two- and three-particle scattering amplitudes from lattice QCD

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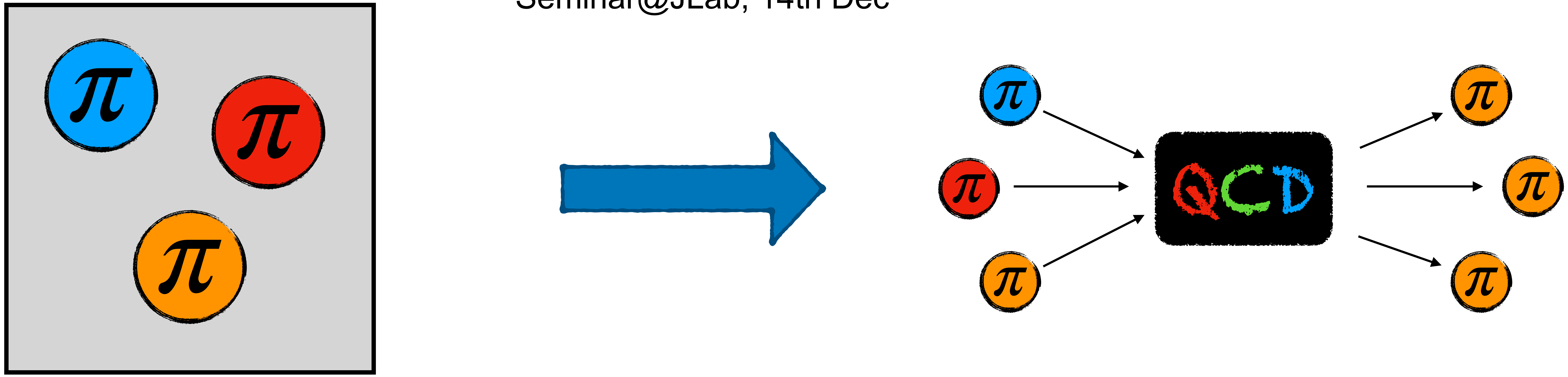
Seminar@JLab, 14th Dec



Acknowledgements

IFIC people

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Three-particle people

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Martin Ueding
Carsten Urbach

VNIVERSITAT DE VALÈNCIA

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"la Caixa"

Outline

1. Introduction
2. Lattice QCD
3. Finite-Volume Spectrum
4. Two-particle scattering
5. Three particles in finite volume
6. Conclusions

Introduction

Quantum Chromodynamics

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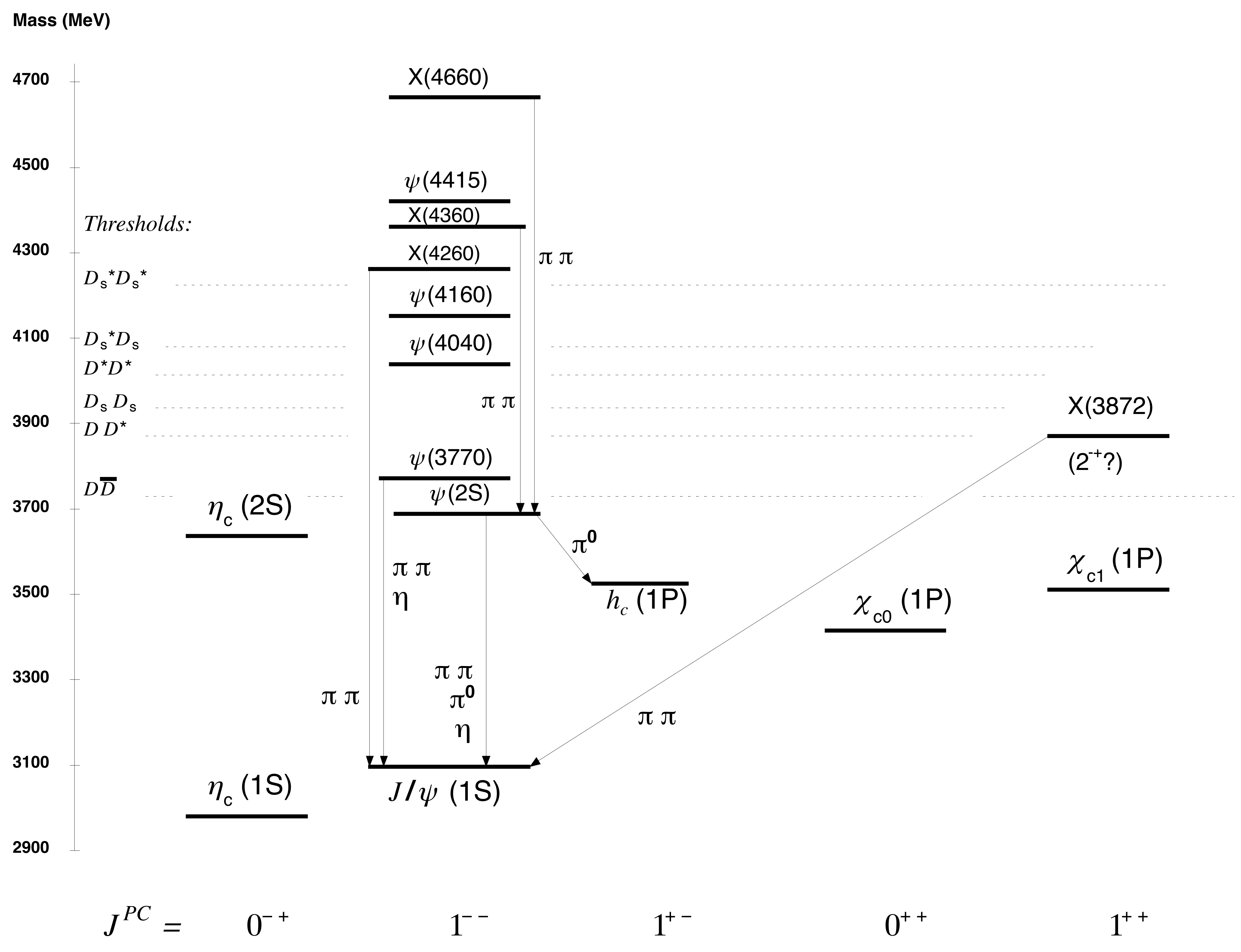
Charmonium Spectrum (PDG)



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- QCD effects in experimental signatures of particle decays
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  - XYZ, charmonium, bottomonium
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- First-principles nuclear interactions:
  - $2N$ & $3N$ interactions: Input for neutron stars and larger nuclei EFT treatment

Towards the QCD S-Matrix

The S-Matrix contains the physical information of the theory:

$$S_{ab}(E) \equiv \langle \mathrm{out}|\mathcal{S}|\mathrm{in}\rangle$$

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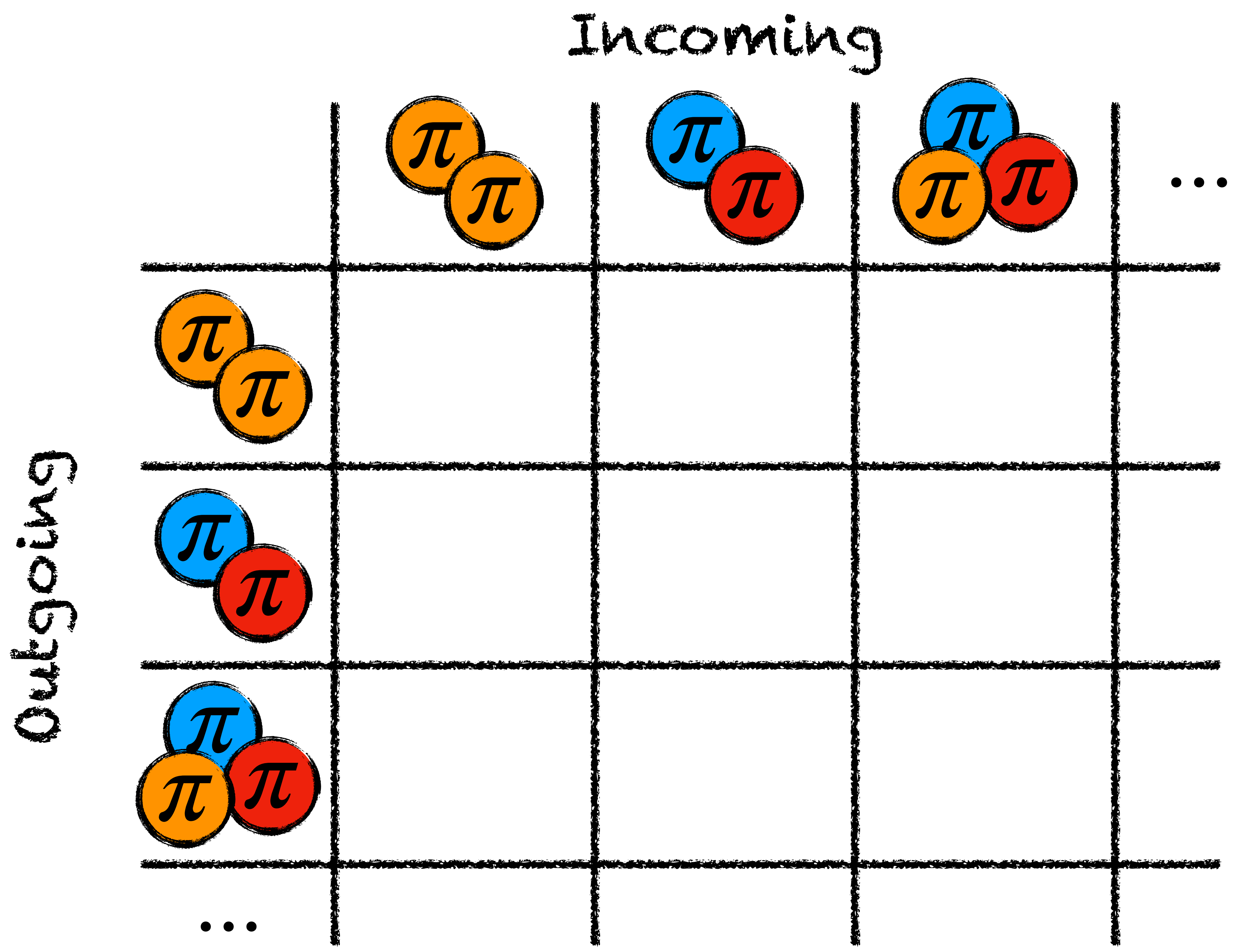

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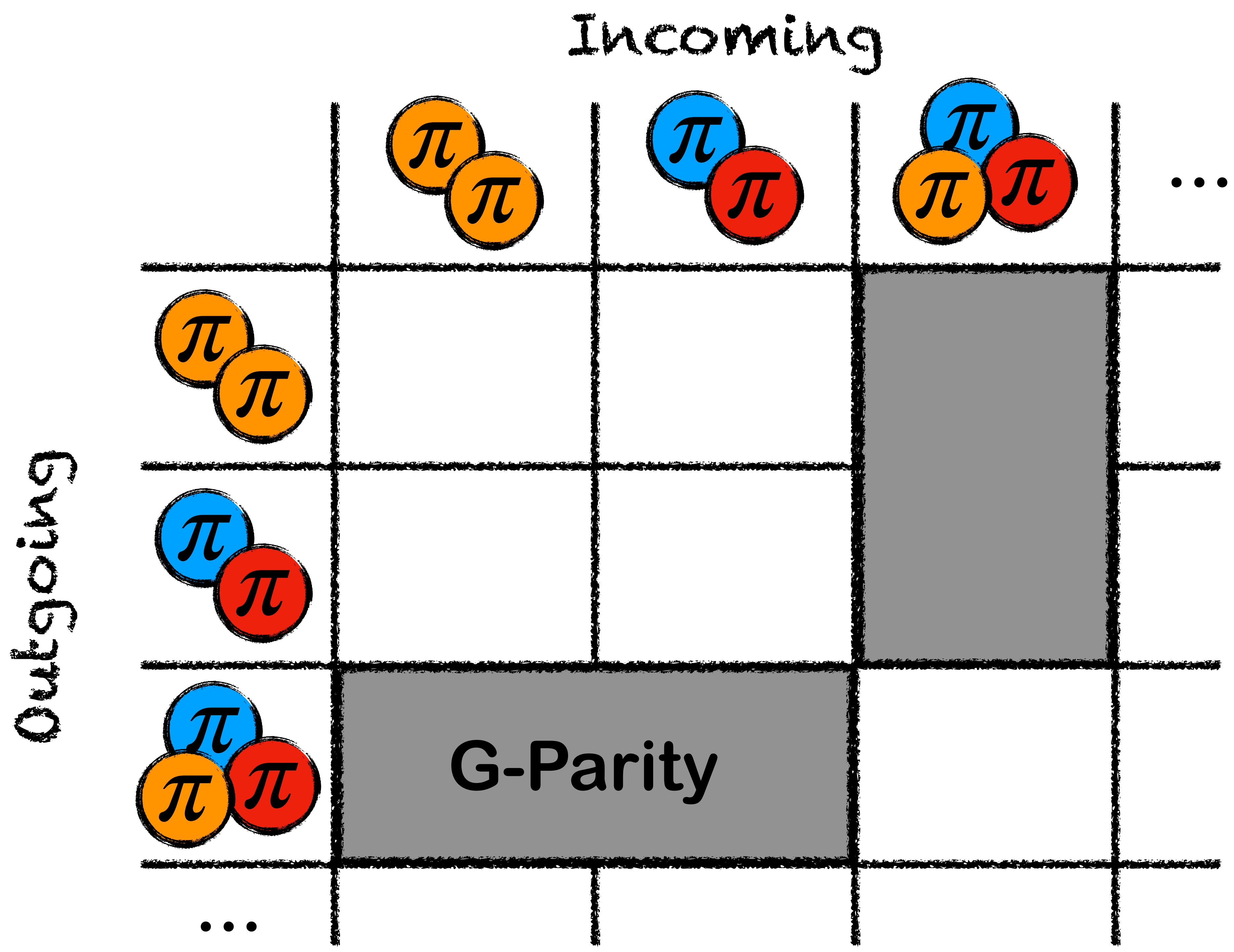

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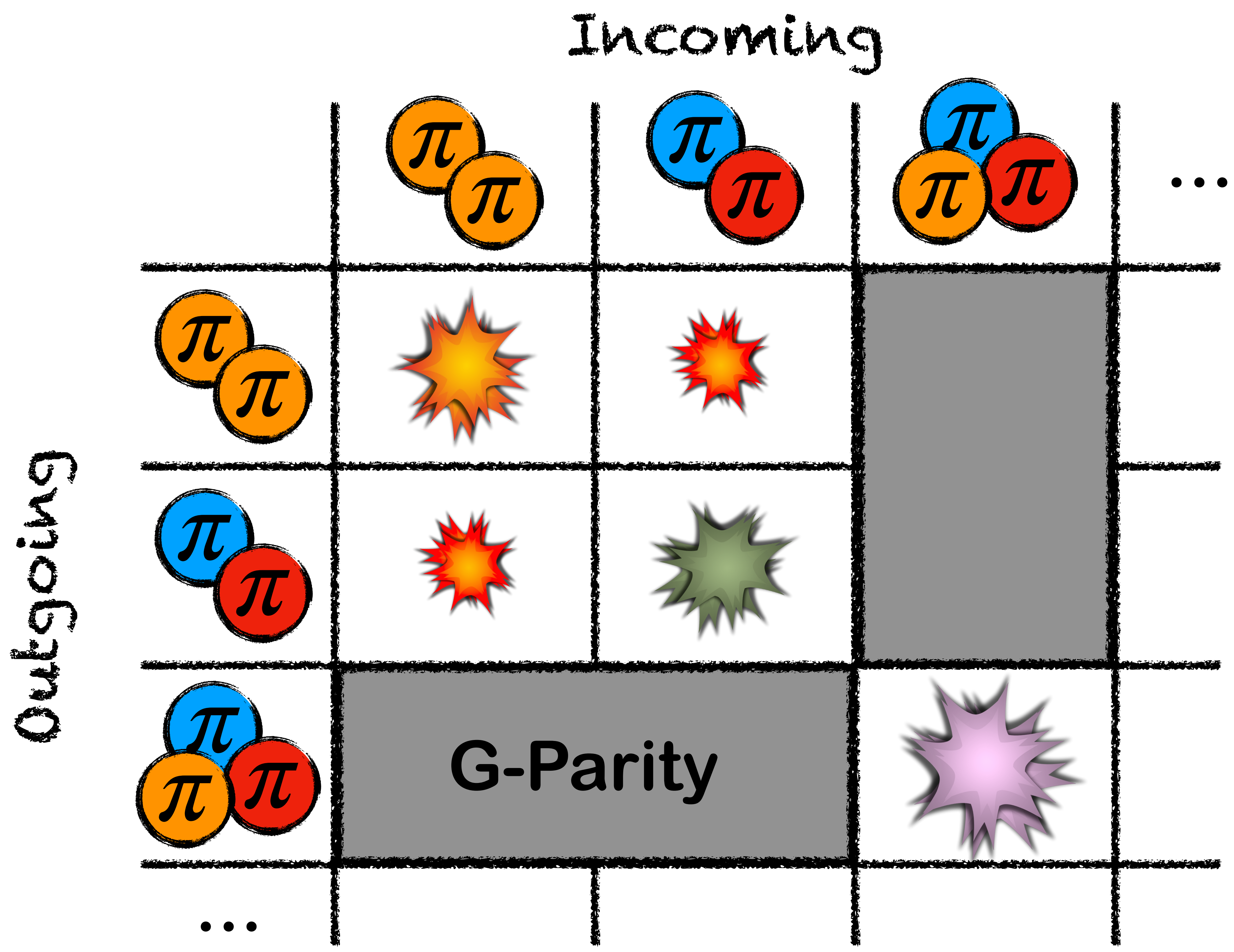

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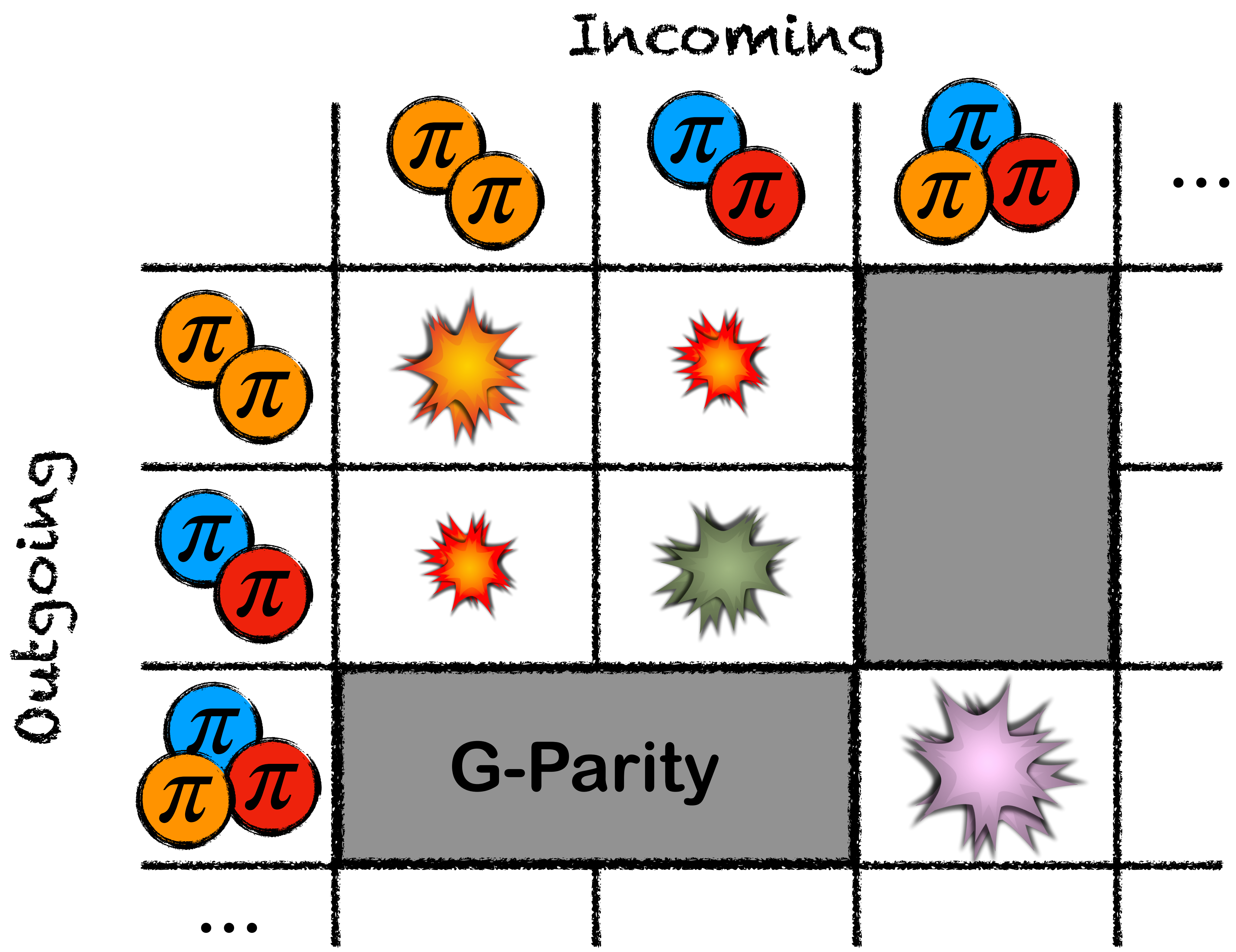
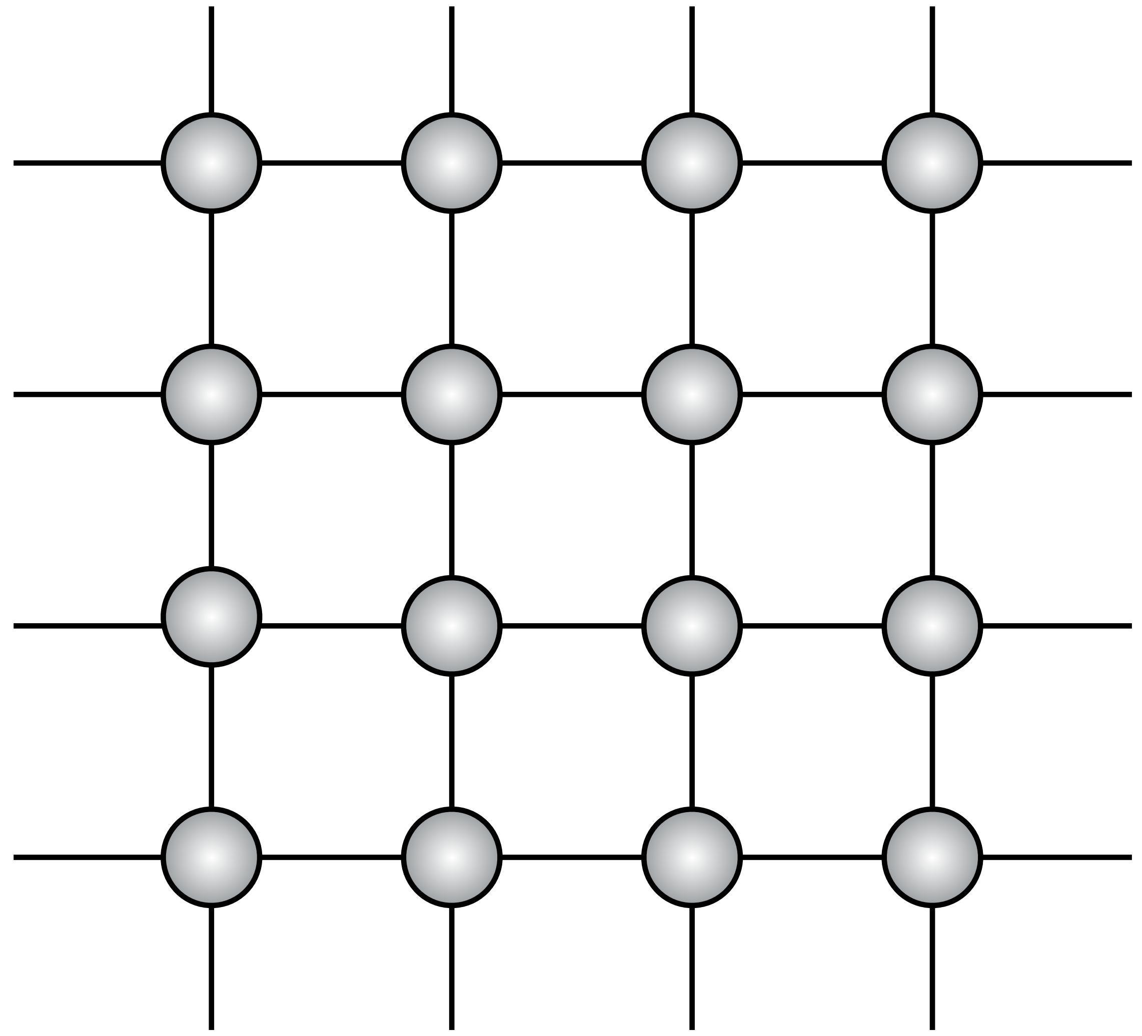


