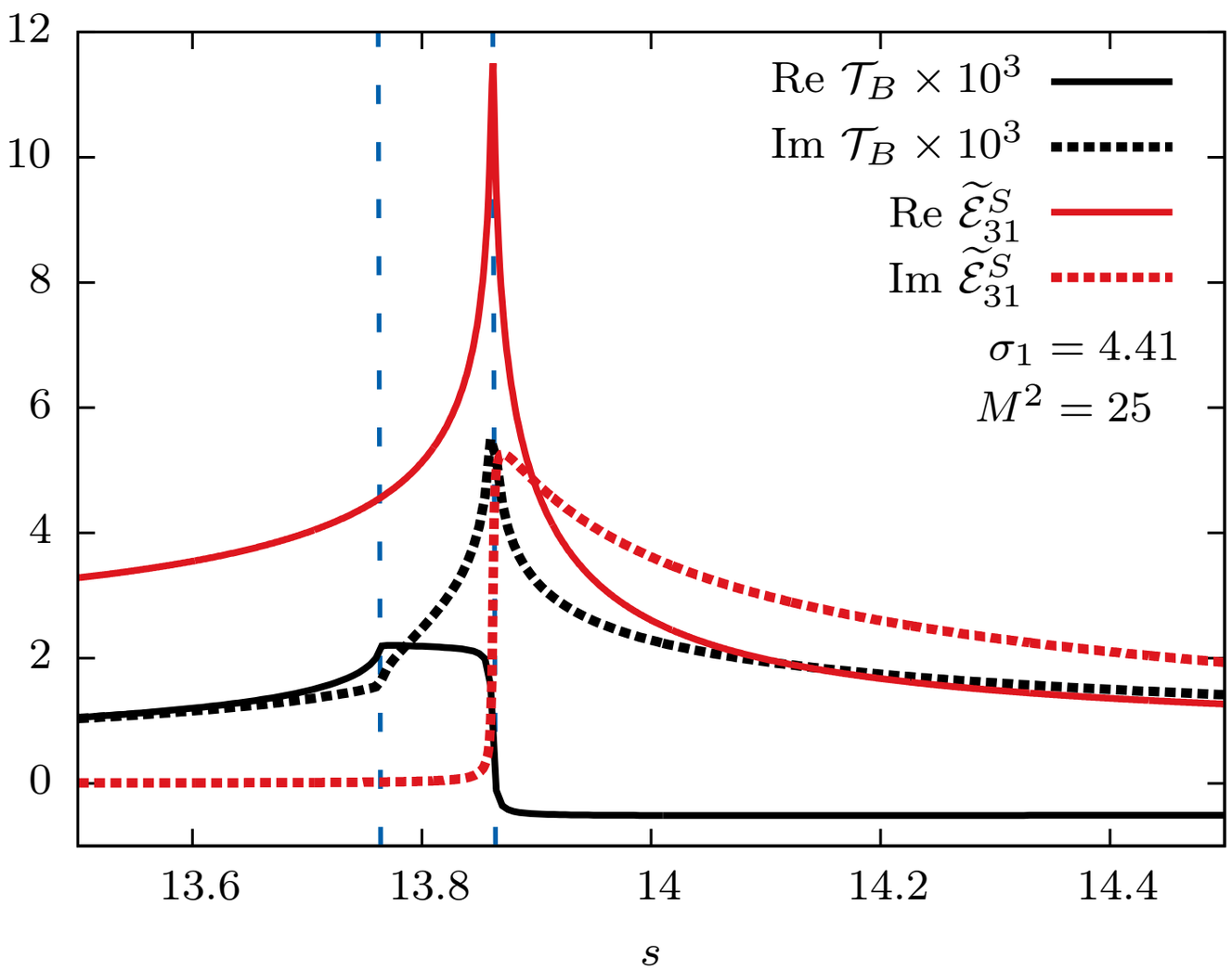
Imaginary parts identical above production threshold

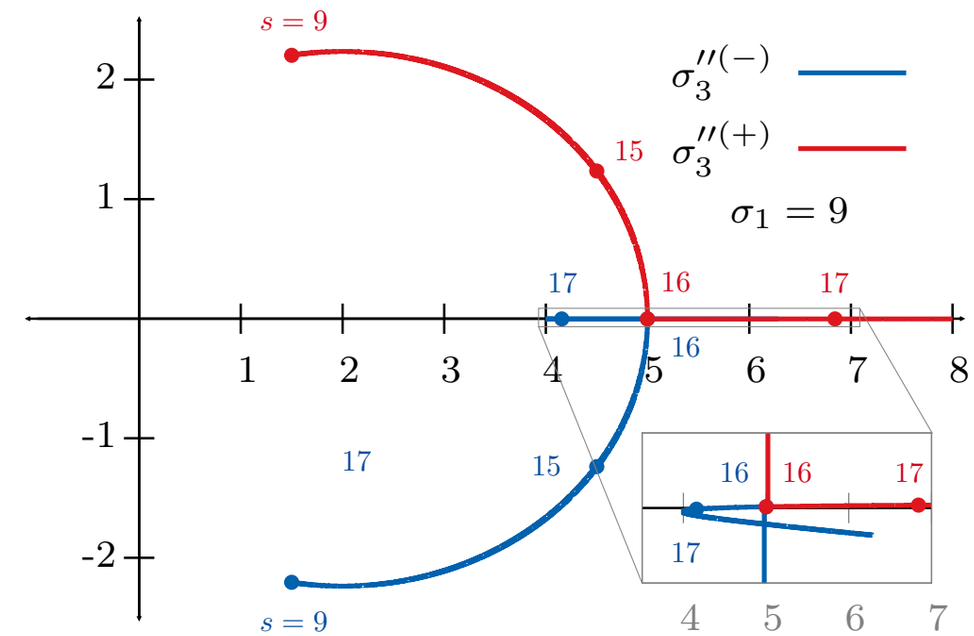




Triangle Singularity

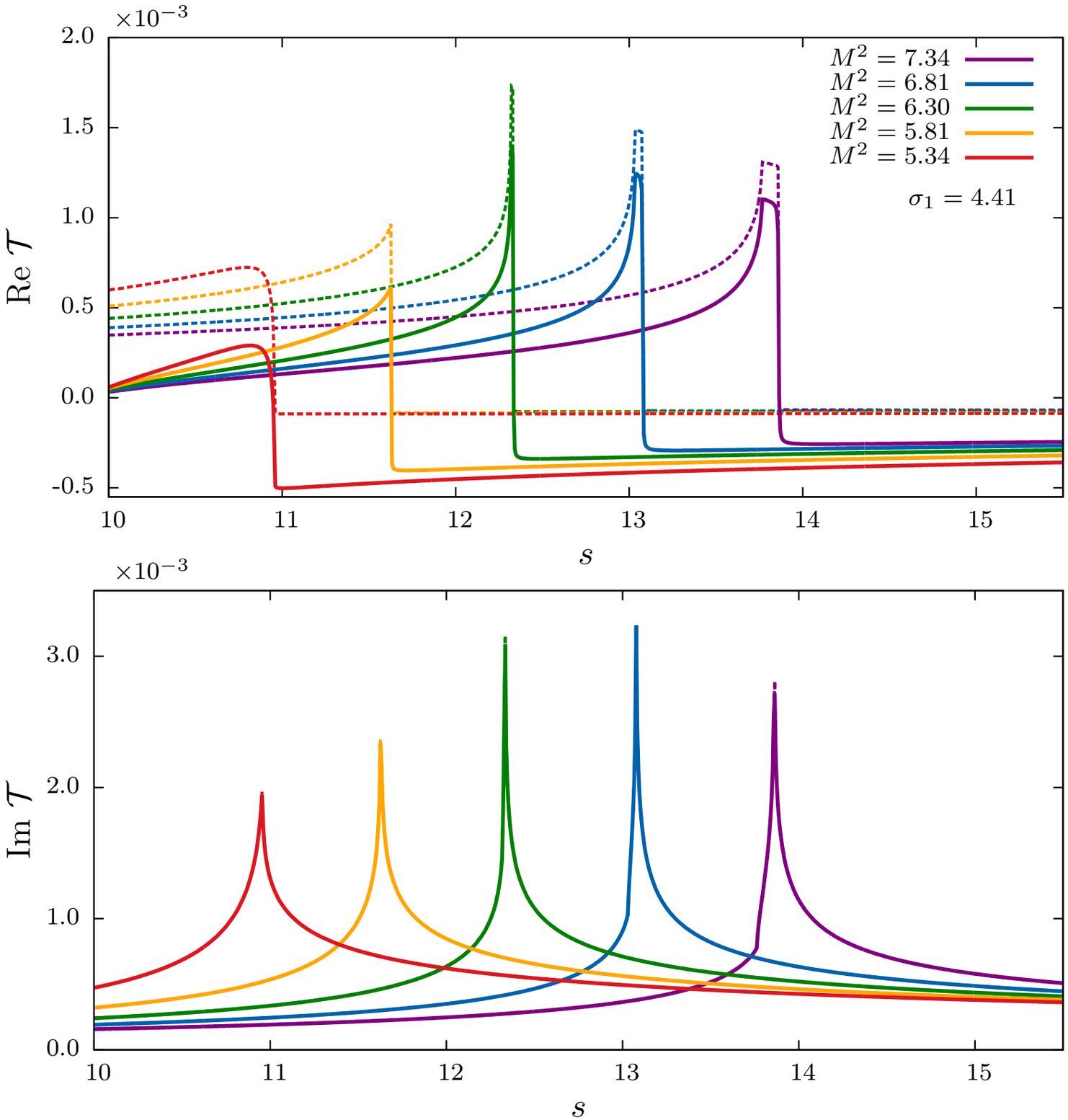
- Triangle singularity develops in B-matrix
  - Real particle exchange physical