This Talk:

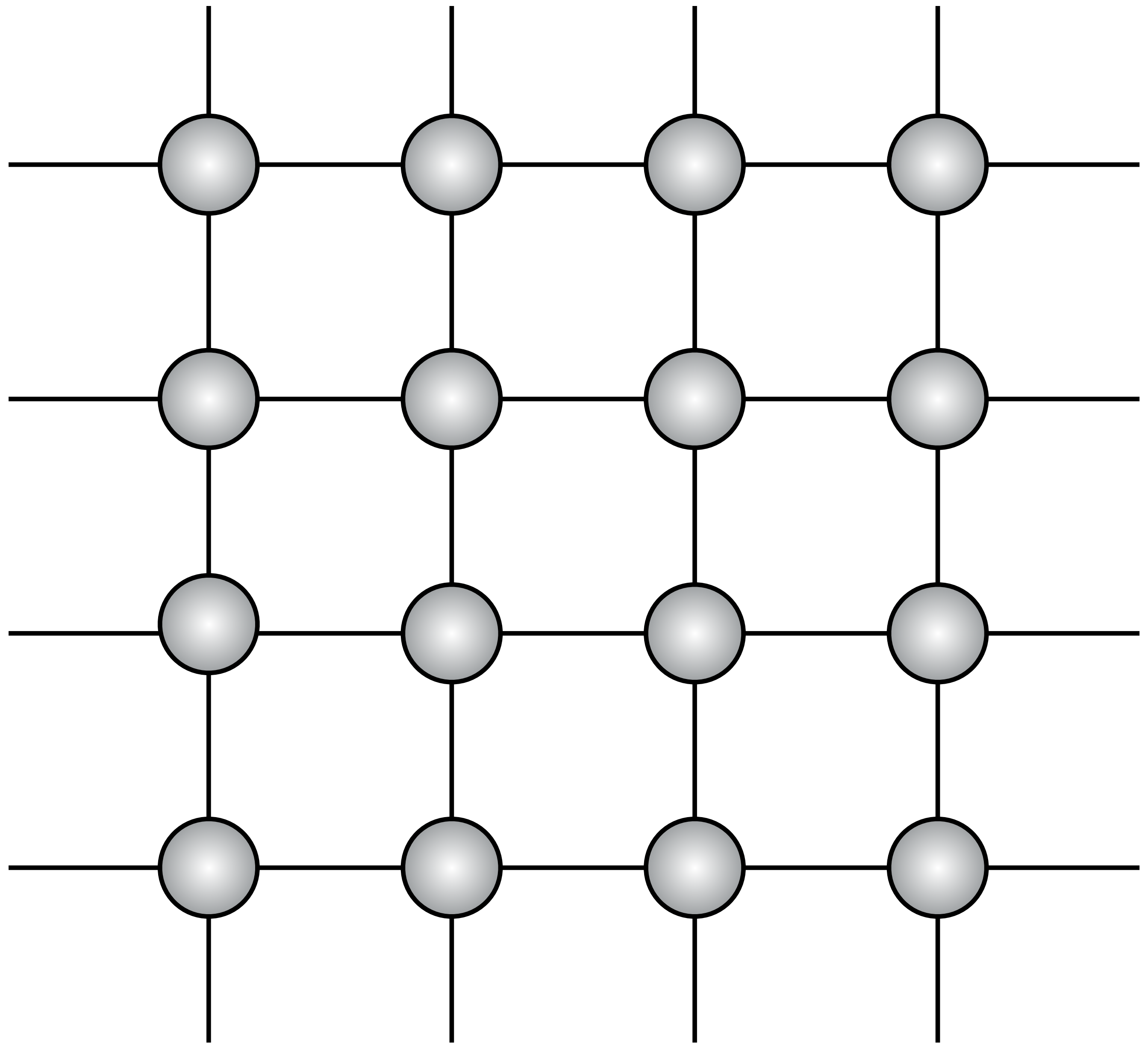
- How to extract S-matrix elements from Lattice QCD:
  - Formalism for $2 \to 2$ scattering
  - Three-particle scattering on the lattice
  - Lattice results three-particle scattering

Lattice QCD basics



Lattice QCD Basics (I)

- Lattice QCD is the state-of-the-art treatment of the strong interaction at hadronic energies



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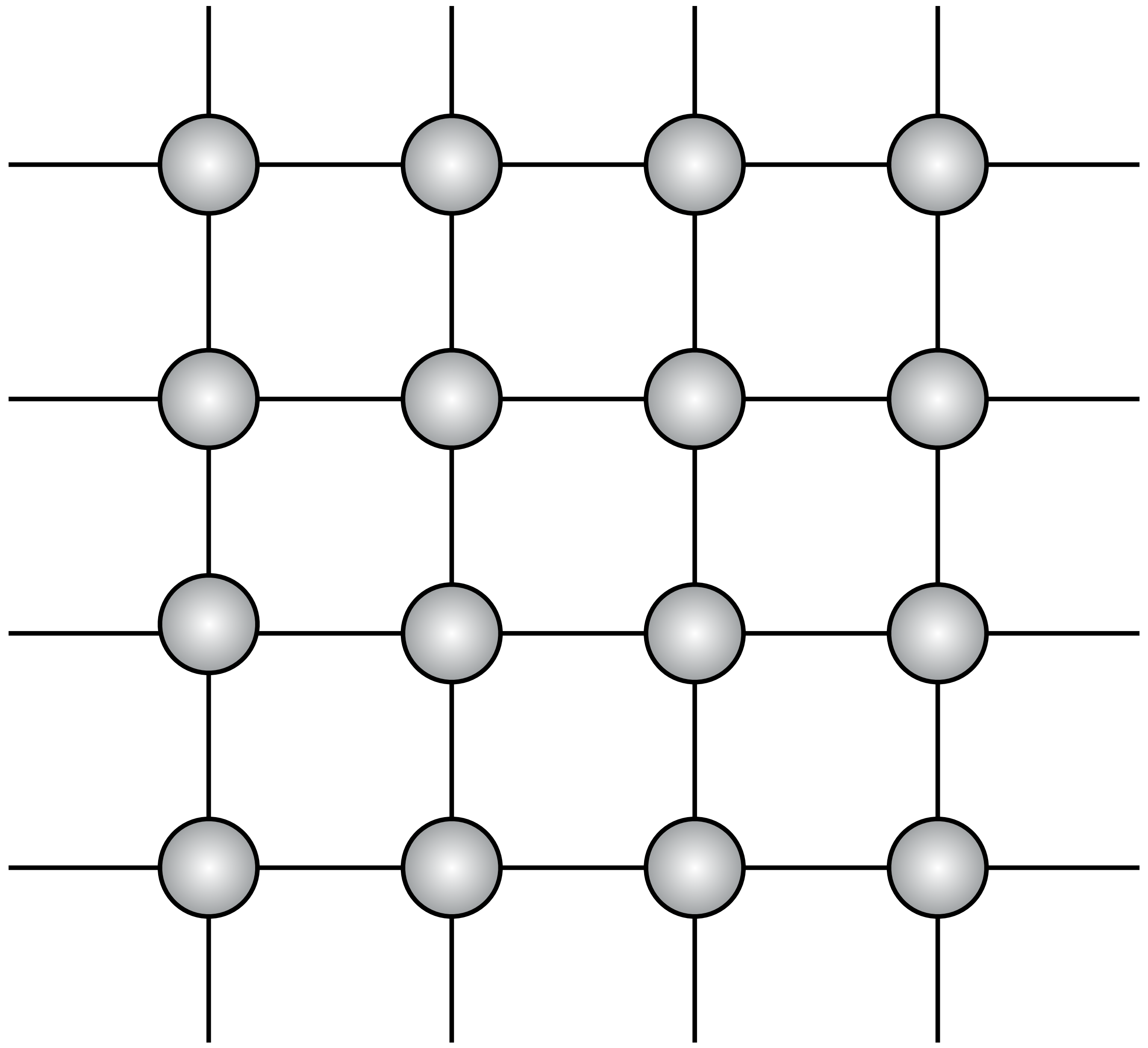
$$\mathcal{Z} = \int D\psi D\bar{\psi} DA e^{-S_E(\psi,\bar{\psi},A_\mu)}$$



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  $\rightarrow$ Under control but technical

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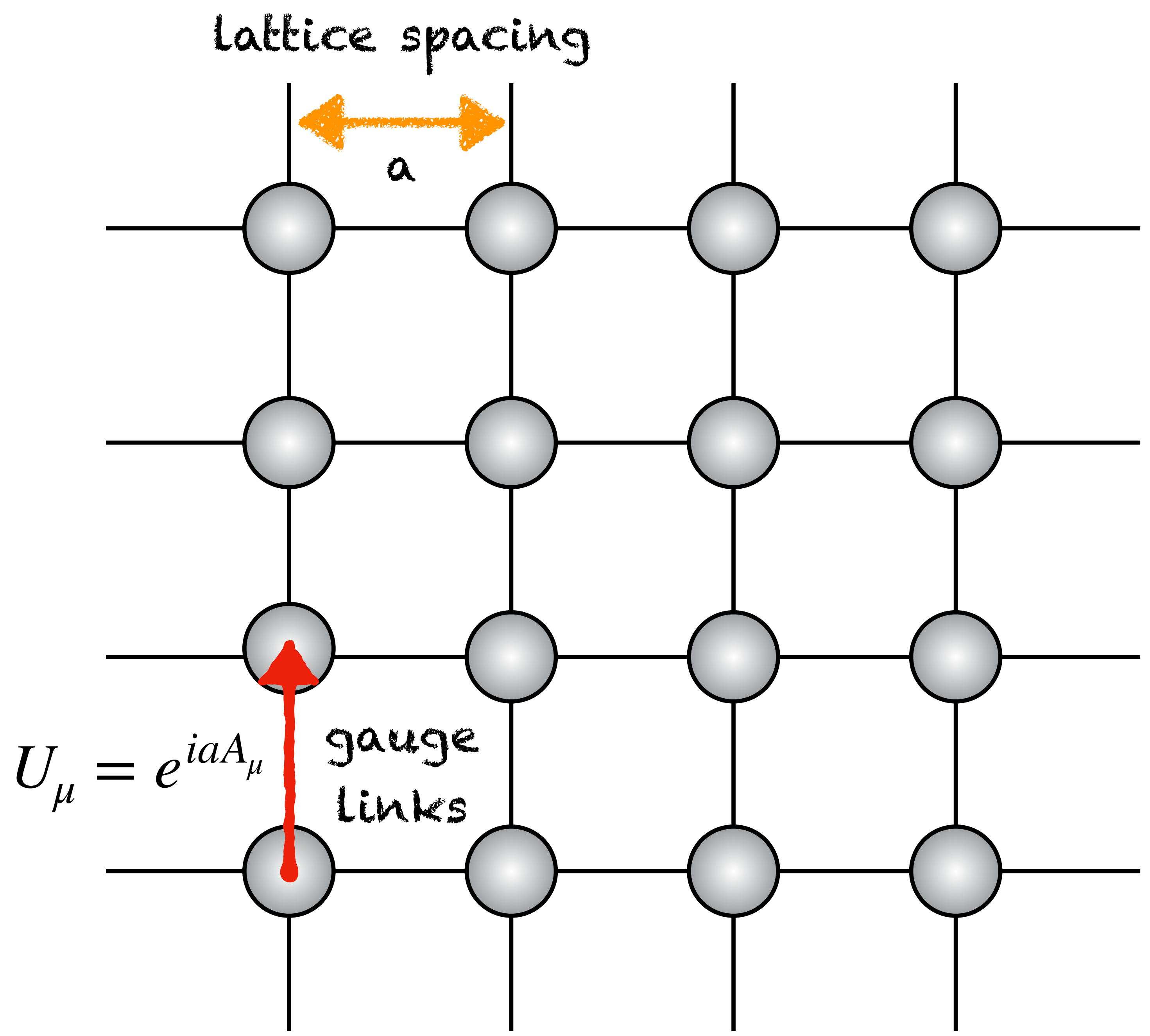

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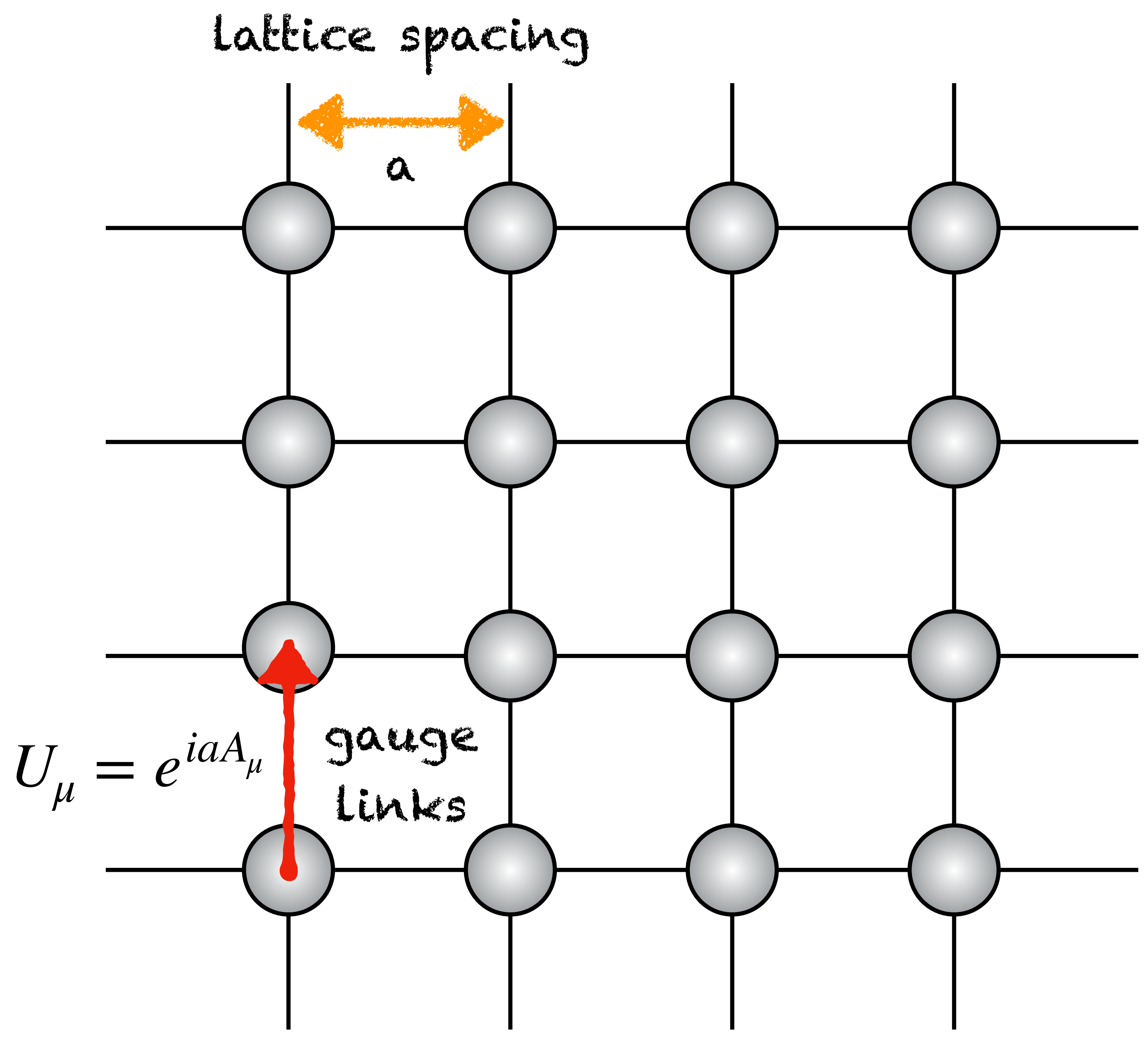
- Compute correlation functions

$$\langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \frac{1}{\mathcal{Z}} \int D\psi D\bar{\psi} DA \, \mathcal{O}(t)\mathcal{O}(0) e^{-S_E(\psi,\bar{\psi},A_\mu)}$$



Lattice QCD Basics (II)

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$$C(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \sum_n \langle 0|\mathcal{O}(t)|n\rangle\langle n|\mathcal{O}(0)|0\rangle$$

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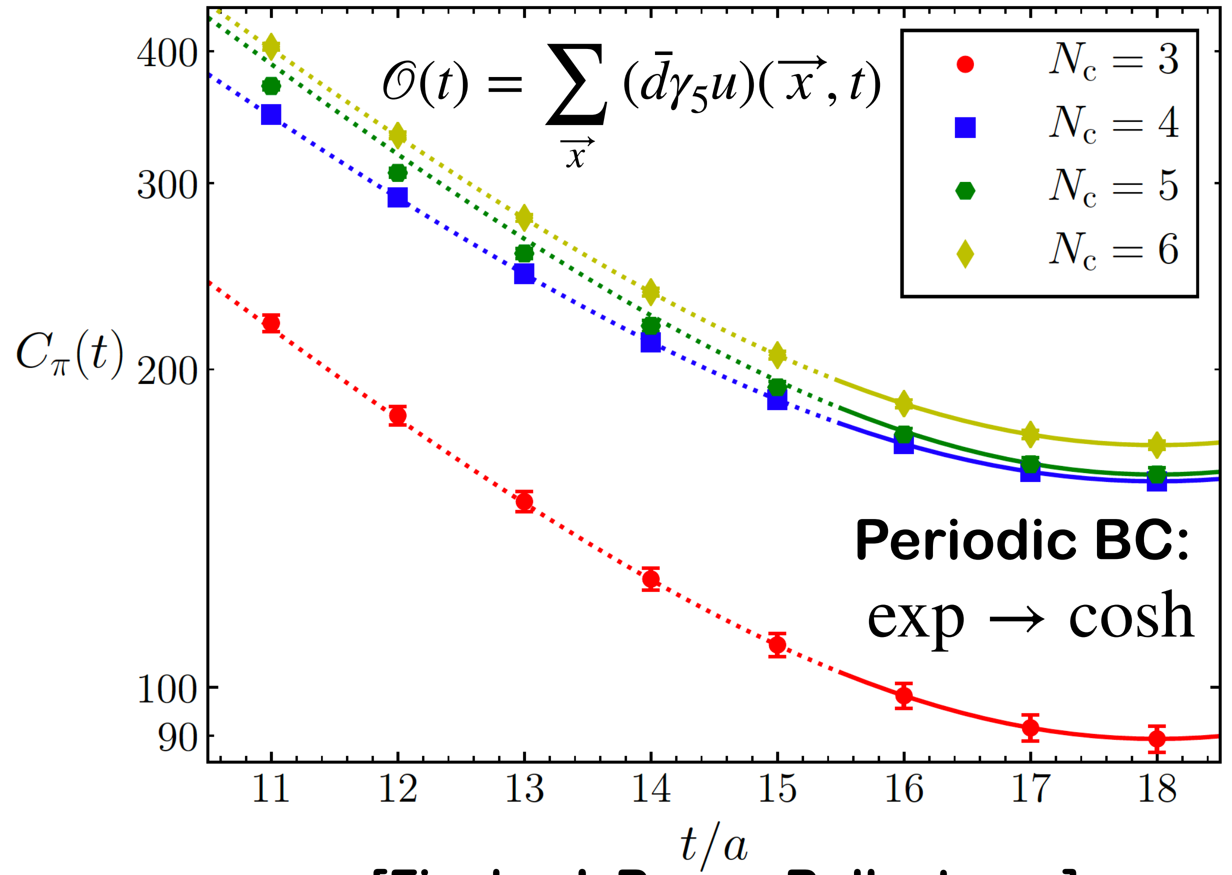

[Fig. by J. Baeza-Ballesteros]

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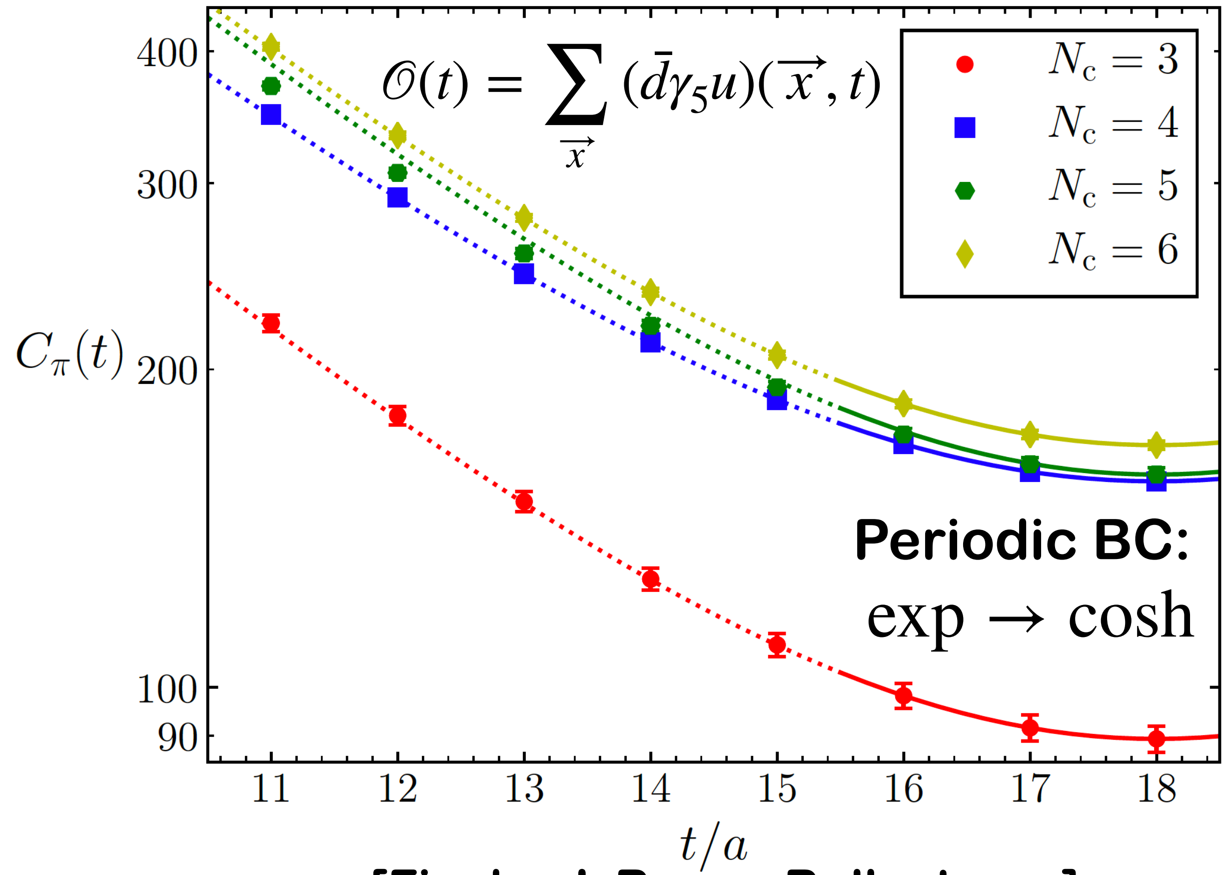
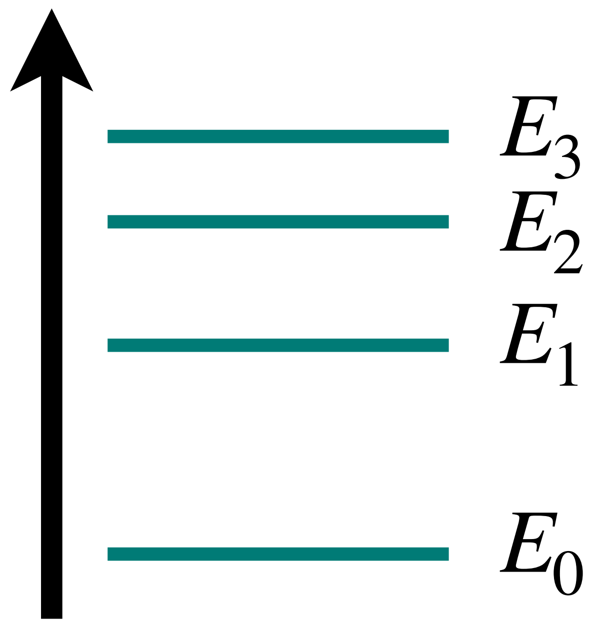