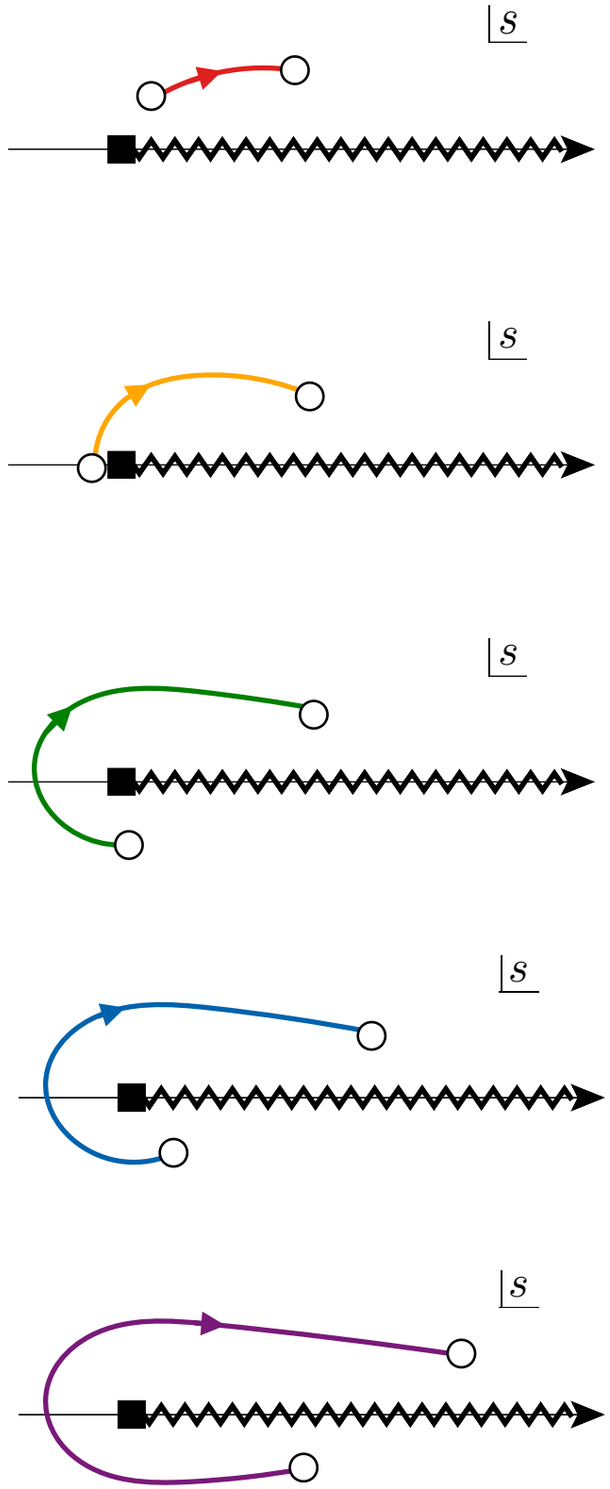


OPE branch points

Triangle singularity



OPE branch points



Curing extra singularities

- Saw triangle amplitude has extra singularities below threshold
- Integrating over isobar shape cures some of them
  - Remaining singularities due to phase space