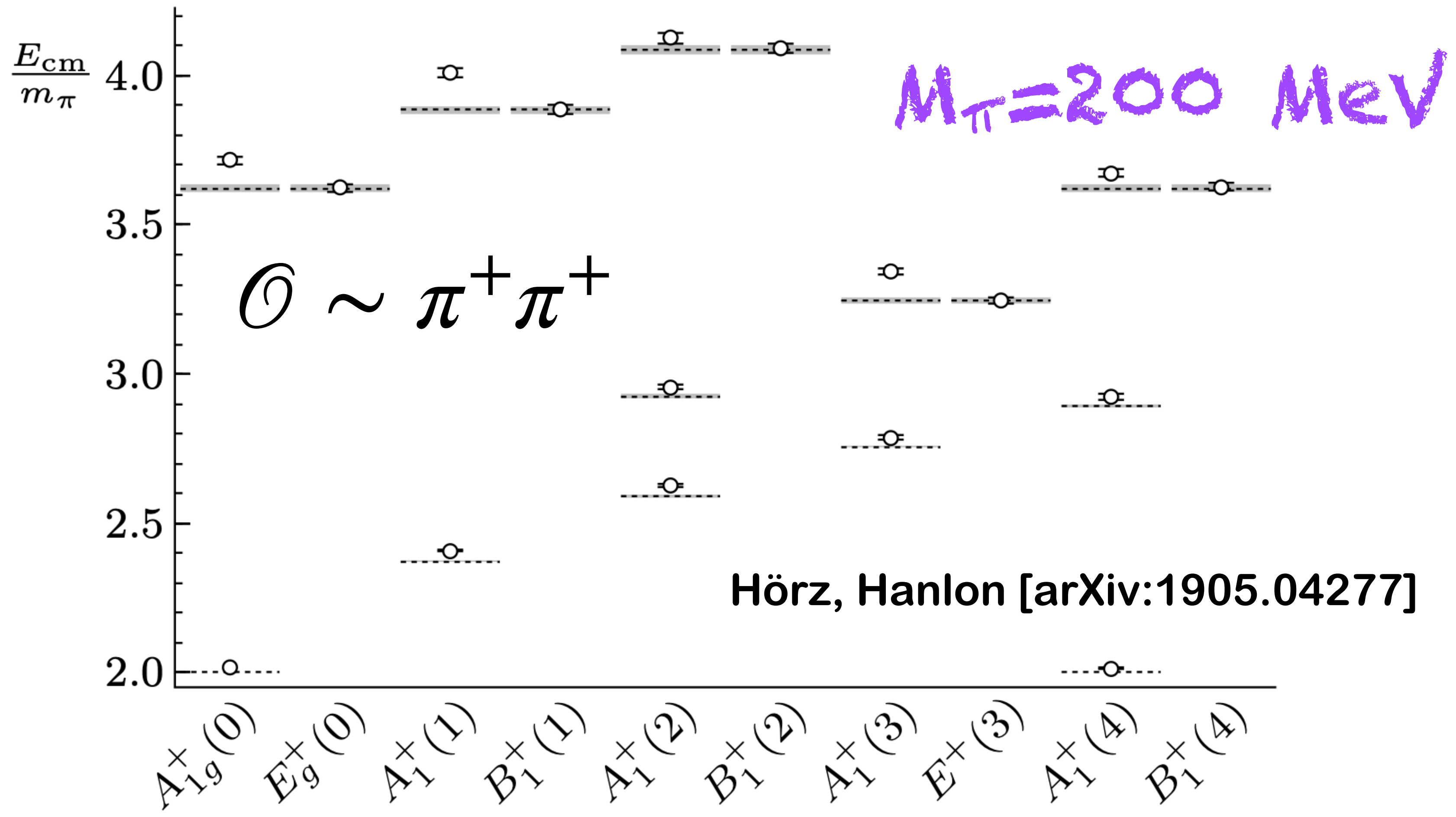
[Fig. by J. Baeza-Ballesteros]

- Multiple operators to obtain several energy levels

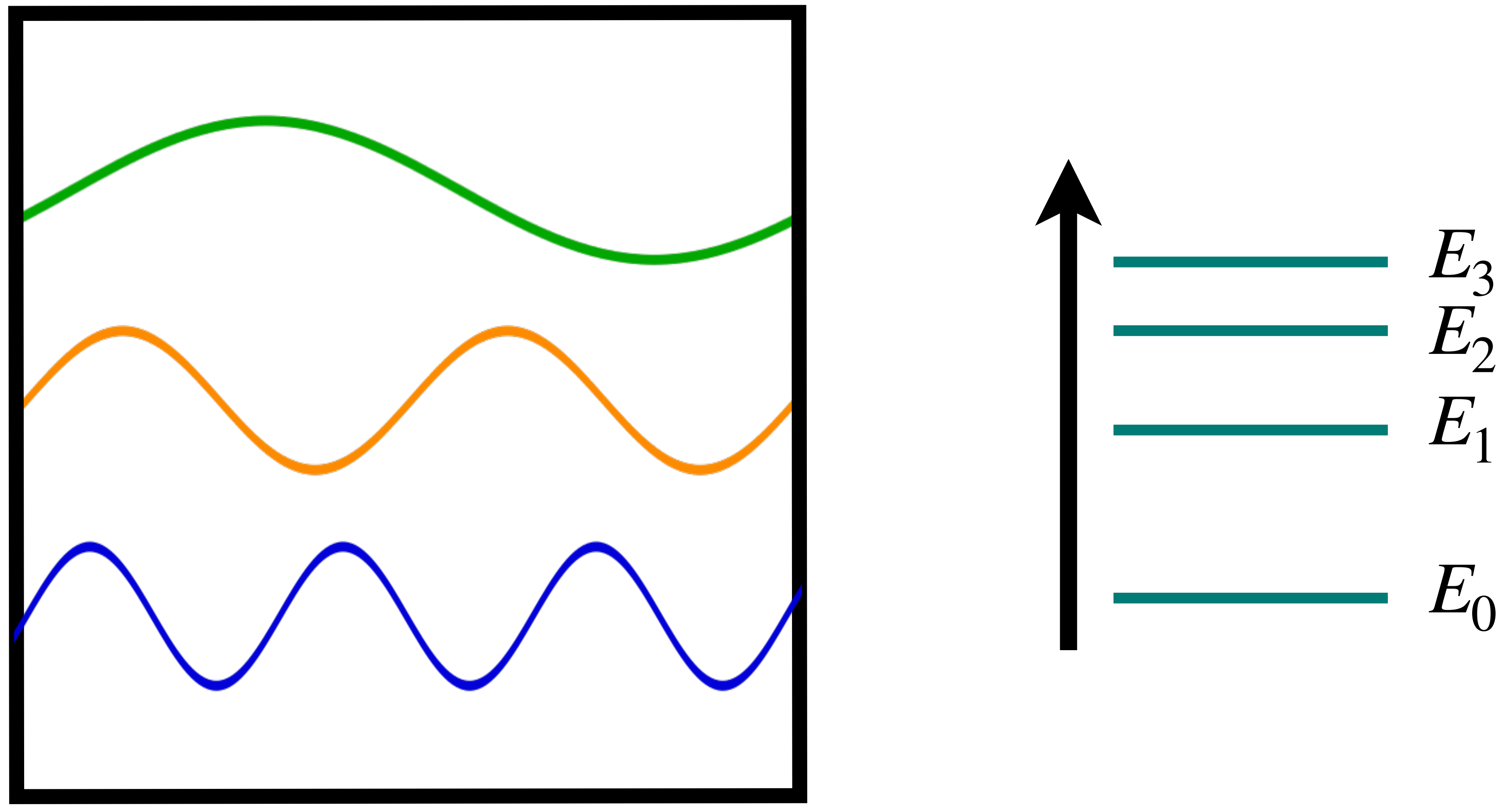
The Spectrum



M_π=200 MeV

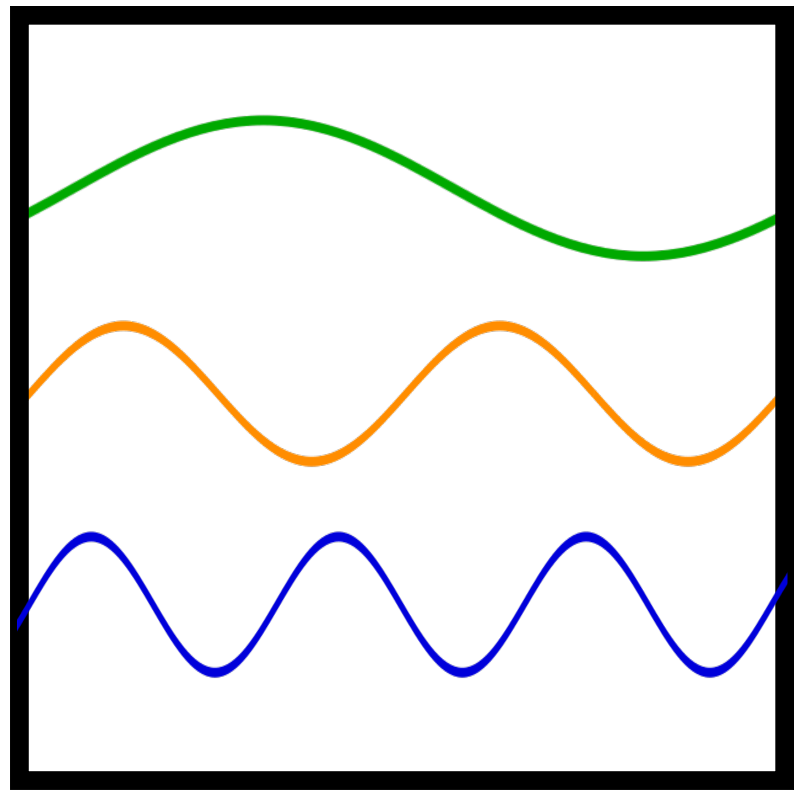


Finite-volume spectrum



Finite-Volume Spectrum

Free scalar particles in finite volume with periodic BC



$$\vec{p} = \frac{2\pi}{L}(n_x, n_y, n_z)$$

Two particles: $E = 2\sqrt{m^2 + \frac{4\pi^2}{L^2}\vec{n}^{\,2}}$

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Interactions change the spectrum:
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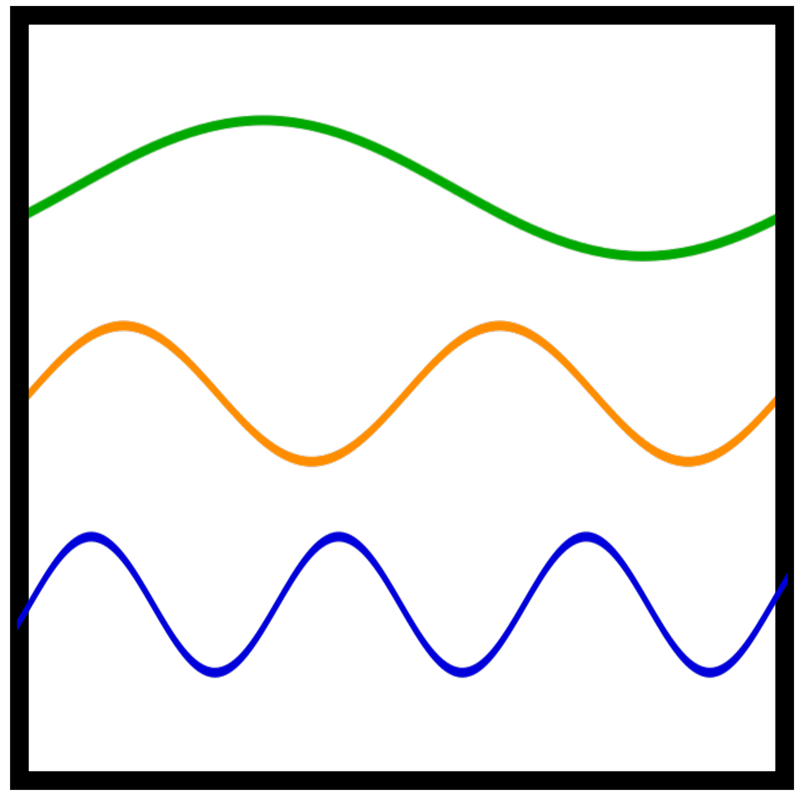

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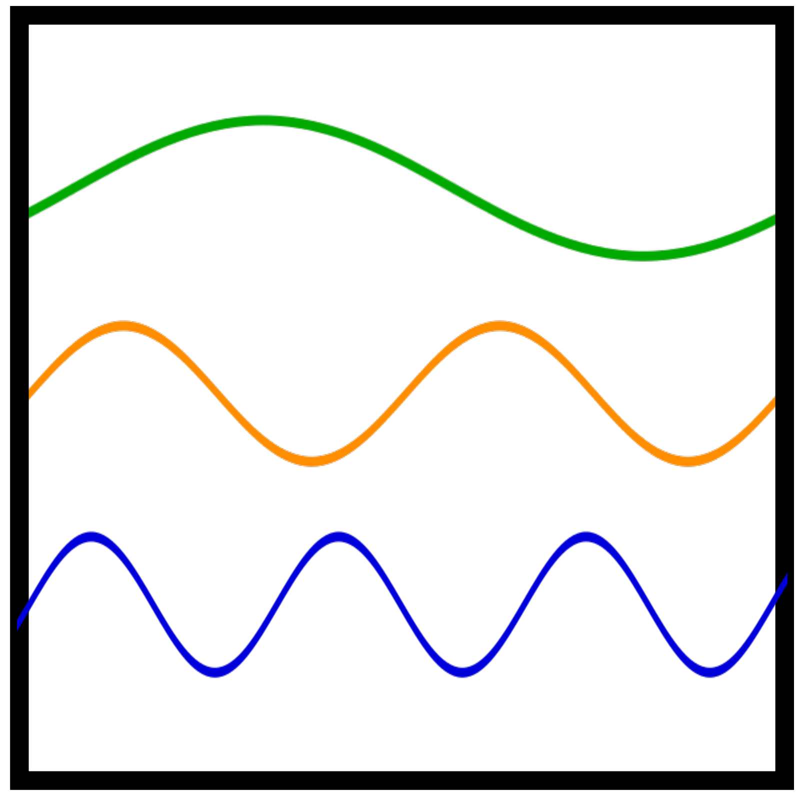

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Ground state to leading order

$$E_2 - 2m = \langle \phi(\vec{0})\phi(\vec{0}) \,|\, \mathbf{H_I} \,|\, \phi(\vec{0})\phi(\vec{0}) \rangle$$

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[Huang, Yang, 1958]

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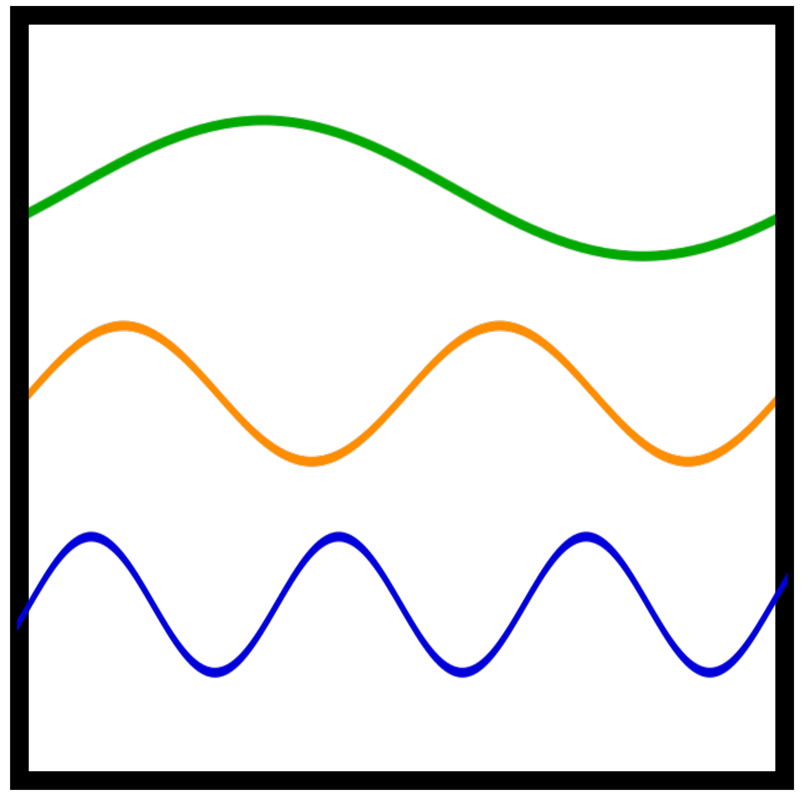

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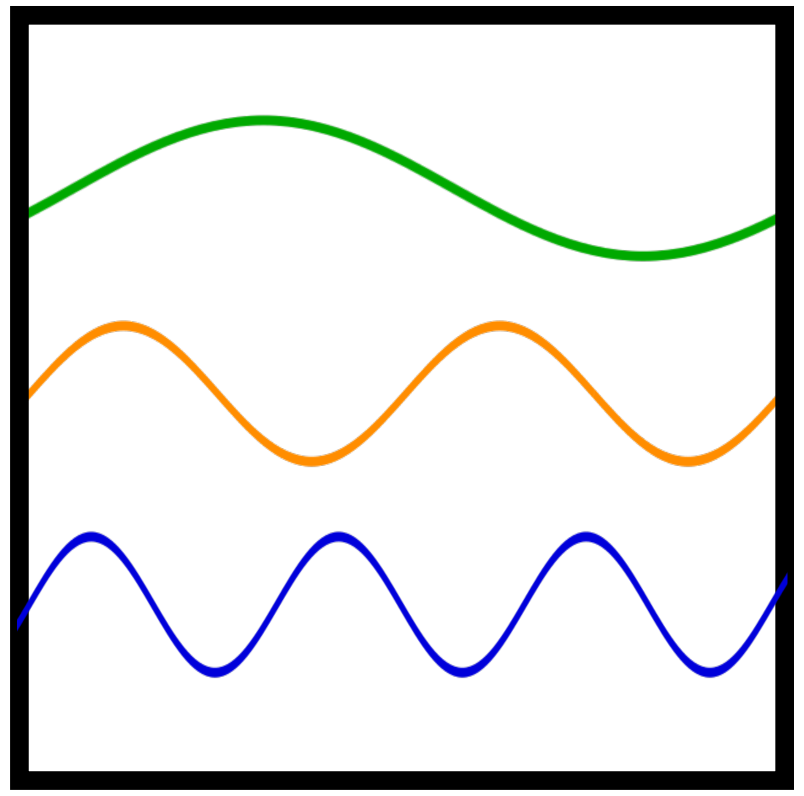

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In general a problem of
Quantum Field Theory
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Threshold expansion

- The 1/L expansion was worked out by M. Lüscher to NNLO:

$$\Delta E_2 = \frac{4\pi a_0}{mL^3}\left[1 + c_1\left(\frac{a_0}{L}\right) + c_2\left(\frac{a_0}{L}\right)^2\right] + O(L^{-6})$$

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Extensions:
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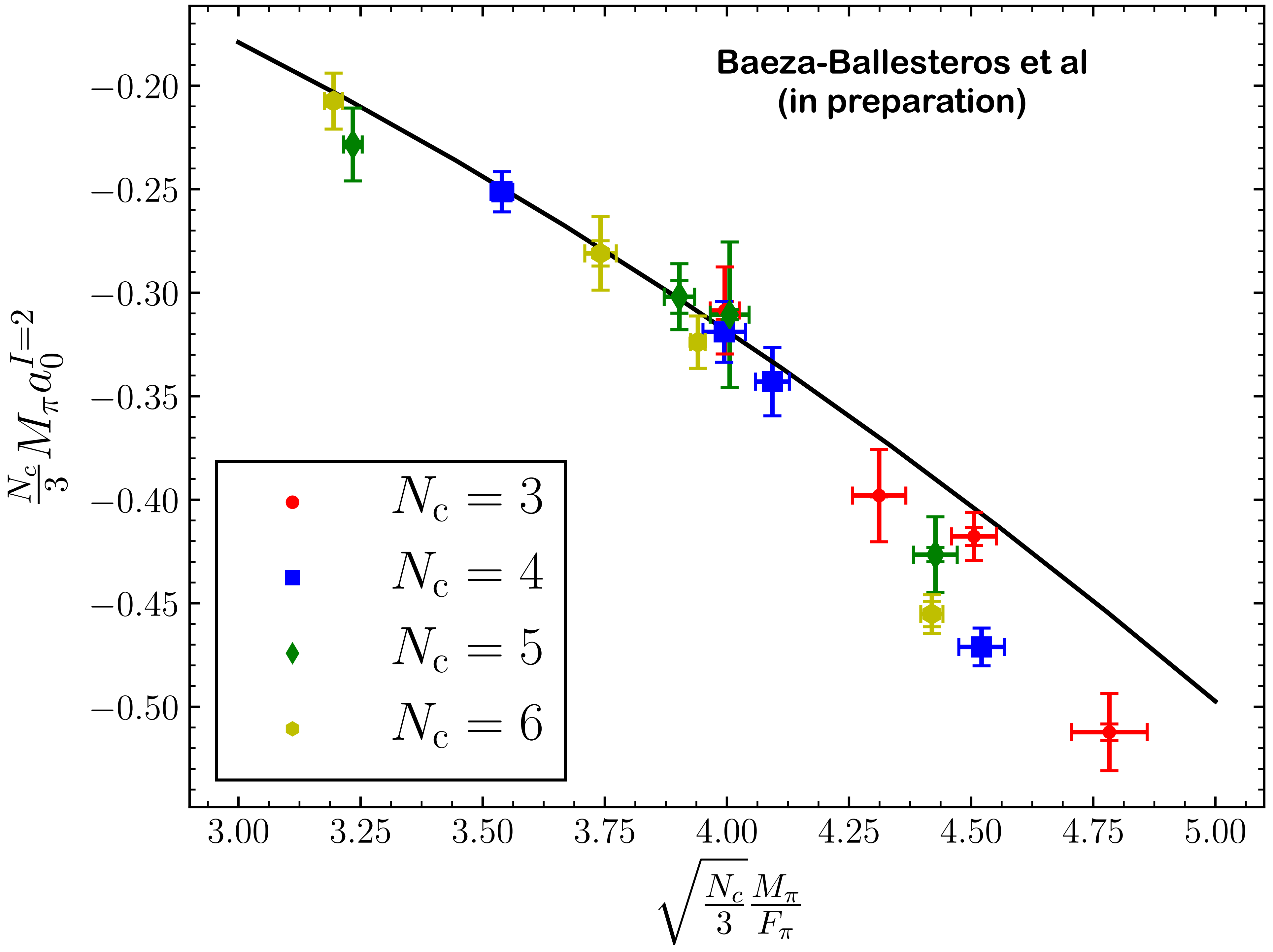
isospin-2 ππ scattering length as a function of the number of colors



Perturbative expansions

- **Non-relativistic EFT in finite volume:**

$$\mathcal{L} = \psi^\dagger \left(i\partial^0 - m + \frac{\nabla^2}{2m} + \frac{\nabla^4}{8m^3} \right) \psi - \frac{g_1}{4}(\psi^\dagger\psi)^2 - g_2(\psi^\dagger\psi^\dagger)\nabla^2(\psi\psi)$$

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$$- \left(\frac{a_0}{\pi L}\right)^3\left[\mathcal{I}^3 + (2N-7)\mathcal{I}\mathcal{J} + (5N^2 - 41N + 63)\mathcal{K} + 8(N-2)(2\mathcal{Q} + \mathcal{R})\right]$$
$$\left. + (4N-9)\frac{\pi a_0}{m^2L^3} + (4N-6)\frac{\pi a_0^2 r_0}{L^3}\right\}$$
$$+ \binom{N}{3}\left\{\frac{32\pi a_0^4}{mL^6}\left(3\sqrt{3} - 4\pi\right)\left(2\ln(mL) - \Gamma'(1) - \ln 4\pi\right) - \frac{\bar{\mathcal{T}}}{6L^6}\right\}.$$