$$\mathcal{T}_B(s) = \frac{1}{\pi}\int_{\sigma_3^{(\mathrm{th})}}^{\infty} d\widehat{\sigma}\,\mathrm{Im}\,f_3(\widehat{\sigma})\int_{\sigma_3^{(\mathrm{th})}}^{\left(\sqrt{s}-m_3\right)^2} d\sigma_3''\,\frac{\rho_3\left(s,\sigma_3''\right)\widetilde{\mathcal{E}}_{31}^{S}\left(\sigma_3'',s,\sigma_1\right)}{\widehat{\sigma}-\sigma_3''-i\epsilon}$$

Isobar shape

$$f_3\left(\sigma_3''\right) = \frac{1}{\pi}\int_{\sigma_3^{(\mathrm{th})}}^{\infty} d\widehat{\sigma}\,\frac{\mathrm{Im}\,f_3(\widehat{\sigma})}{\widehat{\sigma}-\sigma_3''-i\epsilon}$$

Singularities due to σ integration are moved to second sheet

- What about dispersing kernel in s ?

$$\mathcal{K}_{33}^{F}(s) = \int_{\mathrm{th}}^{\infty} ds'\,\frac{\widetilde{\mathcal{B}}_{31}\left(s'\right)\rho_3\left(s'\right)\widetilde{\mathcal{B}}_{13}\left(s'\right)}{s'-s-i\epsilon}$$

Persistent singularities due to phase space

$$\tau_n = \rho_3(s,\sigma_n)f_n(\sigma_n)$$

$$\widetilde{\mathcal{A}}_{33} = \mathcal{K}_{33}^{F}(s) + \mathcal{K}_{33}^{F}(s)\tau_3(s)\mathcal{K}_{33}^{F}(s) + \cdots$$

Comparing to Lattice Equations

- Interesting to compare B-matrix formalism to BHS equations

Full amplitude

$$i\mathcal{M}_3(\vec{p},\hat{a}'^*;\vec{k},\hat{a}^*) = i\mathcal{D}(\vec{p},\hat{a}'^*;\vec{k},\hat{a}^*) + \mathcal{S}\left\{\int_s\int_r \mathcal{L}^{(u,u)}(\vec{p},\vec{s})i\mathcal{T}(\vec{s},\vec{r})\mathcal{R}^{(u,u)}(\vec{r},\vec{k})\right\}$$

M. T. Hansen and S. R. Sharpe, Phys. Rev. D92, 114509 (2015)

R. A. Briceño, M. T. Hansen, and S. R. Sharpe, Phys. Rev. D95, 074510 (2017)

$$\mathcal{R}^{(u,u)}(\vec{p},\vec{k}) = \left(\frac{1}{3} + i\rho(\vec{p})i\mathcal{M}_2(\vec{p})\right)(2\pi)^3\delta^3(\vec{p}-\vec{k}) + \frac{i\rho(\vec{p})}{2\omega_p}i\mathcal{D}^{(u,u)}(\vec{p},\vec{k}).$$

$$i\mathcal{D}^{(u,u)}(\vec{p},\vec{k}) = i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p},\vec{k})i\mathcal{M}_2(\vec{k}) + \int_s \frac{1}{2\omega_s}i\mathcal{M}_2(\vec{p})iG^\infty(\vec{p},\vec{s})i\mathcal{D}^{(u,u)}(\vec{s},\vec{k})$$

Ladder of OPE's

$$\mathcal{L}^{(u,u)}(\vec{p},\vec{k}) = \left(\frac{1}{3} + i\mathcal{M}_2(\vec{p})i\rho(\vec{p})\right)(2\pi)^3\delta^3(\vec{p}-\vec{k}) + i\mathcal{D}^{(u,u)}(\vec{p},\vec{k})\frac{i\rho(\vec{k})}{2\omega_k},$$

"Amputated" amplitude

$$i\mathcal{T}(\vec{p},\vec{k}) = i\mathcal{K}_{\mathrm{df},3}(\vec{p},\vec{k}) + \int_s\int_r i\mathcal{K}_{\mathrm{df},3}(\vec{p},\vec{s})\frac{i\rho(\vec{s})}{2\omega_s}\mathcal{L}^{(u,u)}(\vec{s},\vec{r})i\mathcal{T}(\vec{r},\vec{k})$$