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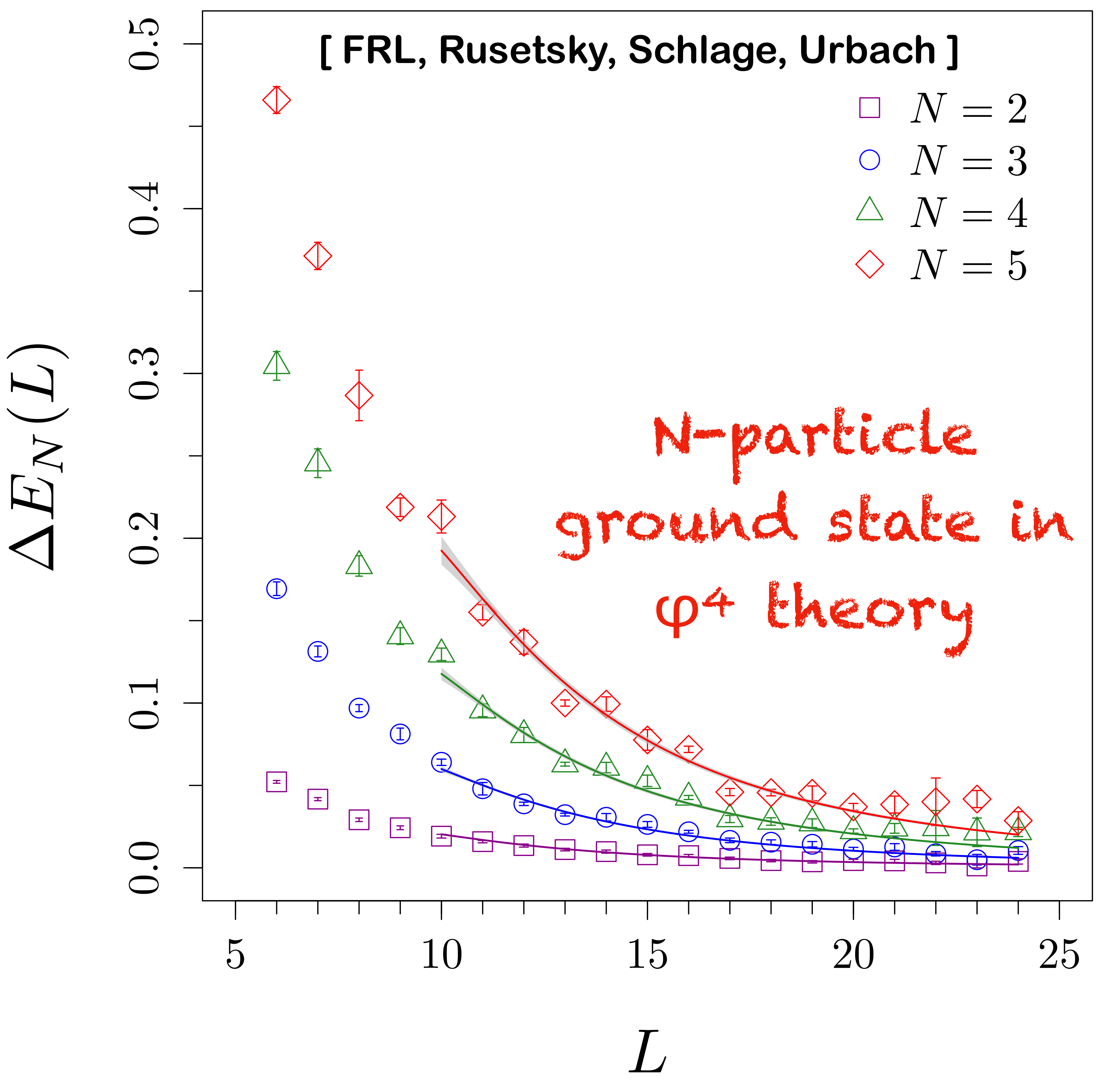

a_0	r_0	$\bar{\mathcal{T}}$	χ^2/ndof
0.439(15)	−227(22)	−189799(42507)	52.29/57

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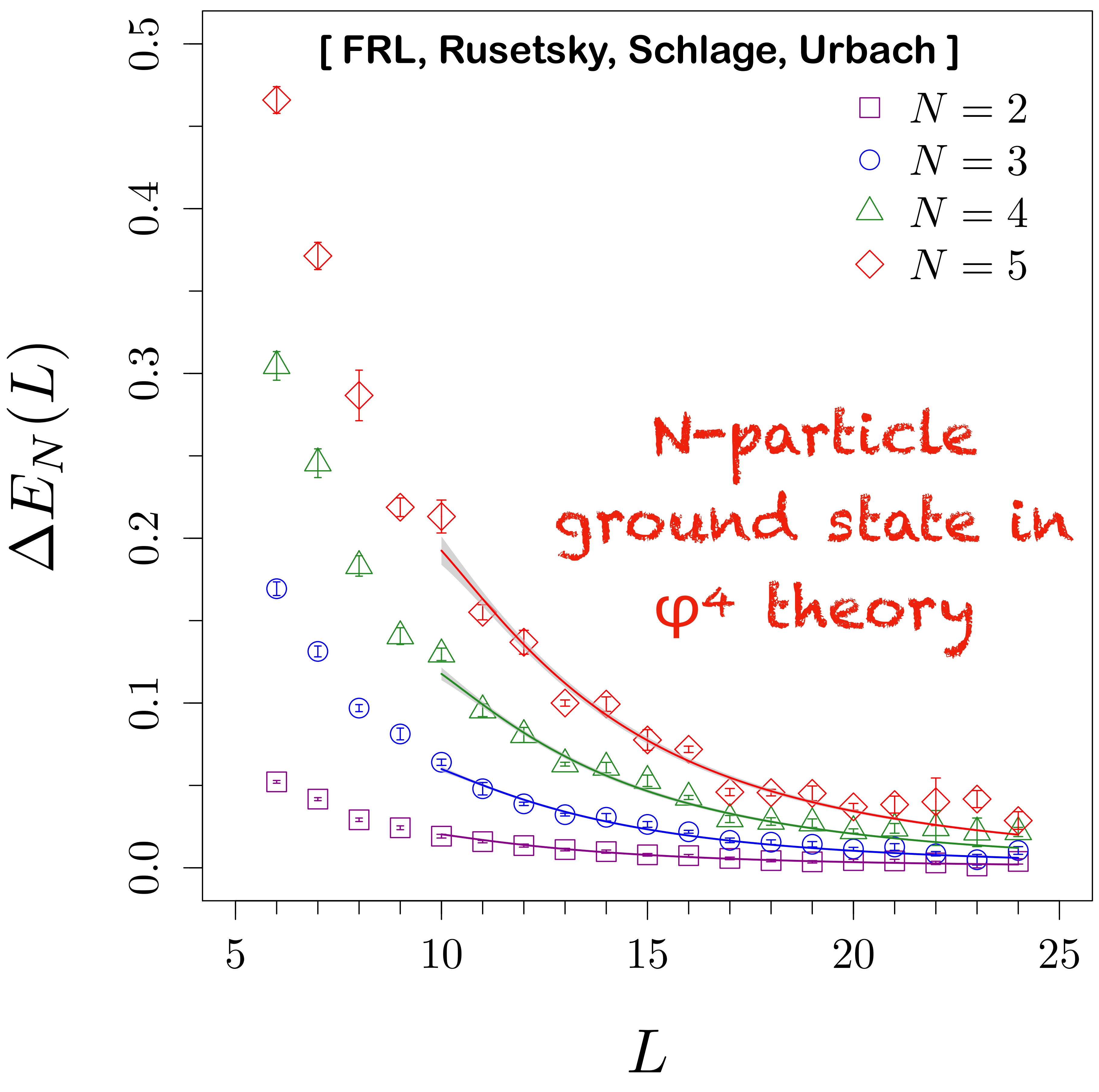
$$- \left(\frac{a_0}{\pi L}\right)^3\left[\mathcal{I}^3 + (2N-7)\mathcal{I}\mathcal{J} + (5N^2 - 41N + 63)\mathcal{K} + 8(N-2)(2\mathcal{Q} + \mathcal{R})\right]$$

$$+ (4N-9)\frac{\pi a_0}{m^2L^3} + (4N-6)\frac{\pi a_0^2 r_0}{L^3}\Bigg\}$$

$$+ \binom{N}{3}\left\{\frac{32\pi a_0^4}{mL^6}\left(3\sqrt{3} - 4\pi\right)\left(2\ln(mL) - \Gamma'(1) - \ln 4\pi\right) - \frac{\bar{\mathcal{T}}}{6L^6}\right\}.$$

two-particle interactions

Three-particle scattering amplitude



a_0	r_0	$\bar{\mathcal{T}}$	χ^2/ndof
0.439(15)	−227(22)	−189799(42507)	52.29/57

Perturbative expansions

- **Non-relativistic EFT in finite volume:**

$$\mathcal{L} = \psi^\dagger\left(i\partial^0 - m + \frac{\nabla^2}{2m} + \frac{\nabla^4}{8m^3}\right)\psi - \frac{g_1}{4}(\psi^\dagger\psi)^2 - g_2(\psi^\dagger\psi^\dagger)\nabla^2(\psi\psi)$$

$$- g_3\left((\psi^\dagger\psi^\dagger)(\psi\overset{\leftrightarrow}{\nabla}{}^2\psi) + \text{ h.c.}\right) - \frac{\eta_3}{6}(\psi^\dagger\psi)^3,$$

- **Compute energy shift in perturba**

$$\Delta E_0 = \binom{N}{2}\frac{4\pi a_0}{mL^3}\Bigg\{1 - \frac{a_0}{\pi L}\mathcal{I} + \left(\frac{a_0}{\pi L}\right)^2\left[\mathcal{I}^2 + (2N-5\right.$$

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Suitable for weakly interacting systems!

two-particle interactions

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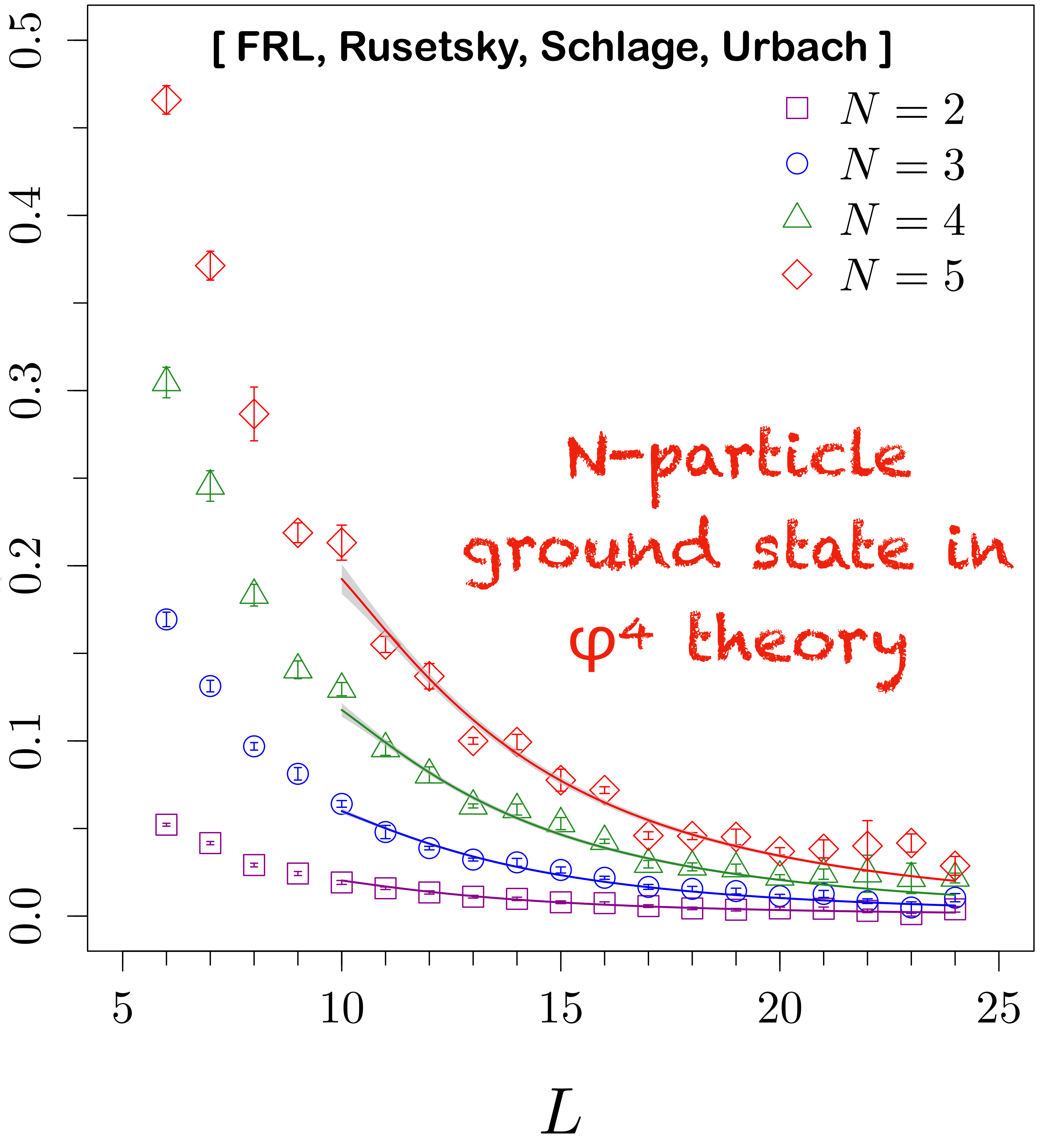

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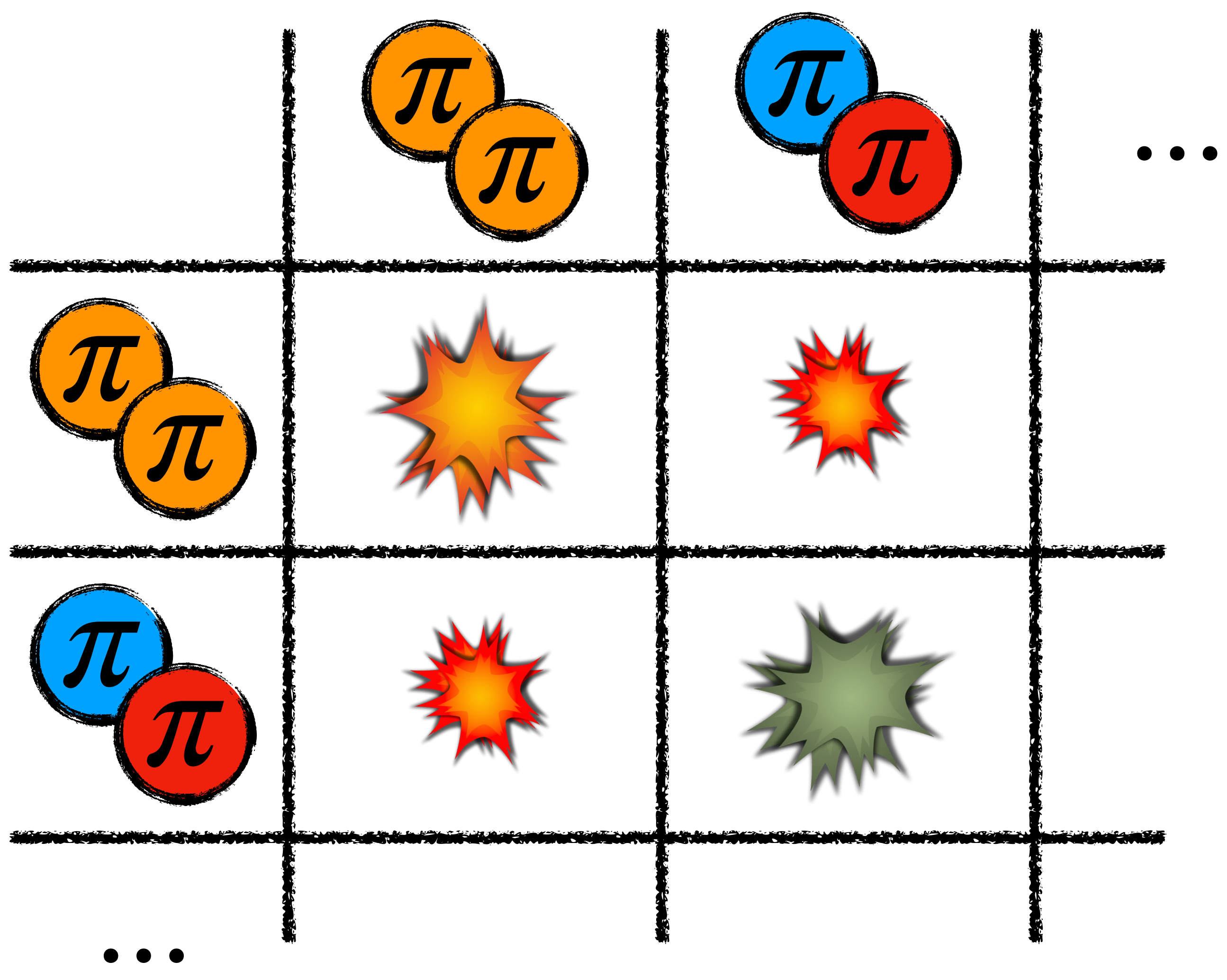
Two particles in finite-volume



The Lüscher Formalism

- A new field was opened by M. Lüscher in ‘86

finite-volume spectrum of two identical scalars → s-wave scattering amplitude

Volume Dependence of the Energy Spectrum in Massive Quantum Field Theories

II. Scattering States

M. Lüscher
Theory Division, Deutsches Elektronen-Synchrotron DESY, D-2000 Hamburg 52, Federal Republic of Germany

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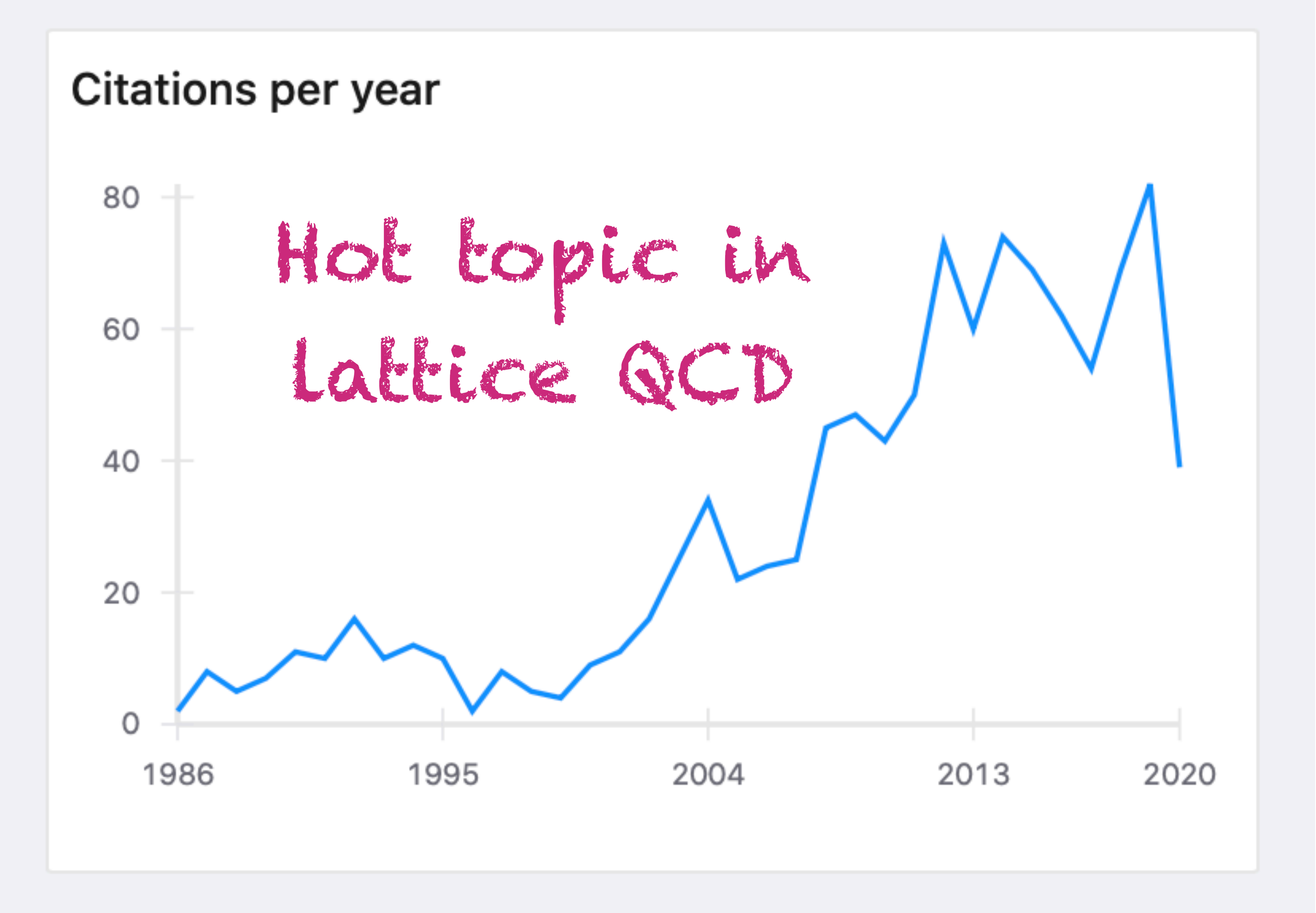

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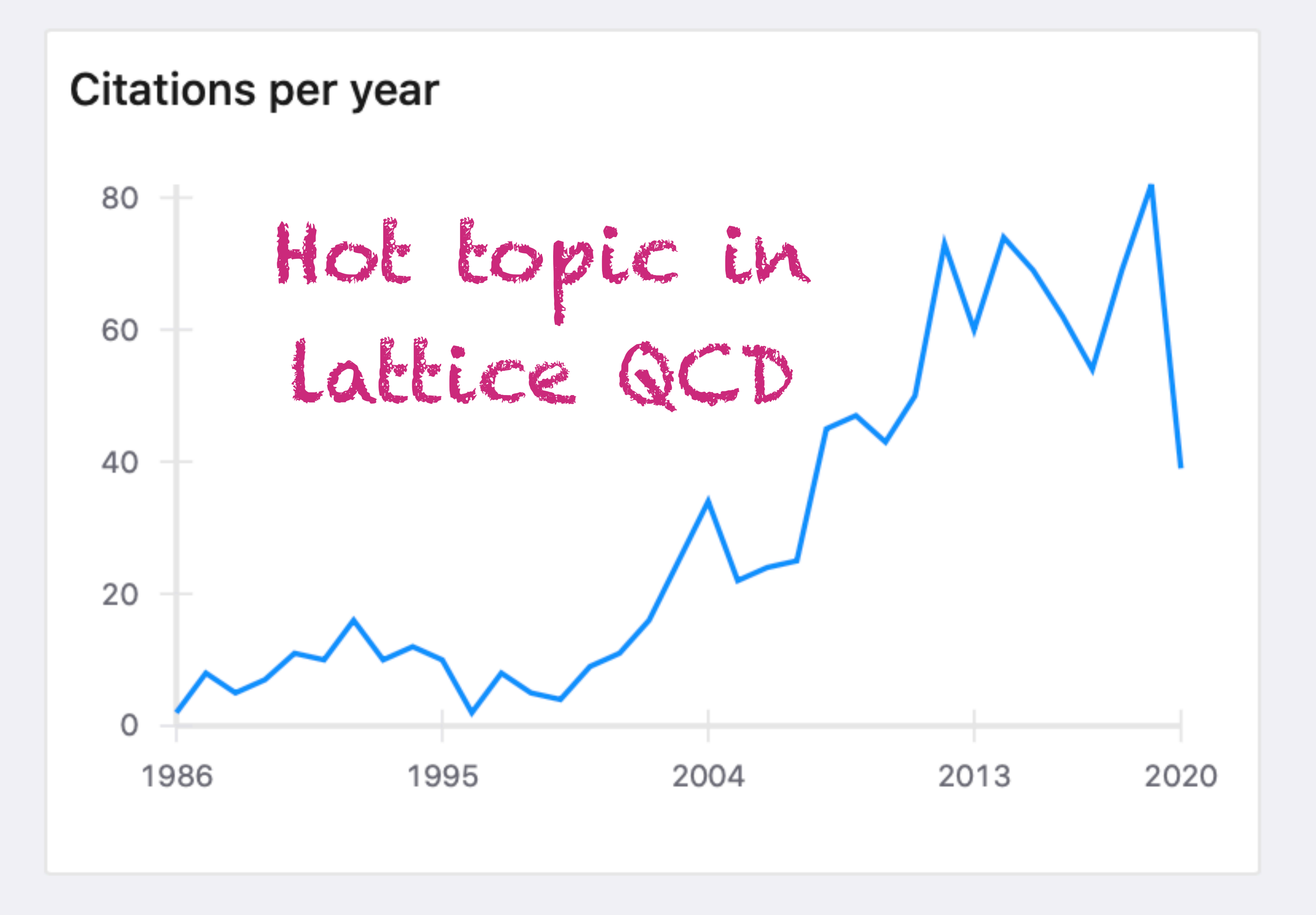