Comparing to Lattice Equations

- Manipulate our formalism - identical particles

We work in PW, extra phase space

$$\widetilde{\mathcal{A}} = \left[1 - (\widetilde{\mathcal{E}} + \widetilde{\mathcal{R}})\rho_3 f\right]^{-1} (\widetilde{\mathcal{E}} + \widetilde{\mathcal{R}})$$

$$\Longrightarrow \qquad \mathcal{A} = f\widetilde{\mathcal{A}}f = \mathcal{D} + \mathcal{L}\left[1 - \widetilde{R}\rho_3\mathcal{L}\right]^{-1}\widetilde{R}\mathcal{L}^\top$$

R* is like *K

$$\mathcal{T} = \widetilde{\mathcal{R}} + \widetilde{\mathcal{R}}\rho_3\mathcal{L}\mathcal{T}$$

$$\mathcal{D} = f\widetilde{\mathcal{E}}f + f\widetilde{\mathcal{E}}\rho_3\mathcal{D}$$

OPE Ladder

$$\mathcal{L} \equiv f + \mathcal{D}\rho_3$$

$$\Longrightarrow \mathcal{A} = \mathcal{D} + \mathcal{L}\mathcal{T}\mathcal{L}^\top$$

- This is a **sketch**, need to do rigorous studies for comparison

$$\mathcal{L}^{(u,u)}(\vec{p},\vec{k}) = \left(\frac{1}{3} + i\mathcal{M}_2(\vec{p})i\rho(\vec{p})\right)(2\pi)^3\delta^3(\vec{p}-\vec{k}) + i\mathcal{D}^{(u,u)}(\vec{p},\vec{k})\frac{i\rho(\vec{k})}{2\omega_k},$$

$$\begin{aligned} i\mathcal{M}_3(\vec{p},\hat{a}'^*;\vec{k},\hat{a}^*) &= i\mathcal{D}(\vec{p},\hat{a}'^*;\vec{k},\hat{a}^*) \\ &\quad + \mathcal{S}\left\{\int_s\int_r \mathcal{L}^{(u,u)}(\vec{p},\vec{s})i\mathcal{T}(\vec{s},\vec{r})\mathcal{R}^{(u,u)}(\vec{r},\vec{k})\right\} \end{aligned}$$