- Fully general formalism exists up to date:
  - Multichannel, non-identical $2 \to 2$ scattering for particles with spin in all partial waves. Including for weak decays, such as $K \to \pi\pi$ (Lellouch-Lüscher)
  - Many people have contributed over the years:
    - Rummukainen and Gottlieb
    - Kim, Sachrajda and Sharpe
    - Göckeler, Horsley, Lage, Meißner, Rakow, Rusetsky, Schierholz, Zanotti
    - Briceño

Quantization Condition(I)

- In order to derive the full relation, consider the finite-volume correlator:

$$C_L(E, \overrightarrow{P}) = \int e^{iPx} \langle \mathcal{O}(x) | \mathcal{O}(0) \rangle =$$

[à la Kim, Sachrajda, Sharpe]

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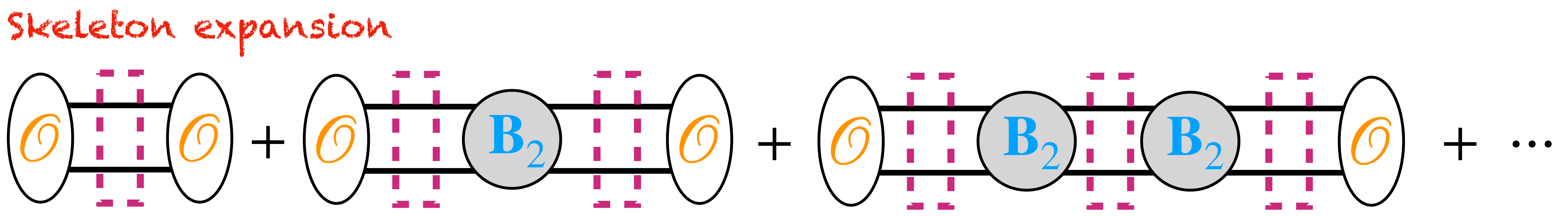

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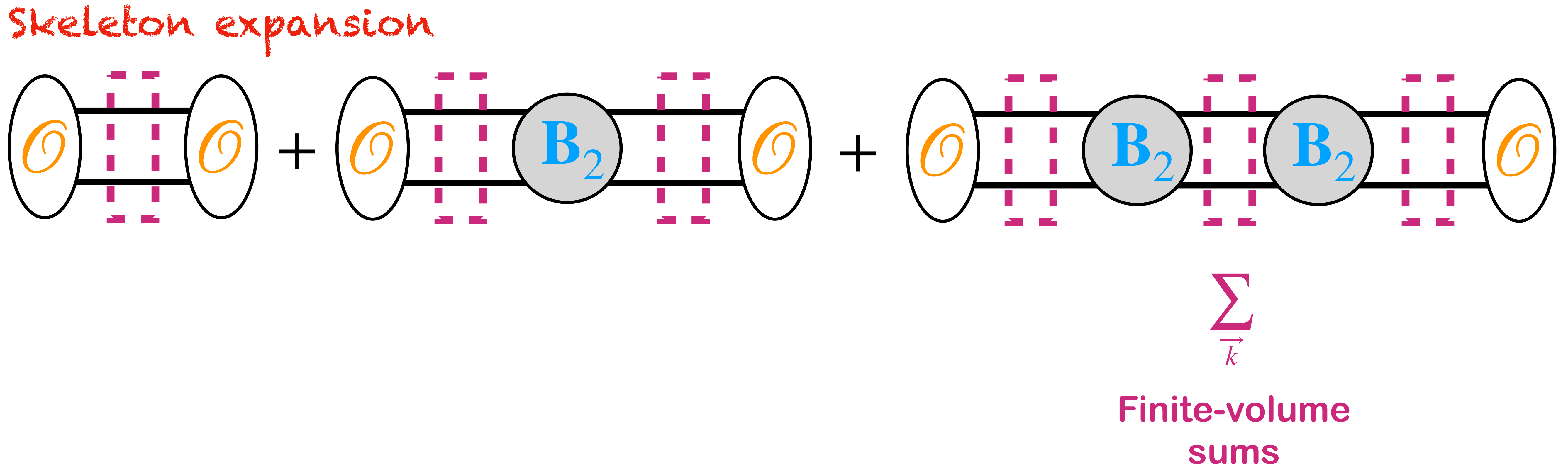

$+ \cdots$

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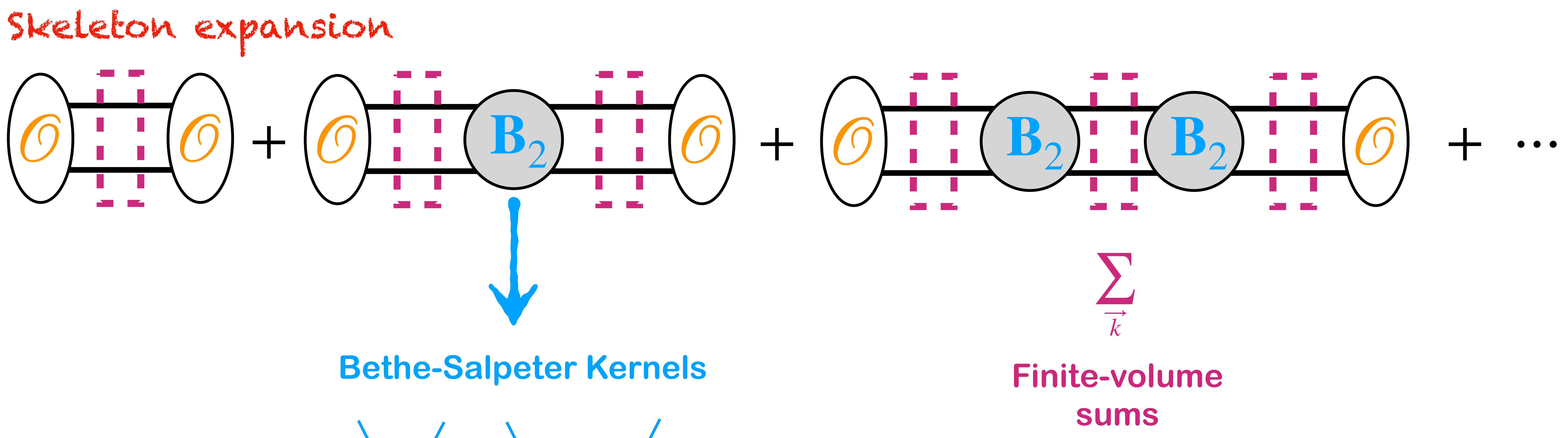

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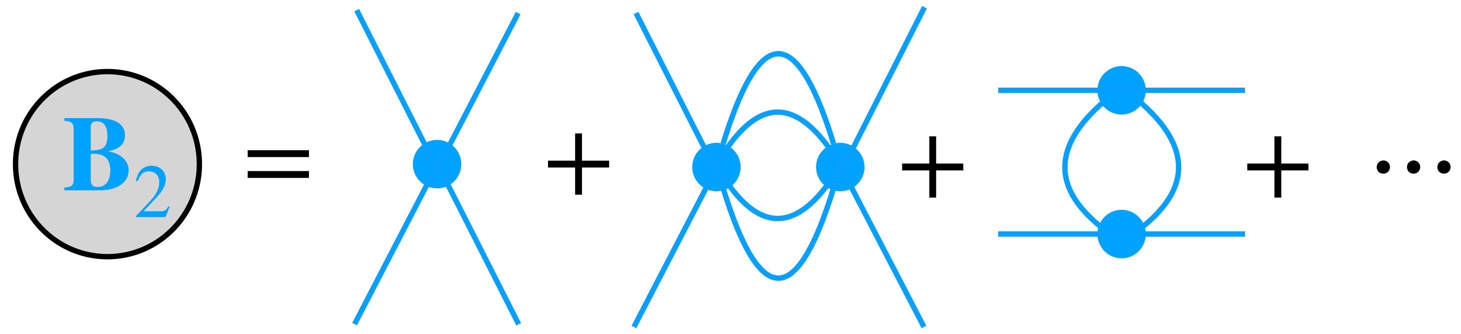

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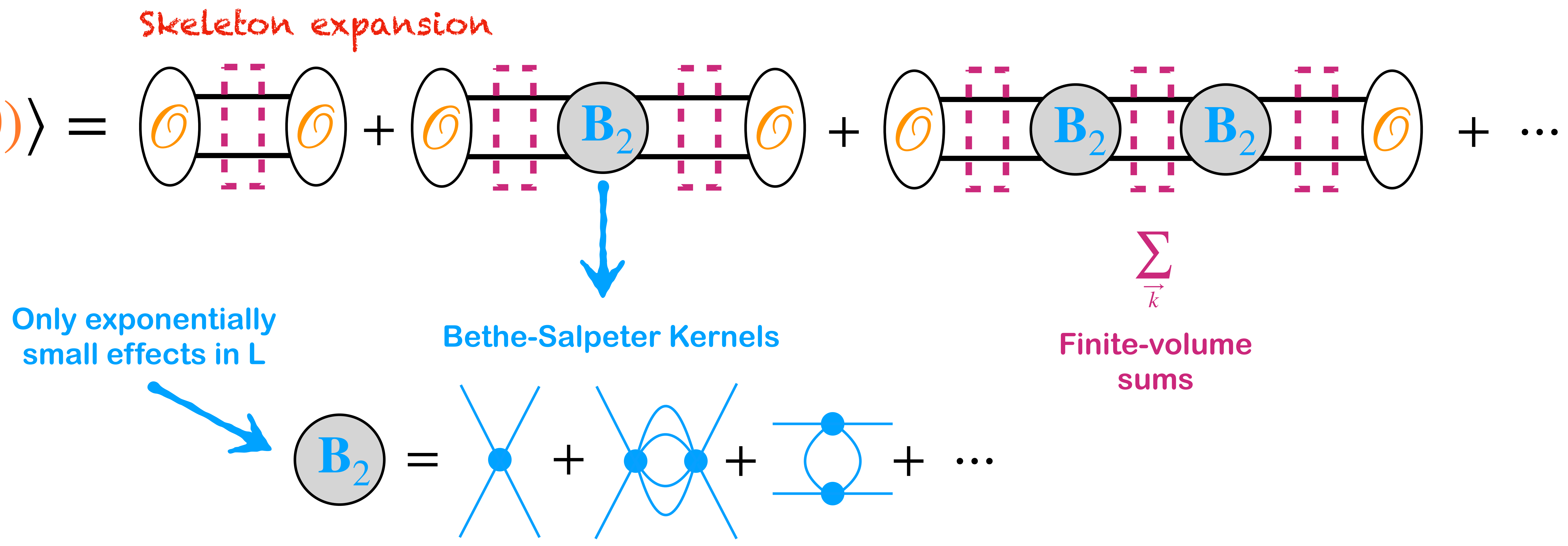

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- In order to derive the full relation, consider the finite-volume correlator:

Skeleton expansion

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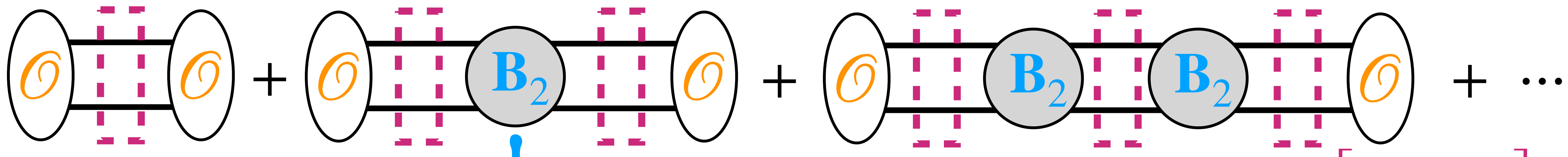


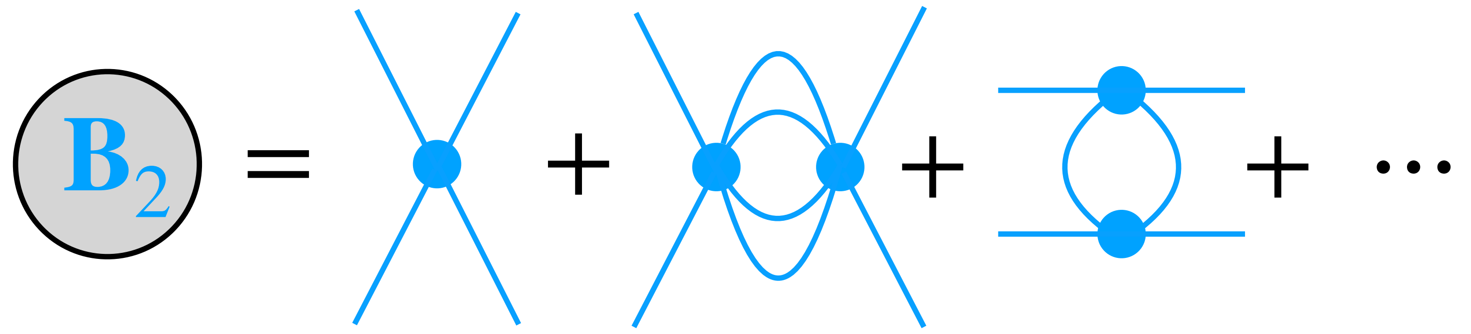
[à la Kim, Sachrajda, Sharpe]

$$\sum_{\vec{k}} \longrightarrow \int d^3k + \left[\sum_{\vec{k}} - \int d^3k \right]$$

Only exponentially small effects in L

Bethe-Salpeter Kernels

Finite-volume sums



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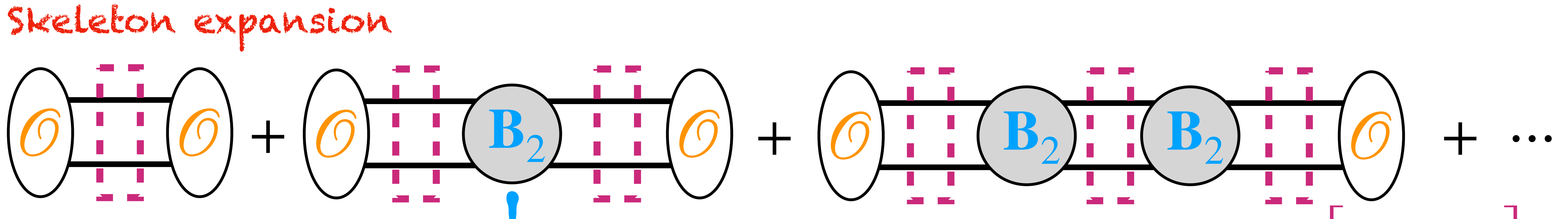

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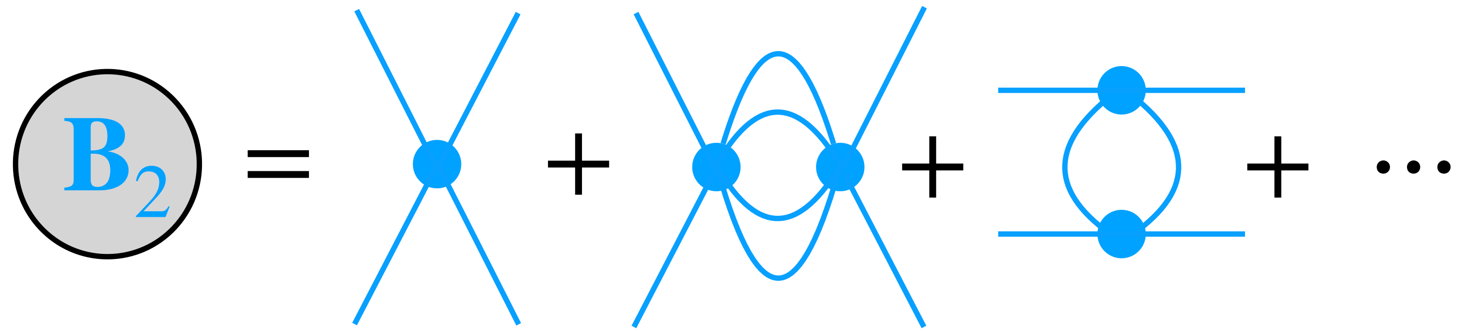

1. Separation of finite-volume effects
2. Resumation of diagrams

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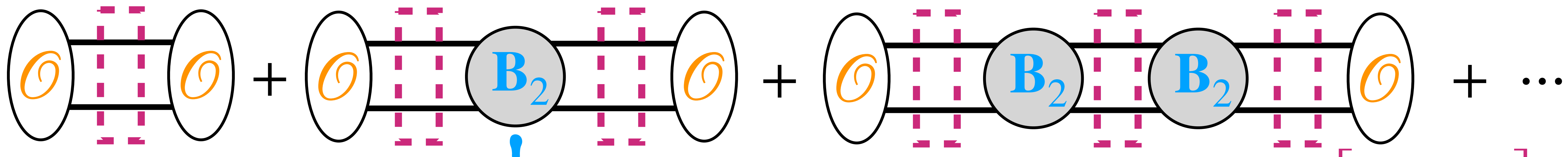

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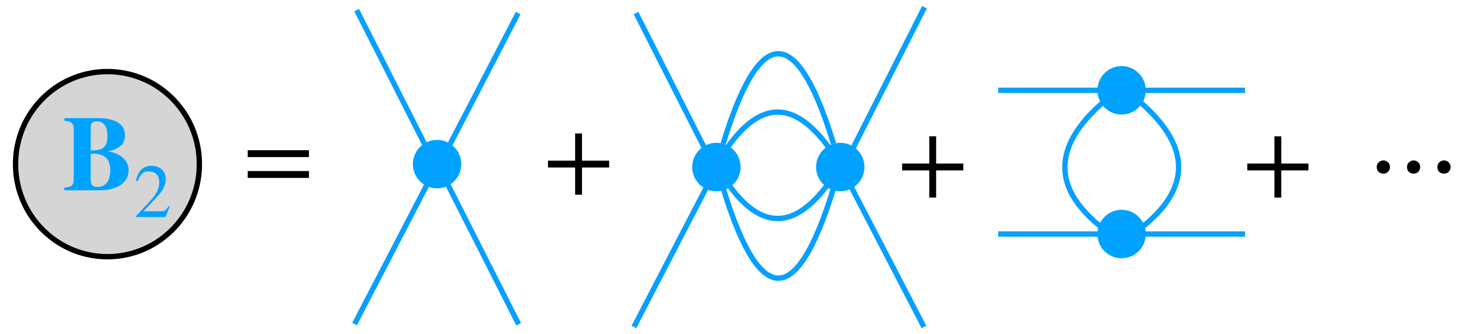

1. Separation of finite-volume effects
2. Resumation of diagrams

$$C_L(E, \vec{P}) = C_\infty(E, \vec{P}) + A^\dagger \frac{1}{\mathcal{K}_2 + F^{-1}} A + O(e^{-mL})$$

Known kinematic function

$$\mathcal{M}_2^{-1} = \mathcal{K}_2^{-1} - i\sqrt{s - 2m^2}$$

Quantization Condition(II)

$$C_L(E, \overrightarrow{P}) = \quad \text{some algebra ...} \quad = C_\infty(E, \overrightarrow{P}) + A^\dagger \frac{1}{\mathcal{K}_2 + F^{-1}} A + O(e^{-mL})$$

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K-matrix parametrized in terms of phase shift

$$\mathcal{K}_2^\ell = \frac{16\pi\sqrt{s}}{q^{2\ell+1}\cot\delta_\ell}$$

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Two-particle Quantization Condition

$$\det\left[\mathcal{K}_2(E_n) + F^{-1}(E_n, \vec{P}, L)\right] = 0$$

Scattering K-Matrix

Known kinematic function

"QC2"

! It holds below $E_{cm} < 4m$

Quantization Condition(III)

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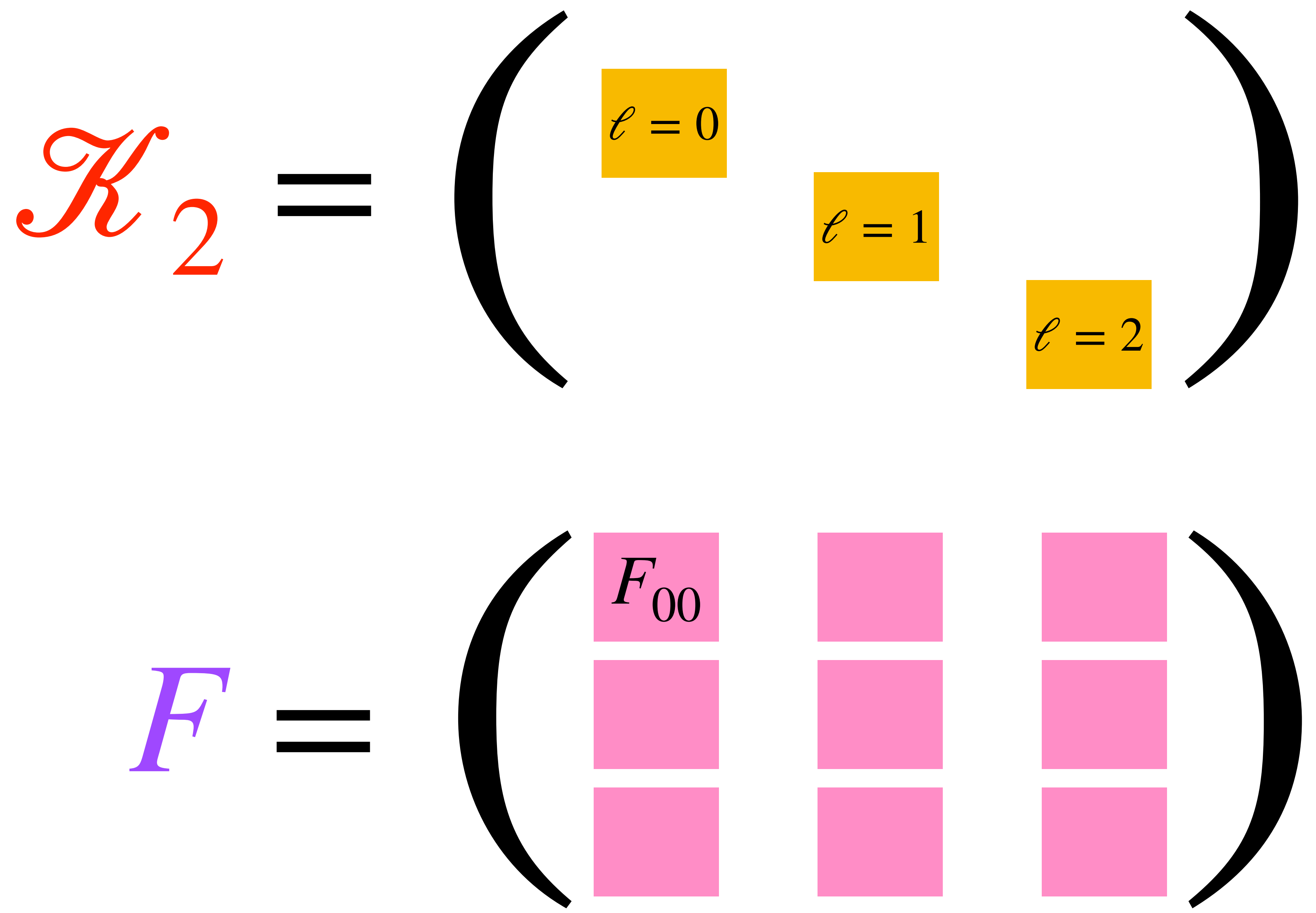
- Matrix indices are angular momentum: ℓm

rotation invariance in infinite volume



finite-volume mixes partial waves

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$$\mathcal{K}_2 = \begin{pmatrix} \ell=0 & & \\ & \ell=1 & \\ & & \ell=2 \end{pmatrix}$$

$$F = \begin{pmatrix} F_{00} & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare \end{pmatrix}$$

finite-volume mixes partial waves

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int \frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2 - q^2}$$

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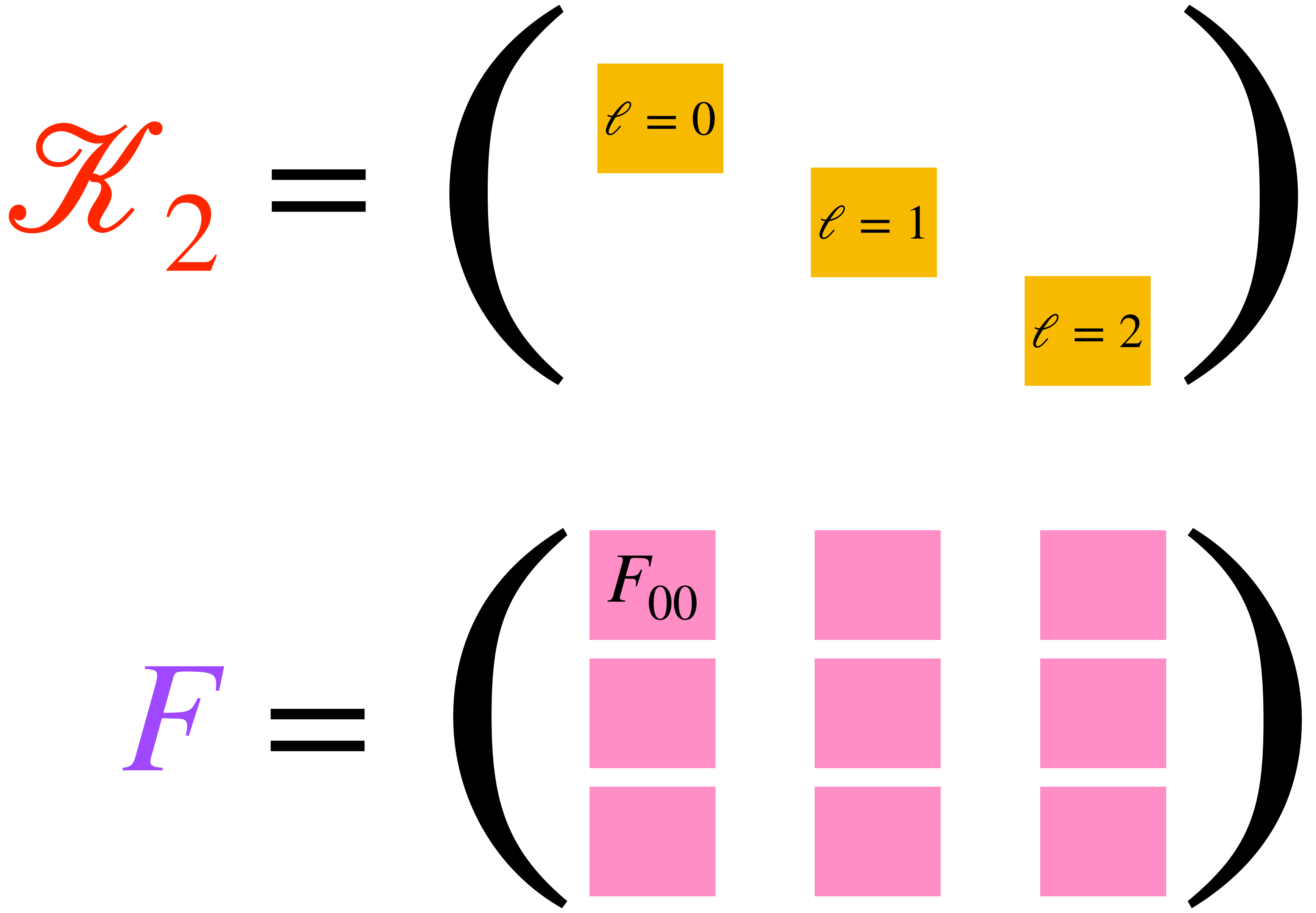
- Matrix indices are angular momentum: ℓm
- Infinite dimensional, need truncation!

Two pions in s-wave

$$\mathcal{K}_2^{s-wave}(E_n) = \frac{-1}{F_{00}(E_n, \vec{P}, L)}$$

rotation invariance in infinite volume



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Isospin-2 ππ scattering

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one energy level → a phase shift point

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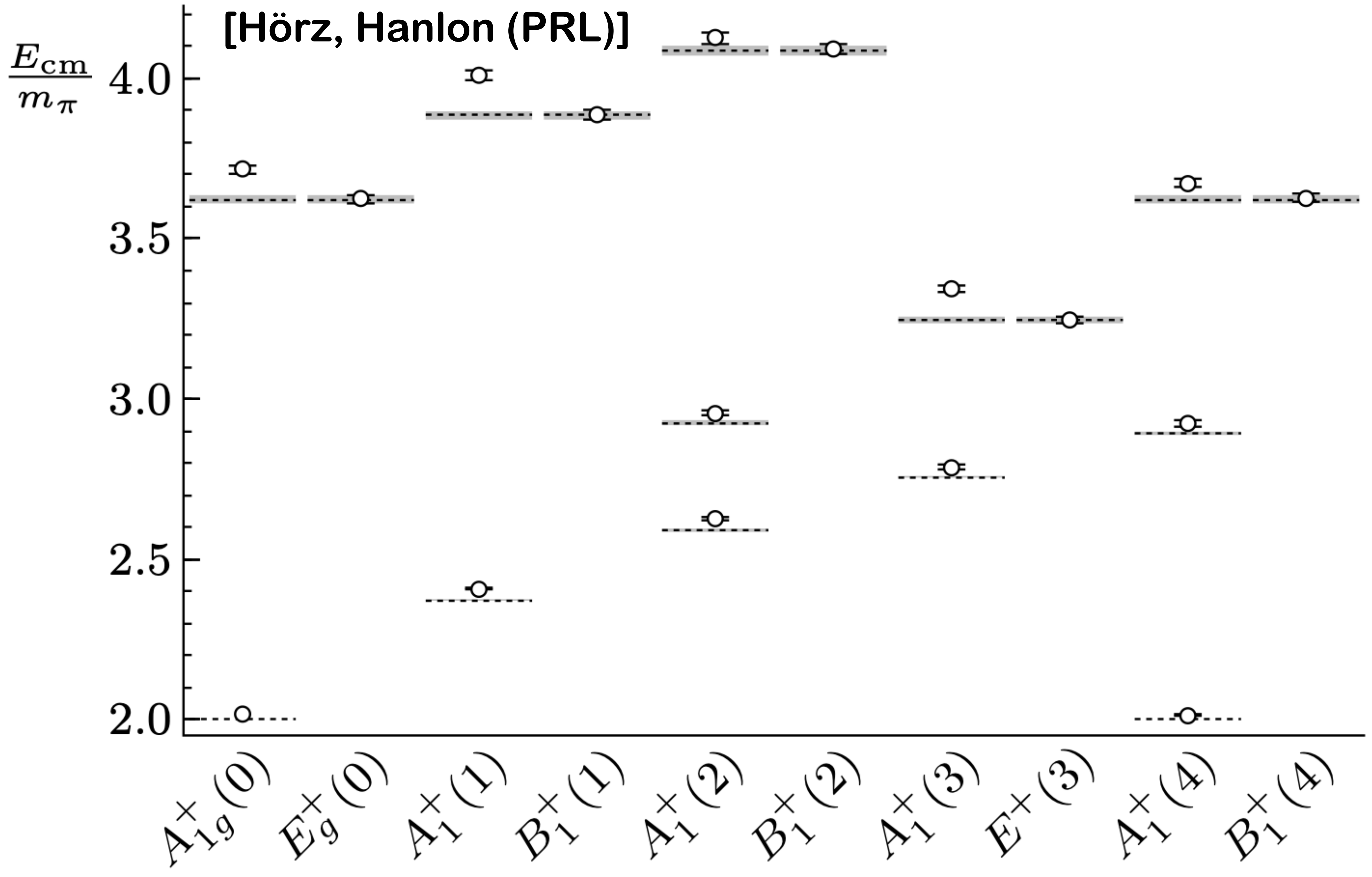

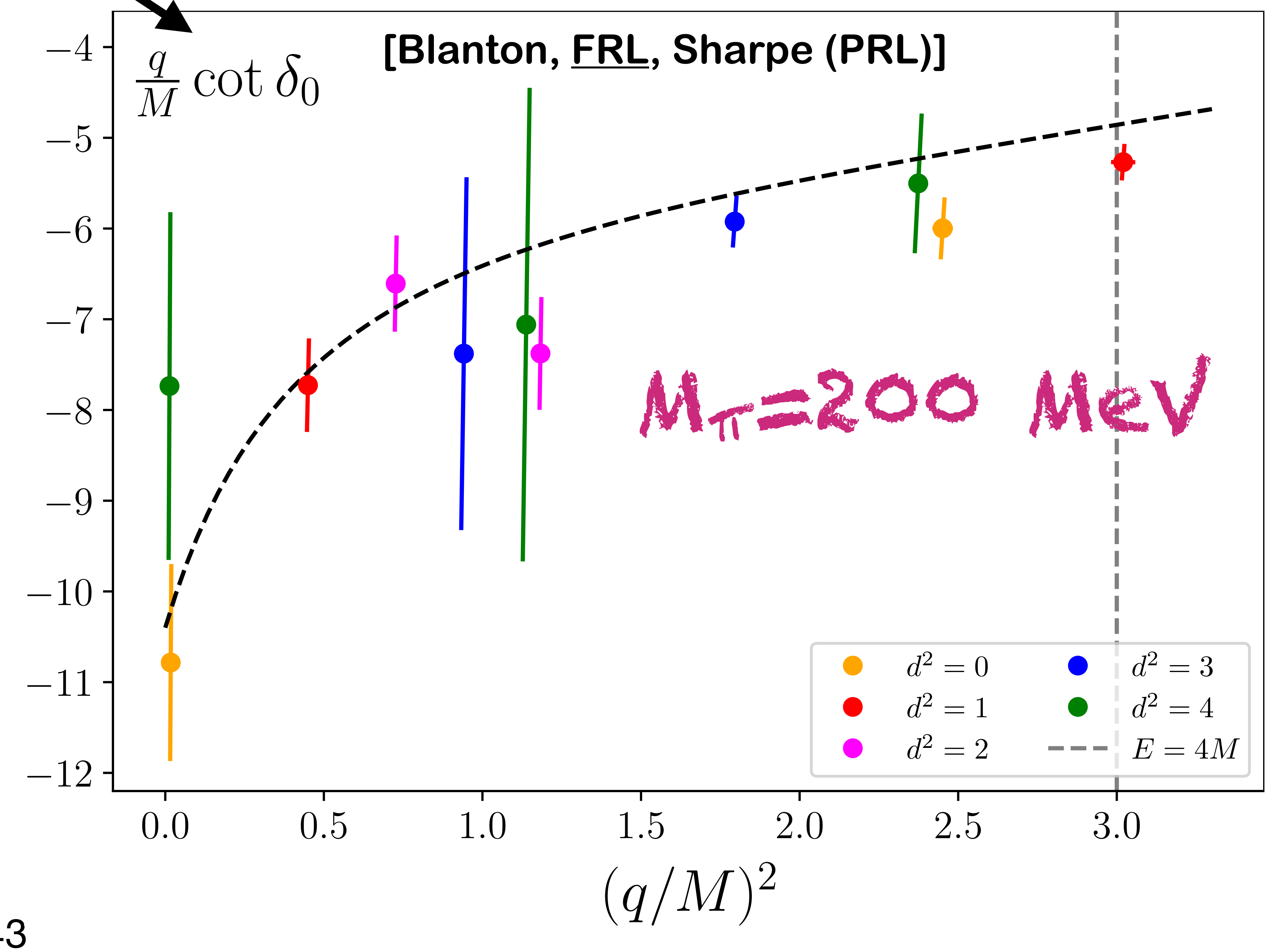

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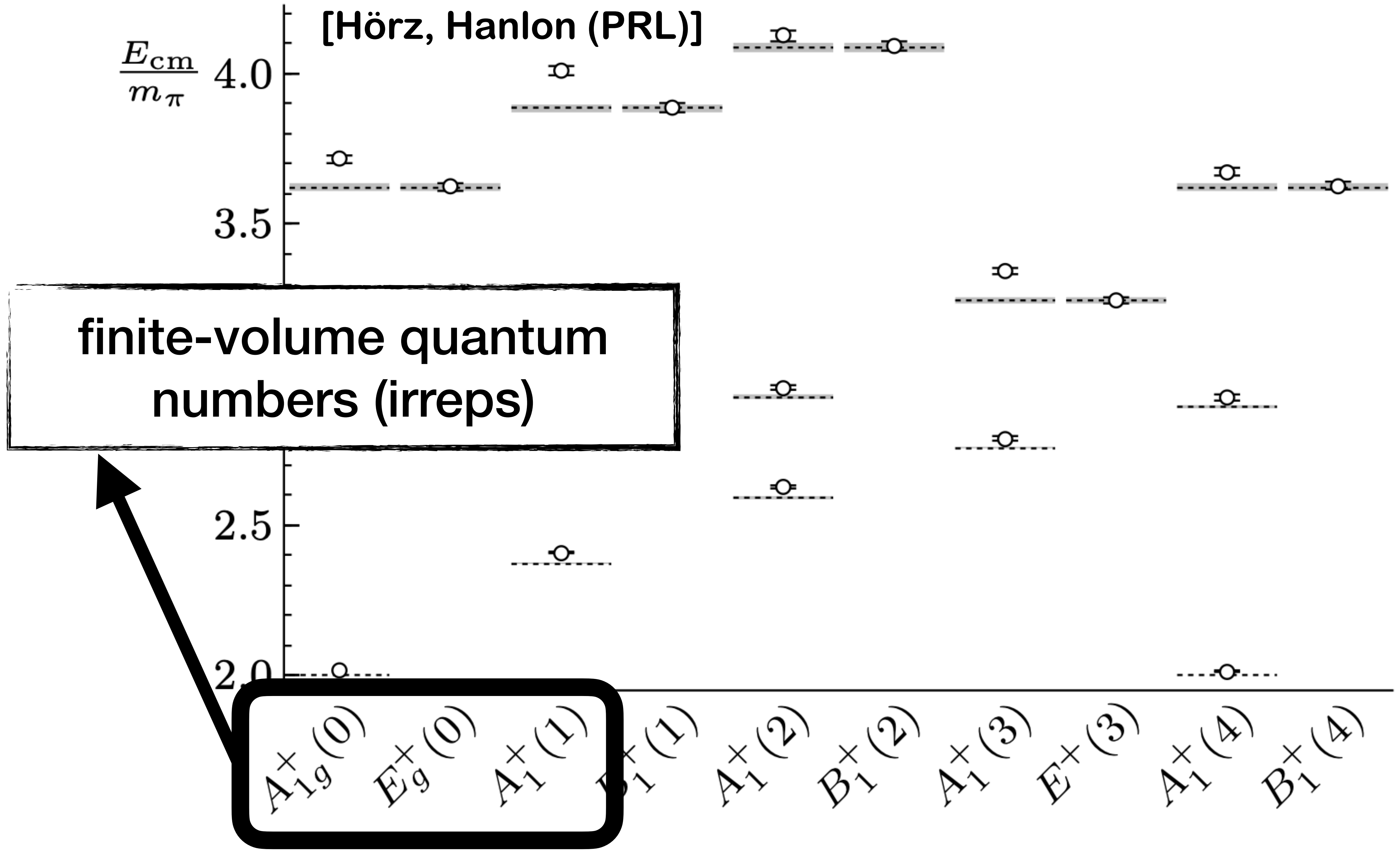

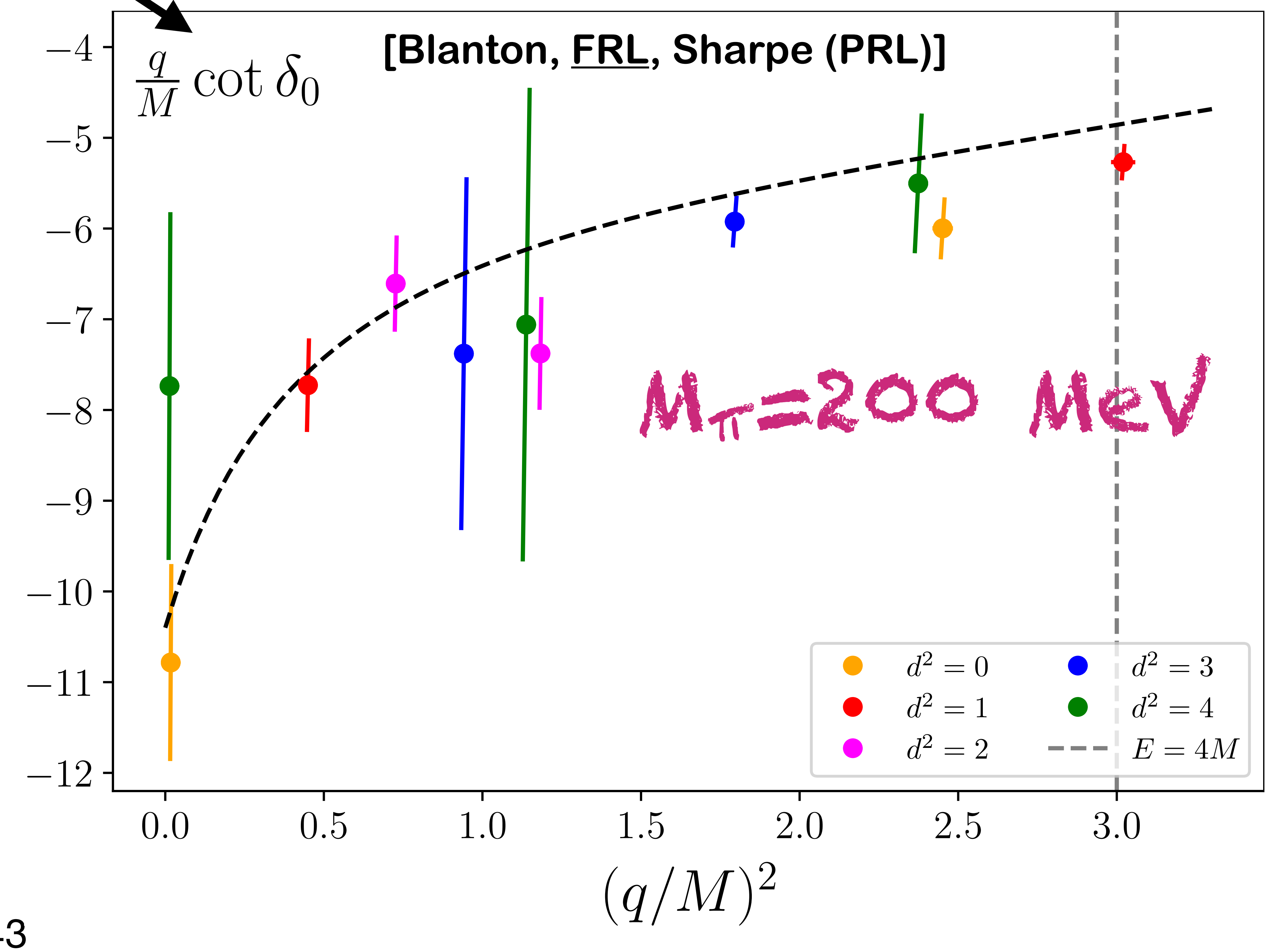

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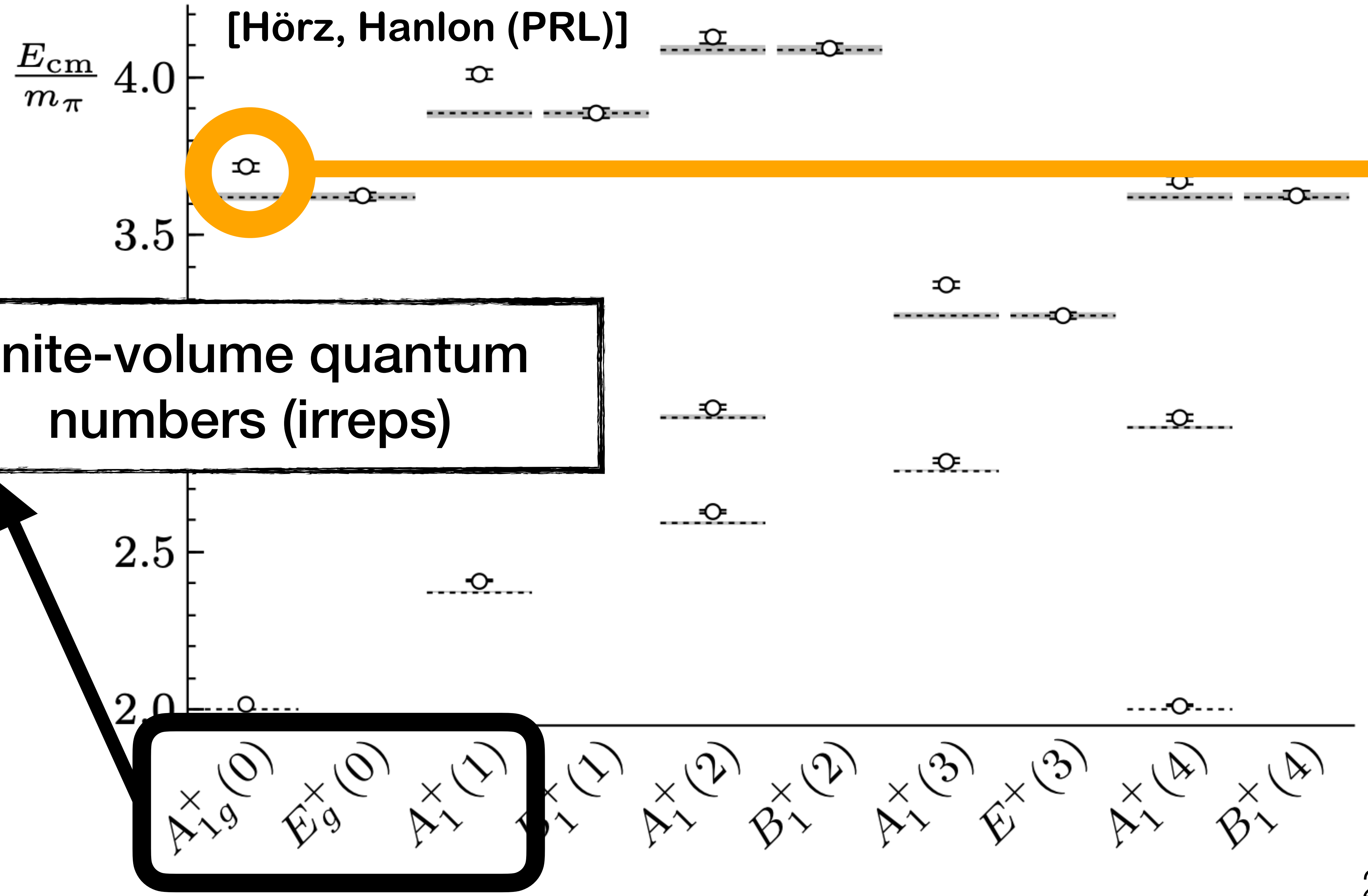