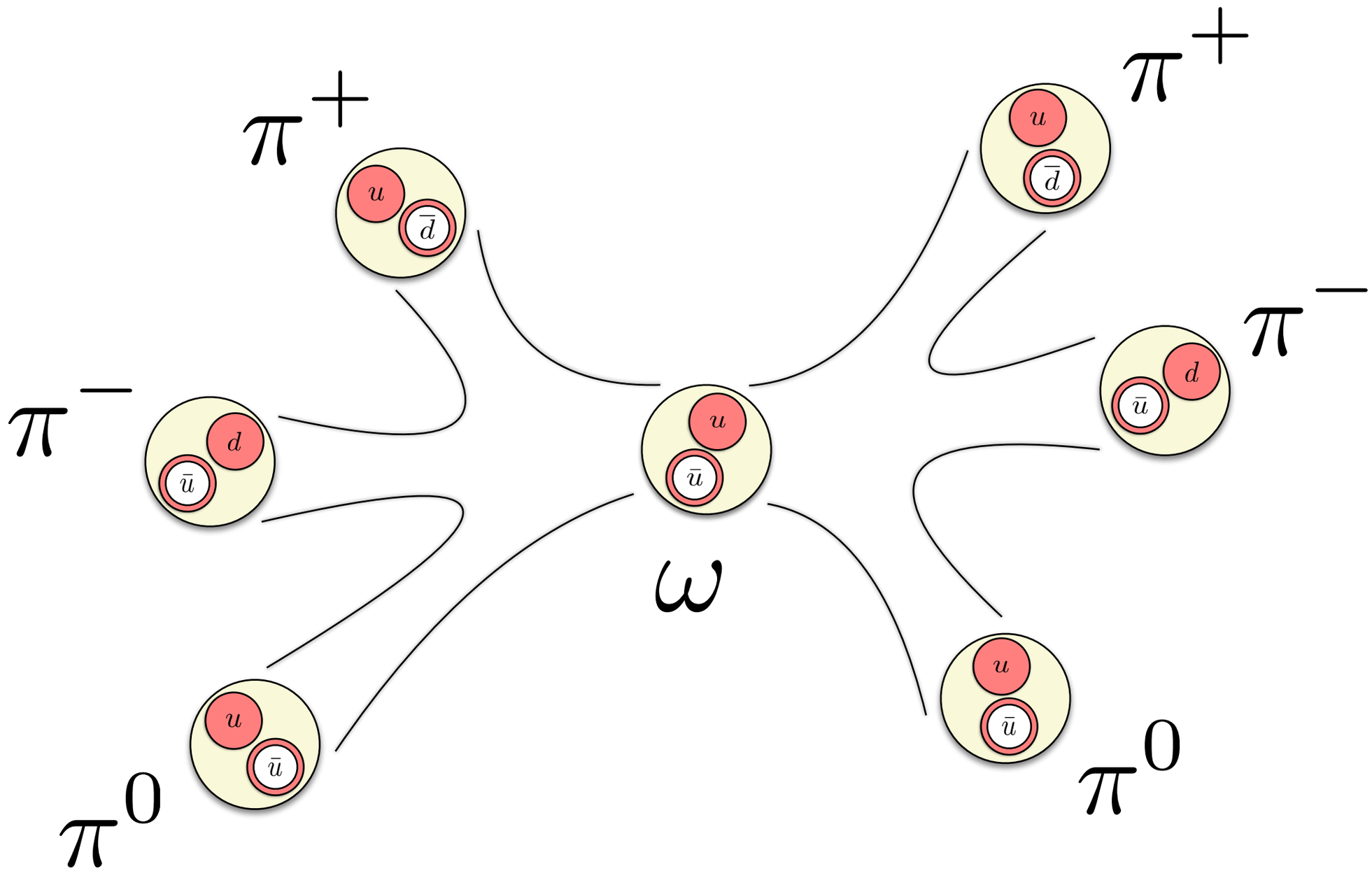
Outlook and Future Directions

- **3→3** unitarity relations within the isobar approximation
  - Spinless external particles
  - Finite number of isobars with spin

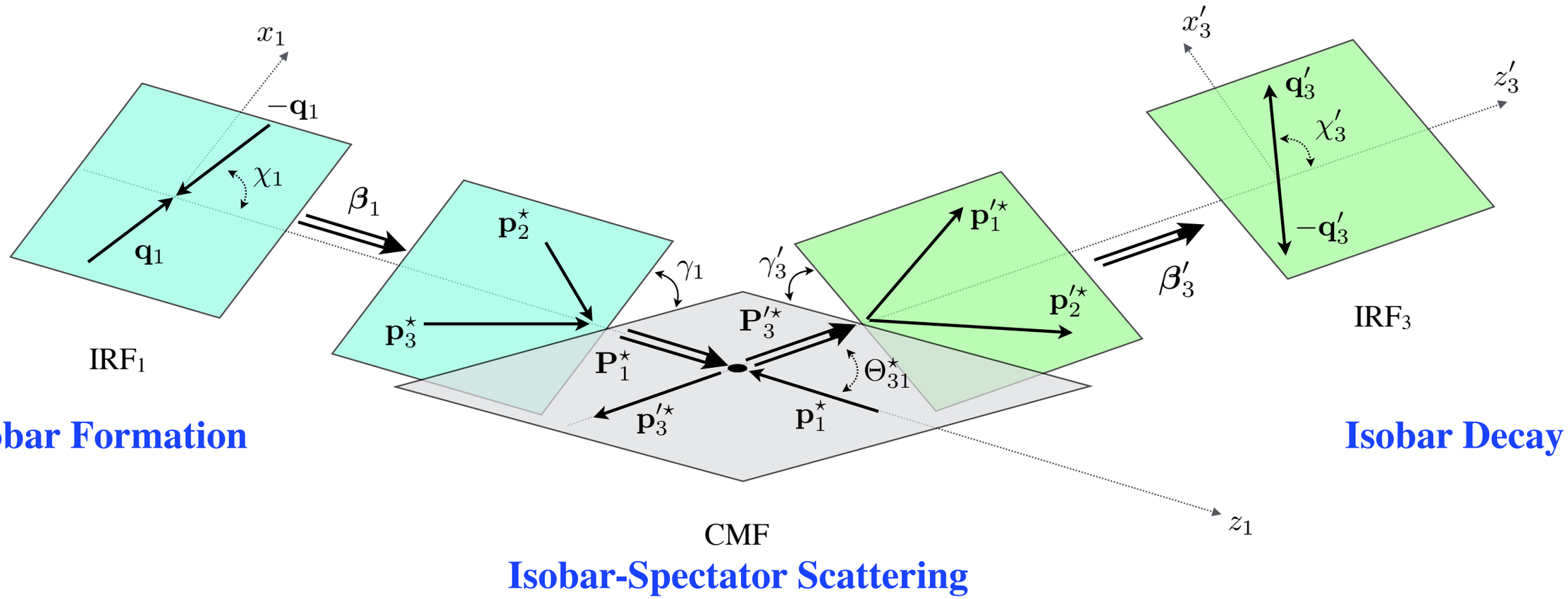
- B-matrix parameterization
  - Linear integral equations — satisfies unitarity relations
  - Iterates triangle and box diagrams