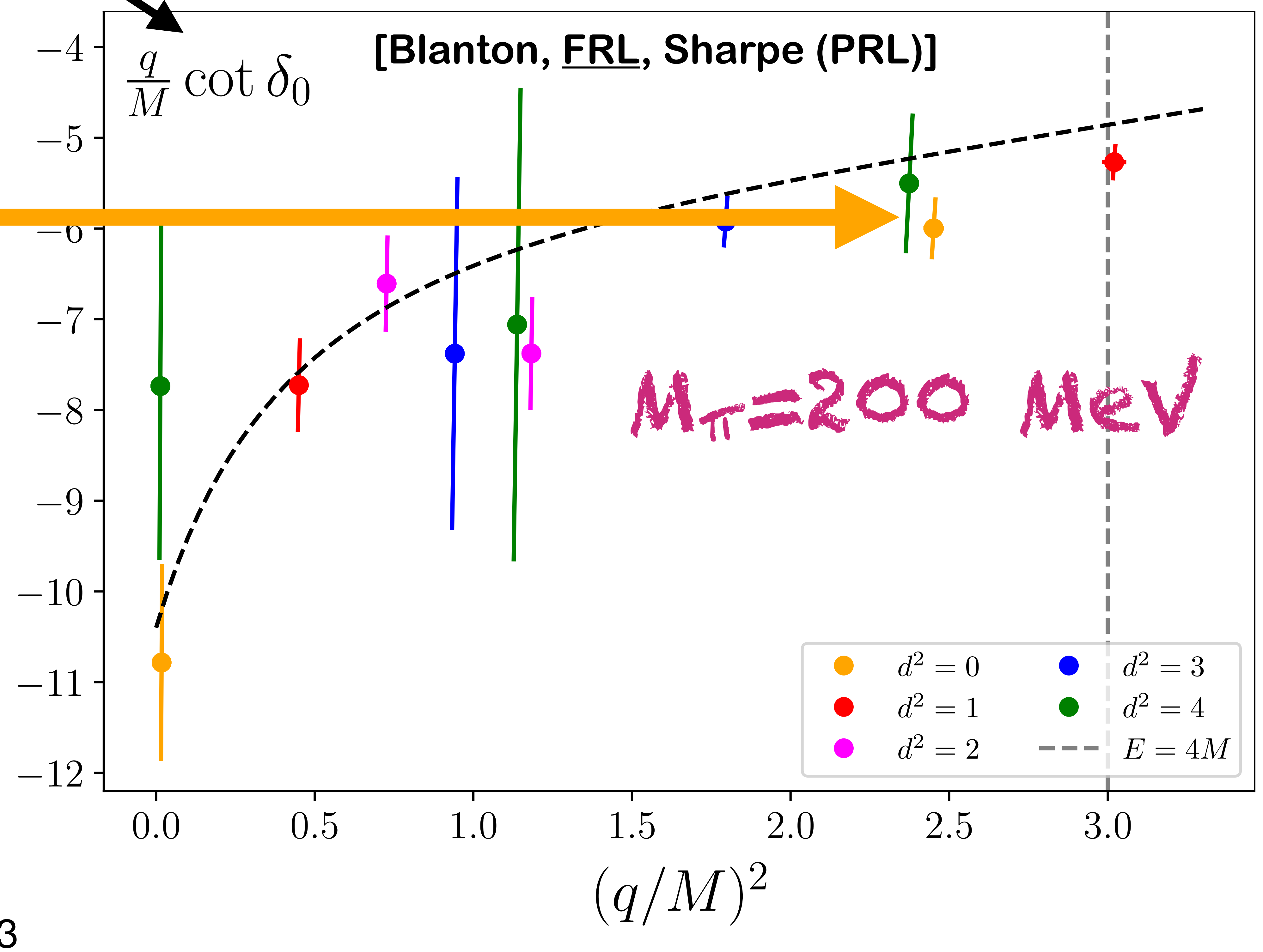

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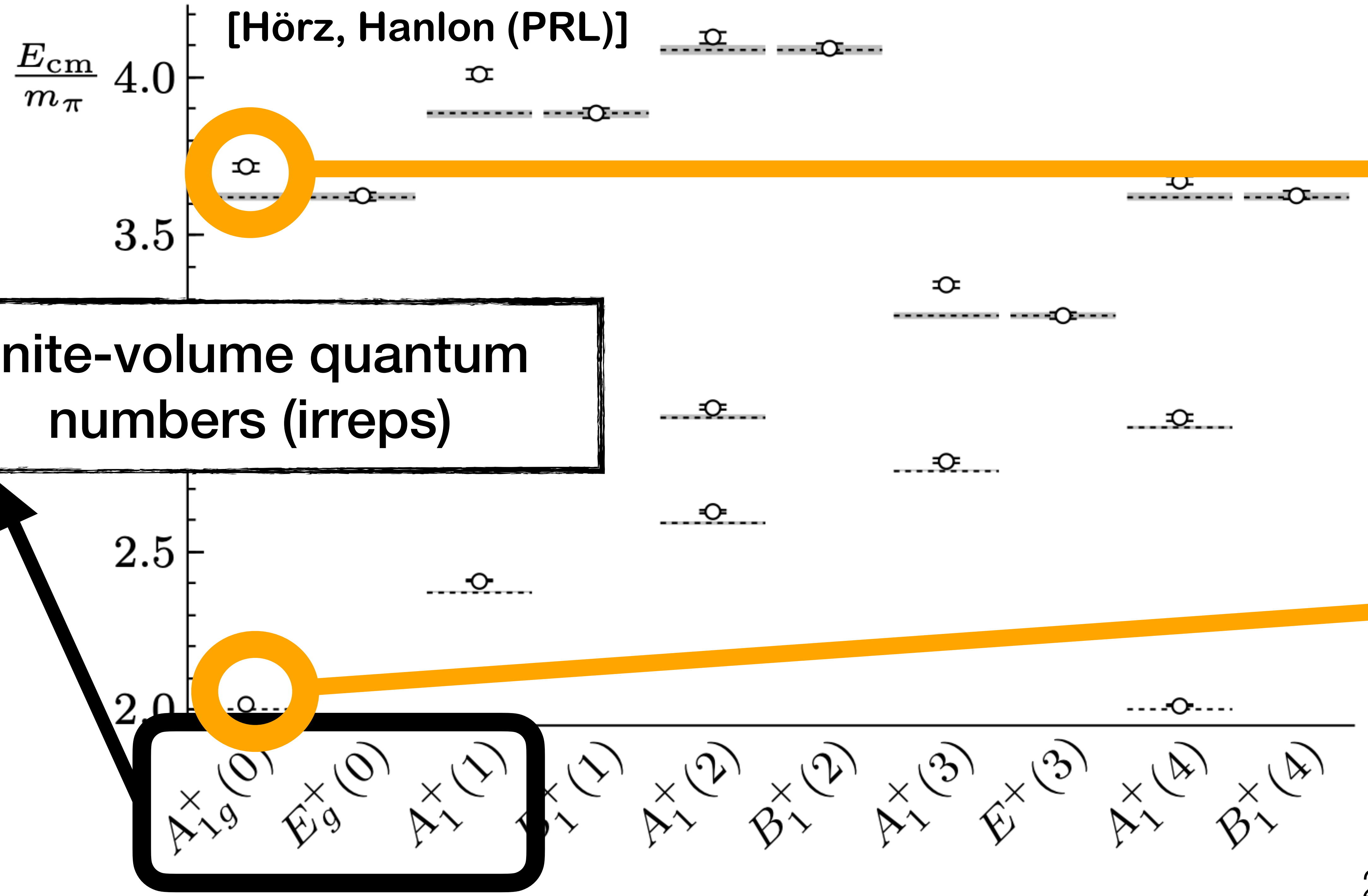
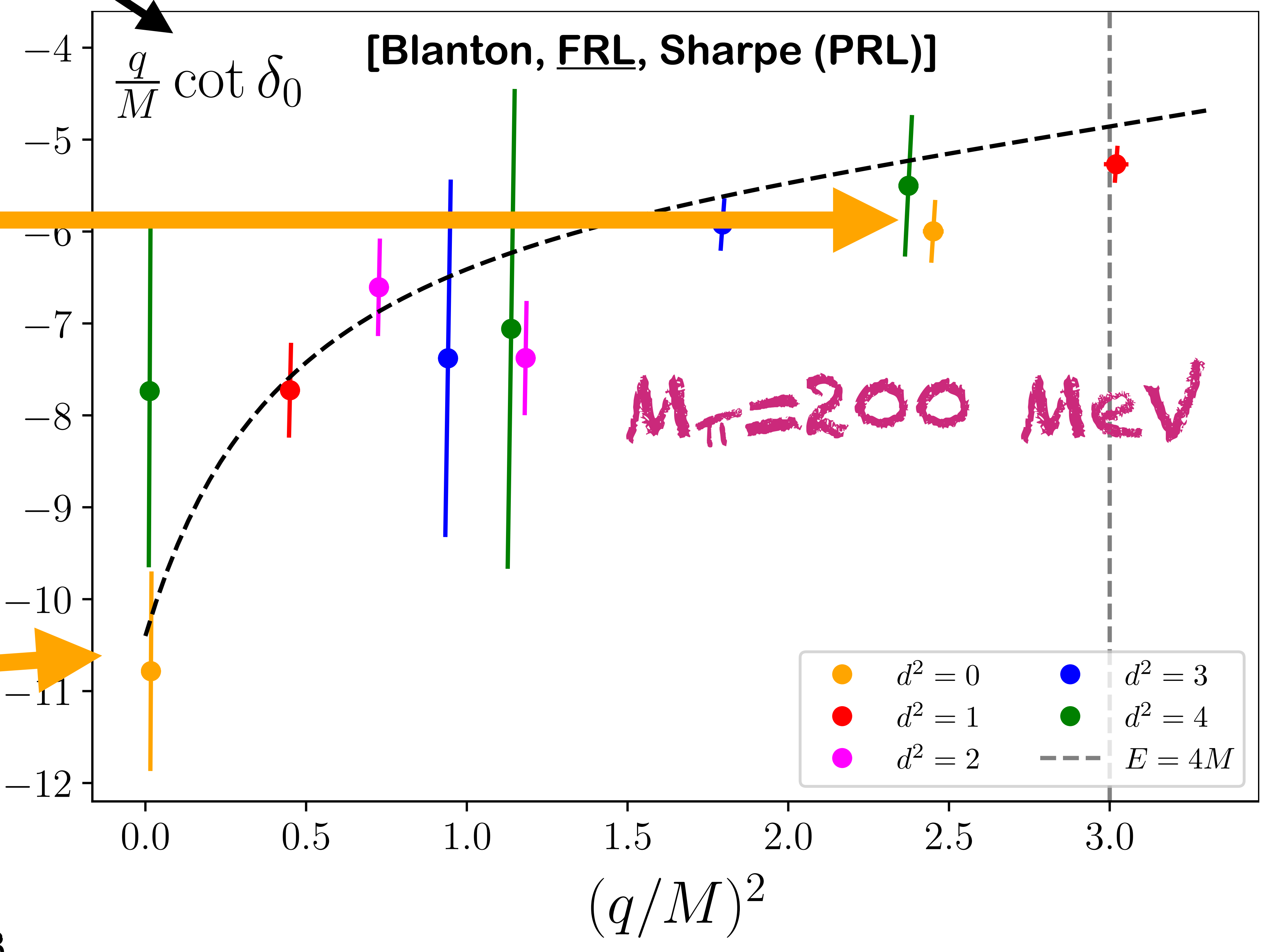


finite-volume quantum numbers (irreps)



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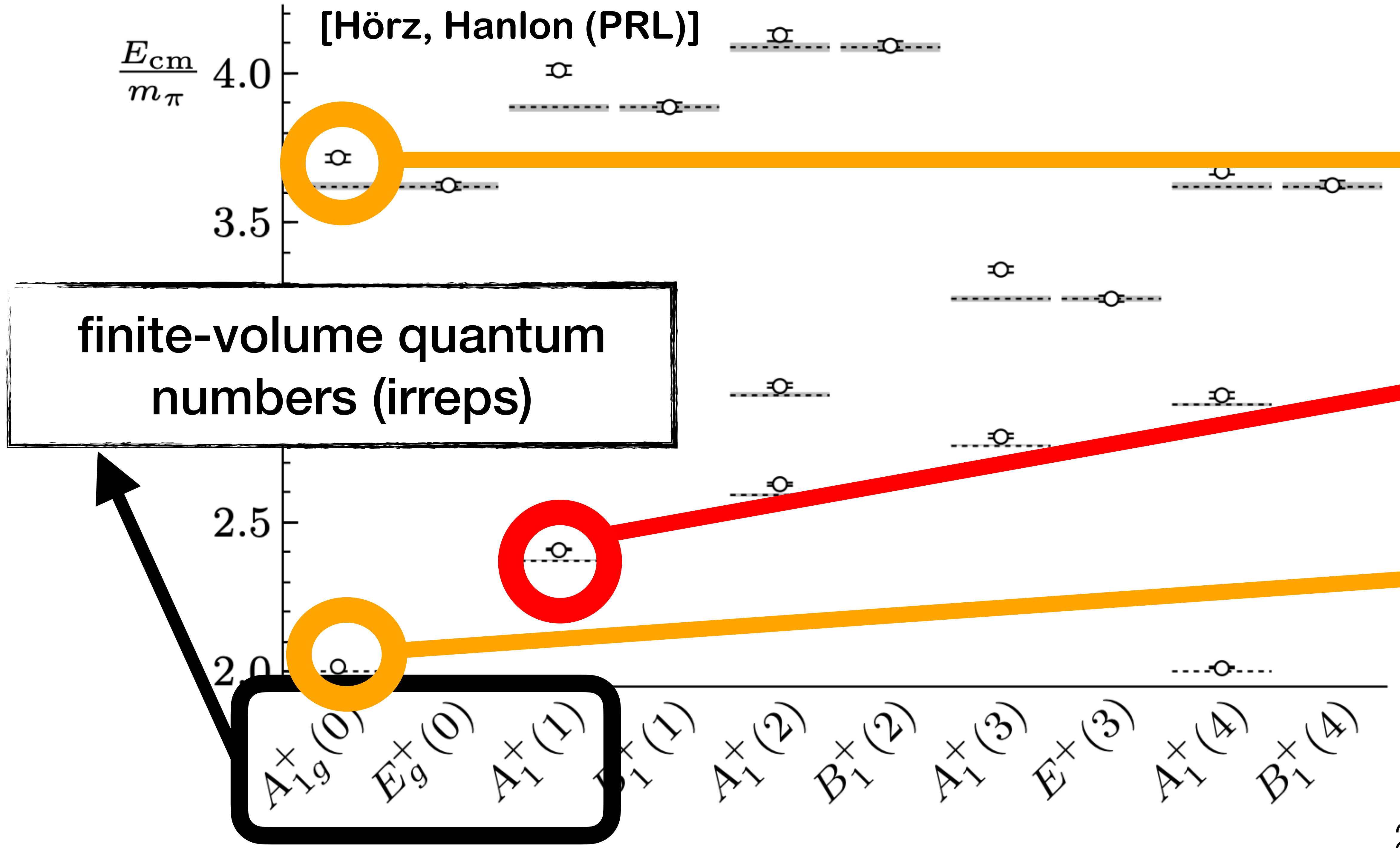
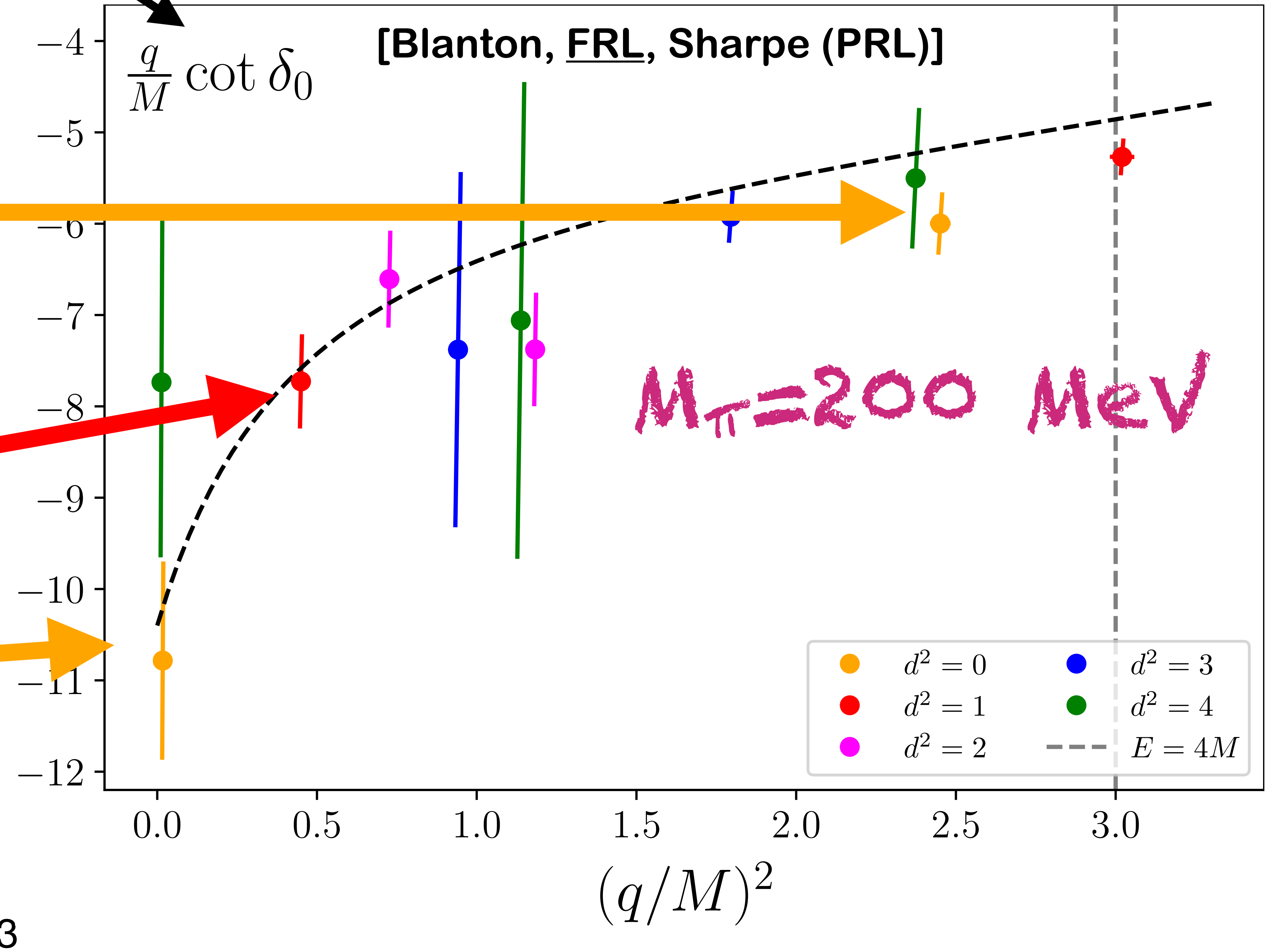
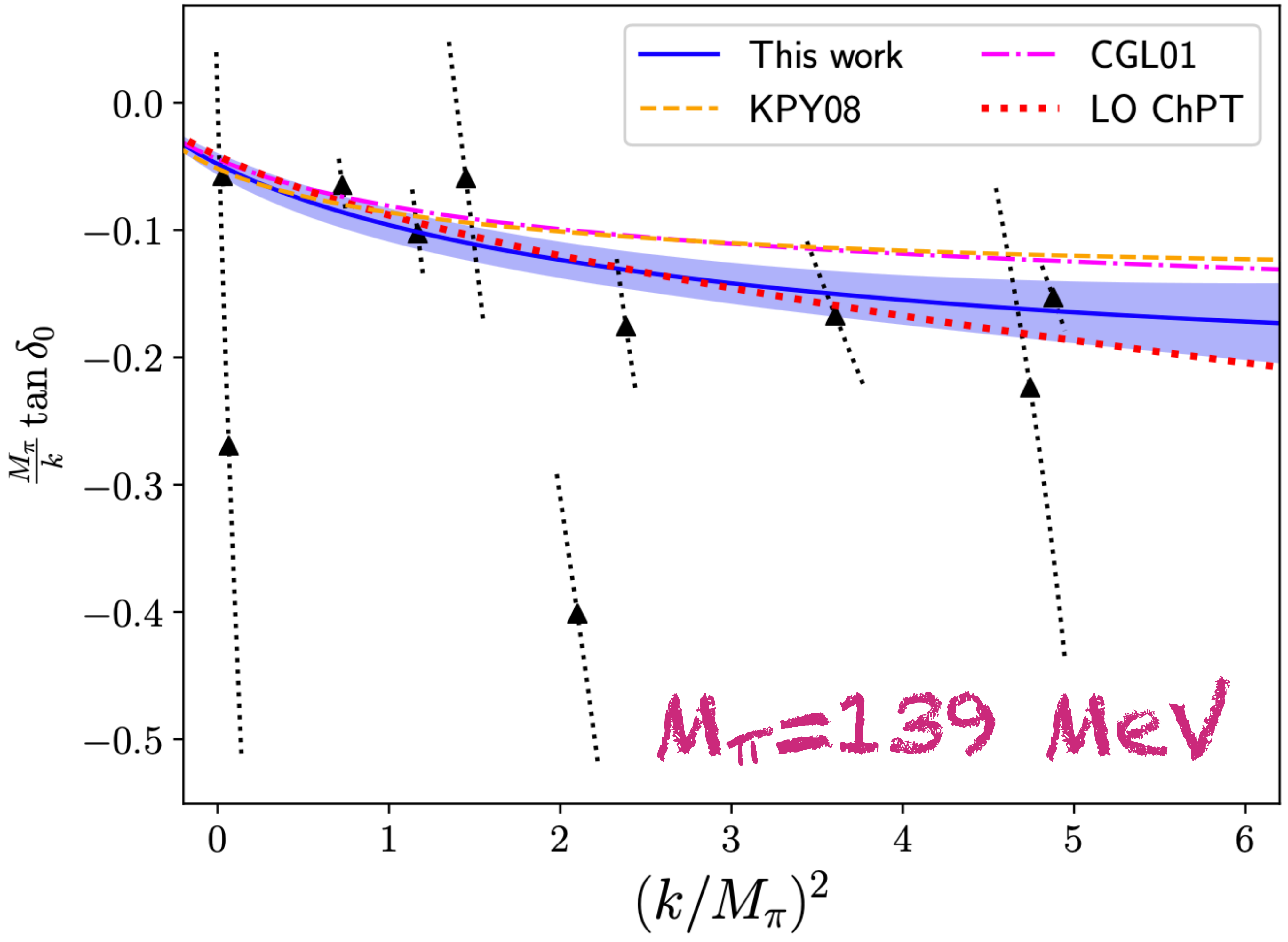




Isospin-2 ππ scattering

s-wave



[Fischer, Kostrzewa, Liu, FRL, Ueding, Urbach (ETMC)]

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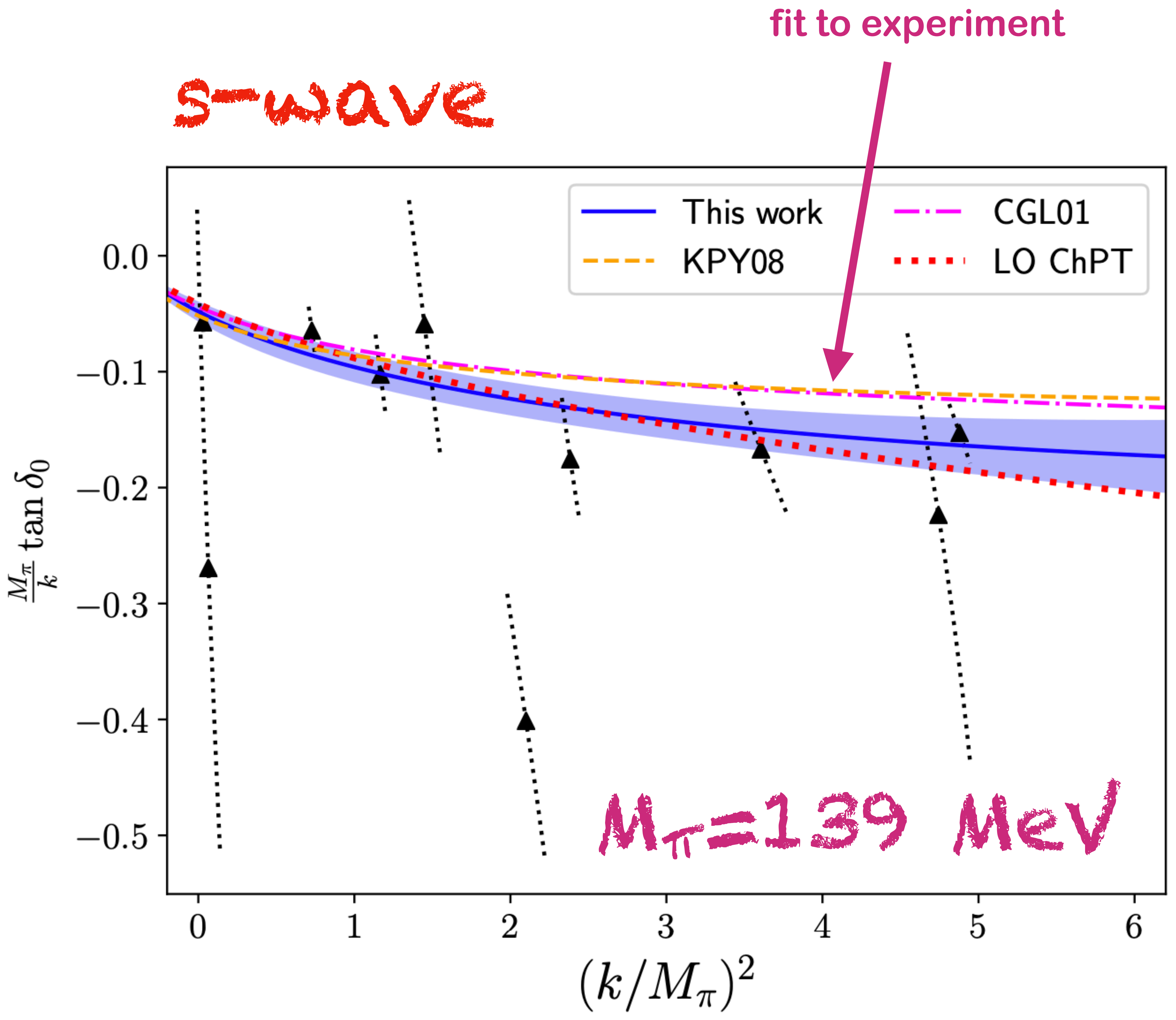