- Future Studies
  - Study box diagram
  - Analytic continuation to complex energy plane - resonance studies
  - Connect our formalism with LQCD formalisms

Back-up



Kinematics


x_1
$-\mathbf{q}_1$
χ_1
$\mathbf{q}_1$
$\boldsymbol{\beta}_1$
IRF$_1$
Isobar Formation
$\mathbf{p}_2^\star$
$\mathbf{p}_3^\star$
γ_1
γ_3'
$\mathbf{P}_3'^\star$
$\mathbf{P}_1^\star$
$\Theta_{31}^\star$
$\mathbf{p}_3'^\star$
$\mathbf{p}_1^\star$
CMF
$\mathbf{p}_1'^\star$
$\mathbf{p}_2'^\star$
x_3'
z_3'
$\mathbf{q}_3'$
χ_3'
$-\mathbf{q}_3'$
$\boldsymbol{\beta}_3'$
IRF$_3$
Isobar Decay
z_1
Isobar-Spectator Scattering

Unitarity Relations

Disconnected 2→2 Unitarity Relation

$$\mathrm{Im}\ \mathcal{F}_j(\mathbf{p}';\mathbf{p}) = \rho_2(\sigma_j)\int d\widehat{\mathbf{q}}''_j\,\mathcal{F}^*_j(\mathbf{p}'';\mathbf{p}')\mathcal{F}_j(\mathbf{p}'';\mathbf{p})$$

Connected 3→3 Unitarity Relation

$$\begin{aligned}\mathrm{Im}\ \mathcal{A}(\mathbf{p}';\mathbf{p}) = \frac{1}{2(2\pi)^5}\int \frac{d^3\mathbf{p}''_1}{2\omega''_1}\frac{d^3\mathbf{p}''_2}{2\omega''_2}\frac{d^3\mathbf{p}''_3}{2\omega''_3}\,\delta^{(4)}(P''-P)\,\mathcal{A}^*(\mathbf{p}'';\mathbf{p}')\mathcal{A}(\mathbf{p}'';\mathbf{p})\\
+\sum_k \rho_2(\sigma'_k)\int d\widehat{\mathbf{q}}''_k\,\mathcal{F}^*_k(\mathbf{p}'';\mathbf{p}')\mathcal{A}(\mathbf{p}'';\mathbf{p})|_{\mathbf{p}''_k=\mathbf{p}'_k}\Theta(\sigma'_k-\sigma_k^{(\mathrm{th})})\\
+\sum_j \rho_2(\sigma_j)\int d\widehat{\mathbf{q}}''_j\,\mathcal{A}^*(\mathbf{p}'';\mathbf{p}')|_{\mathbf{p}''_j=\mathbf{p}_j}\mathcal{F}_j(\mathbf{p}'';\mathbf{p})\Theta(\sigma_j-\sigma_j^{(\mathrm{th})})\\
+\sum_{\substack{j,k\\ j\neq k}}\pi\,\delta(u_{jk}-\mu^2_{jk})\,\mathcal{F}^*_k(\mathbf{p}'';\mathbf{p}')|_{\mathbf{p}''_j=\mathbf{p}_j}\mathcal{F}_j(\mathbf{p}'';\mathbf{p})|_{\mathbf{p}''_k=\mathbf{p}'_k},\end{aligned}$$