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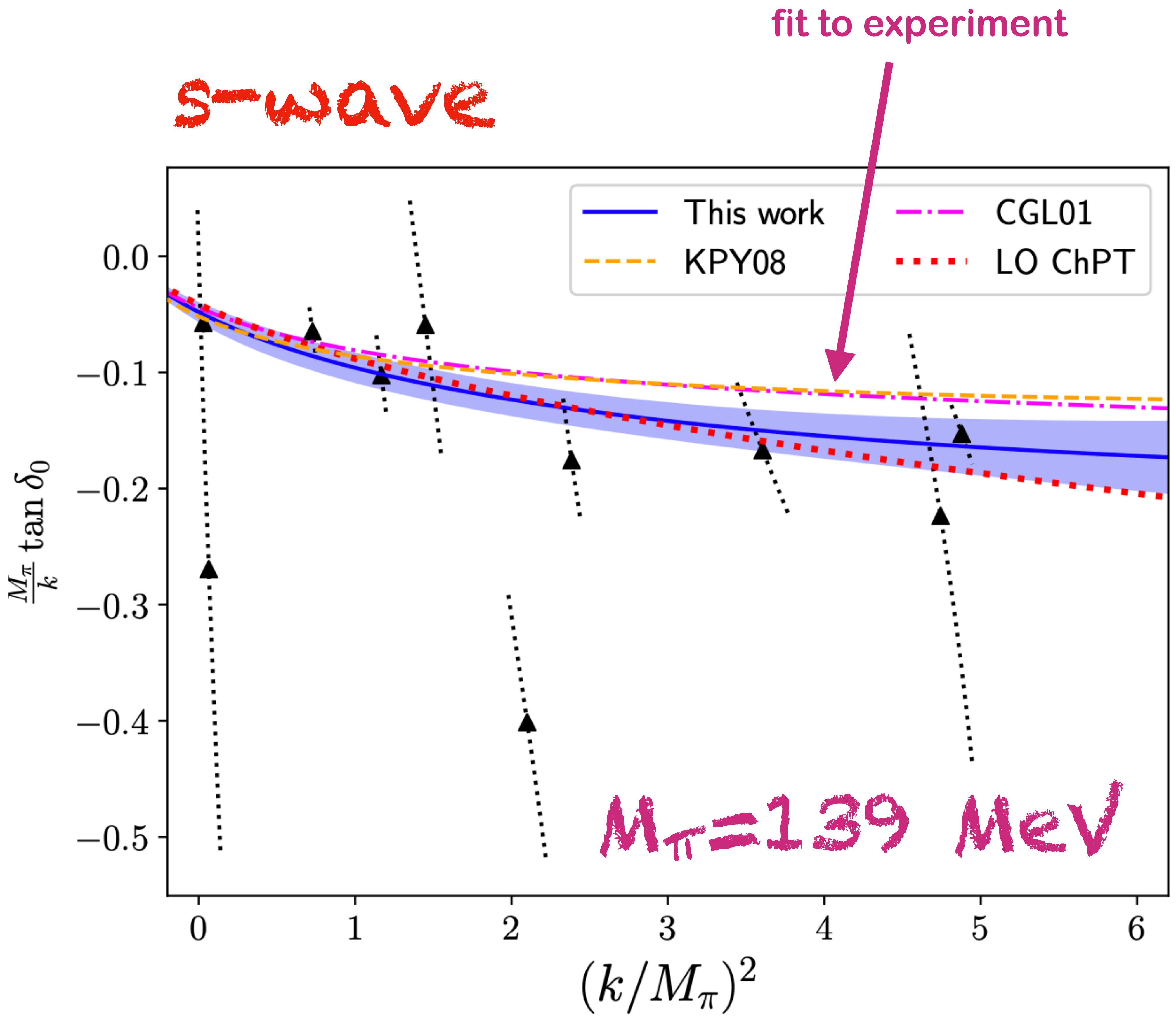

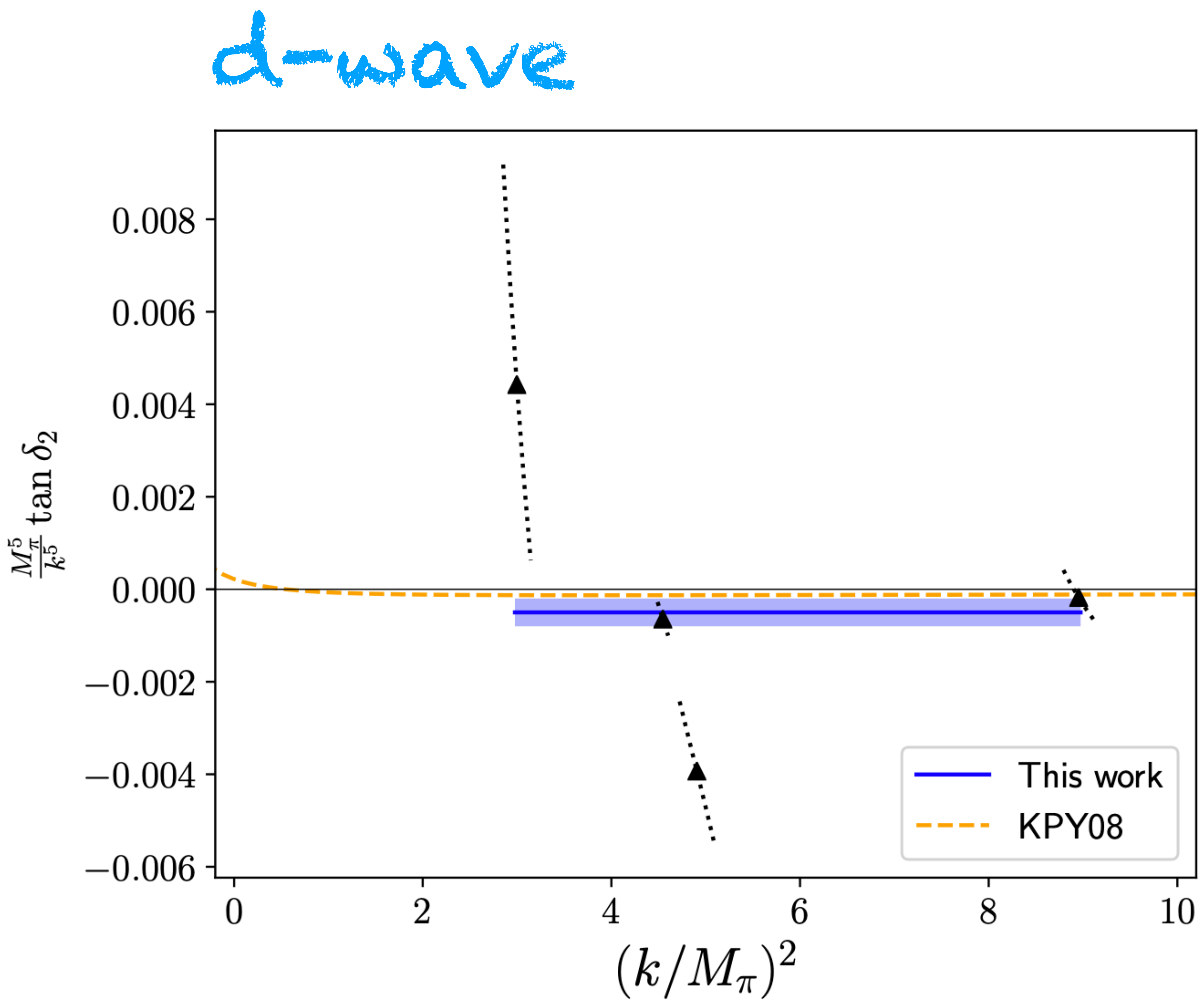

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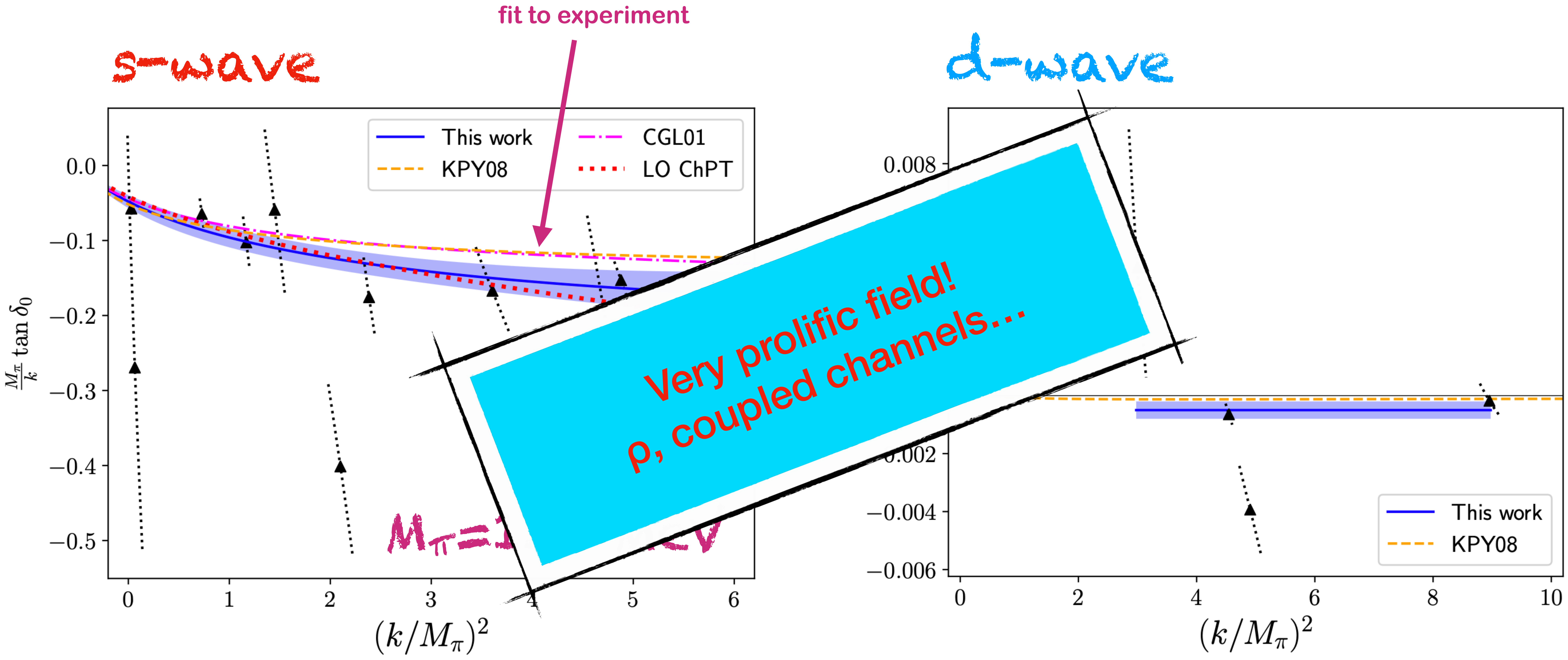

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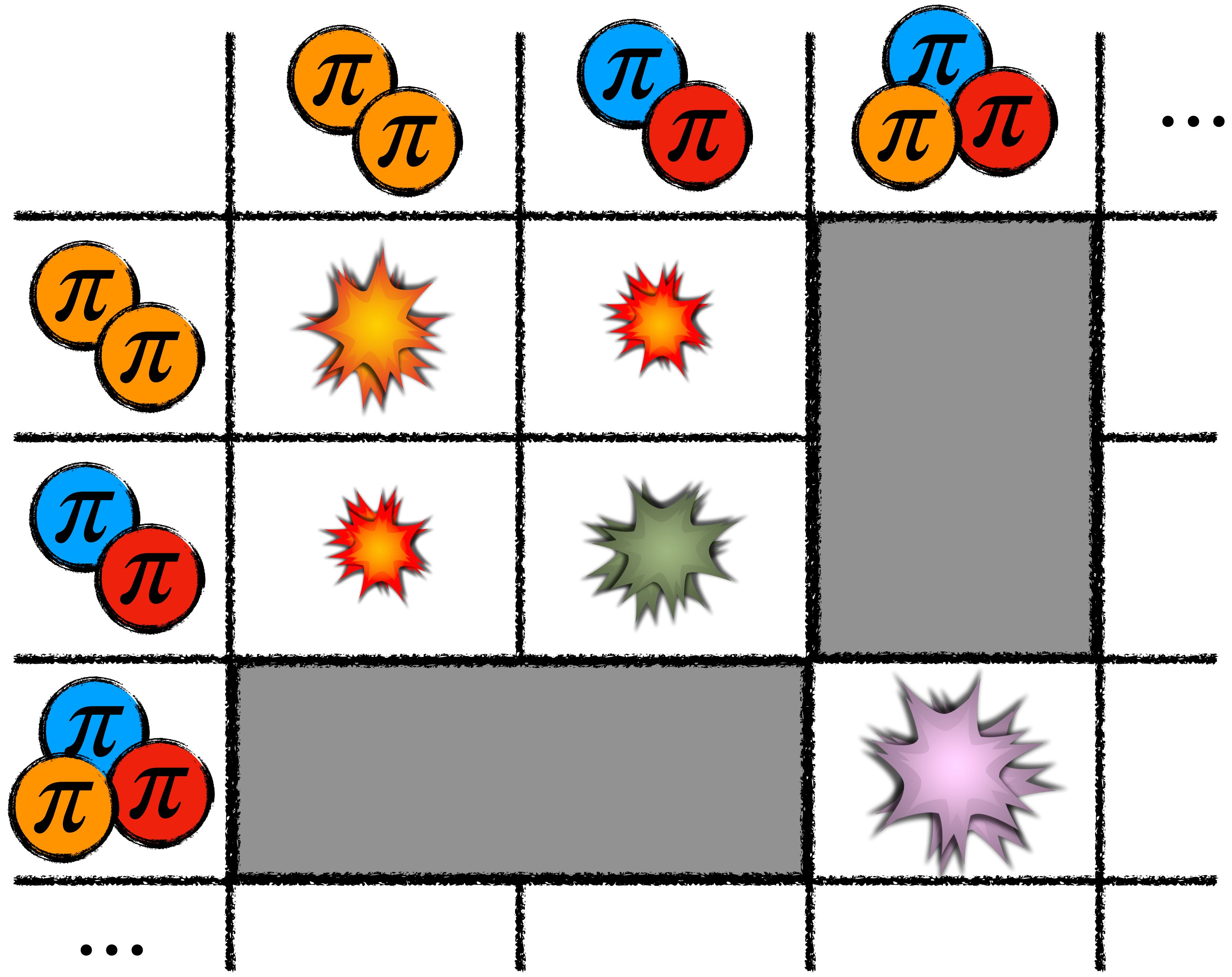
Three particles in finite volume



Why three particles?

- Most resonances have decay modes with more than two-particles

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$$N(1440) \to \Delta\pi \to N\pi\pi$$

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Resonance	$I_{\pi\pi\pi}$	J^P
$\omega(782)$	0	1^-
$h_1(1170)$	0	1^+
$\omega_3(1670)$	0	3^-
$\pi(1300)$	1	0^-
$a_1(1260)$	1	1^+
$\pi_1(1400)$	1	1^-
$\pi_2(1670)$	1	2^-
$a_2(1320)$	1	2^+
$a_4(1970)$	1	4^+

(with 3π decay modes)

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Resonance	$I_{\pi\pi\pi}$	J^P
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$h_1(1170)$	0	1^+
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$\pi(1300)$	1	0^-
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- A necessary step for four or more particles:
  - Many-body nuclear physics
  - CP violation in $D \to 4\pi/2\pi$

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Three-particle formalism

Relativistic, model-independent, three-particle quantization condition

Maxwell T. Hansen[1,*] and Stephen R. Sharpe[1,†]
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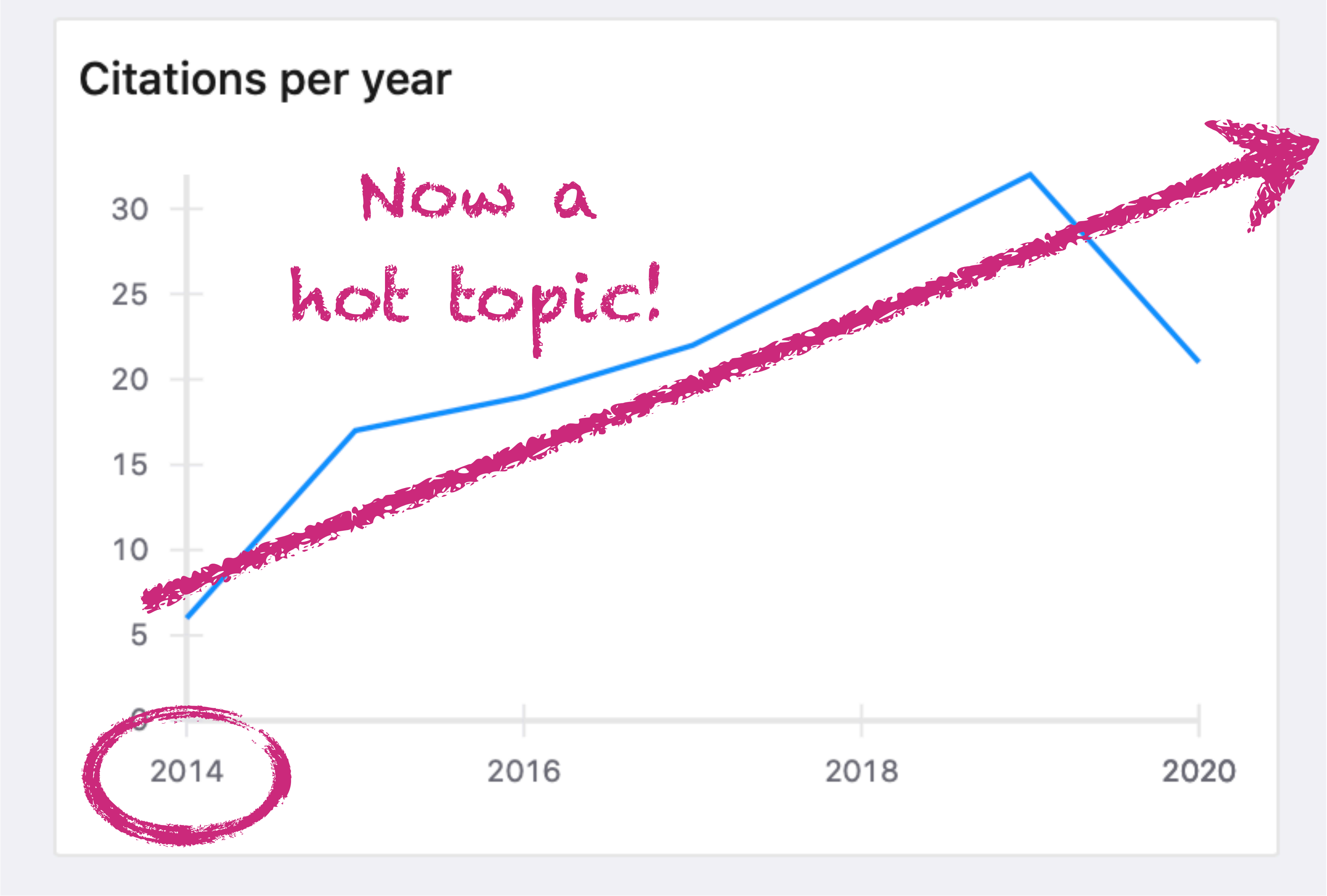

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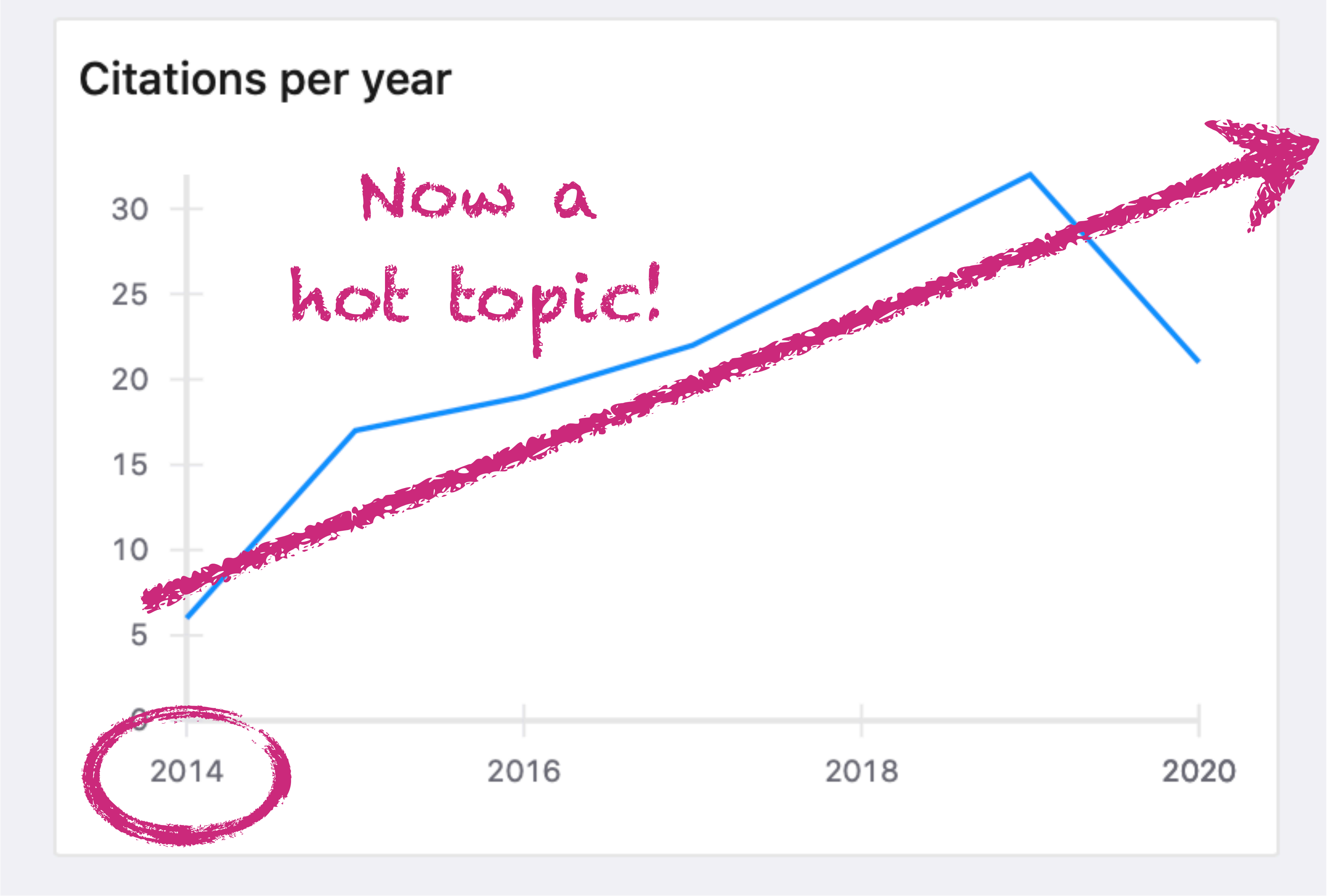

- Currently, three approaches to the formalism
  - Generic Relativistic Field Theory (RFT)
    - Hansen, Sharpe
    - Also: Blanton, Briceño, Jackura, FRL, Szczepaniak
  - Non-Relativistic EFT (NREFT)
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  - Finite-Volume Unitarity (FVU)
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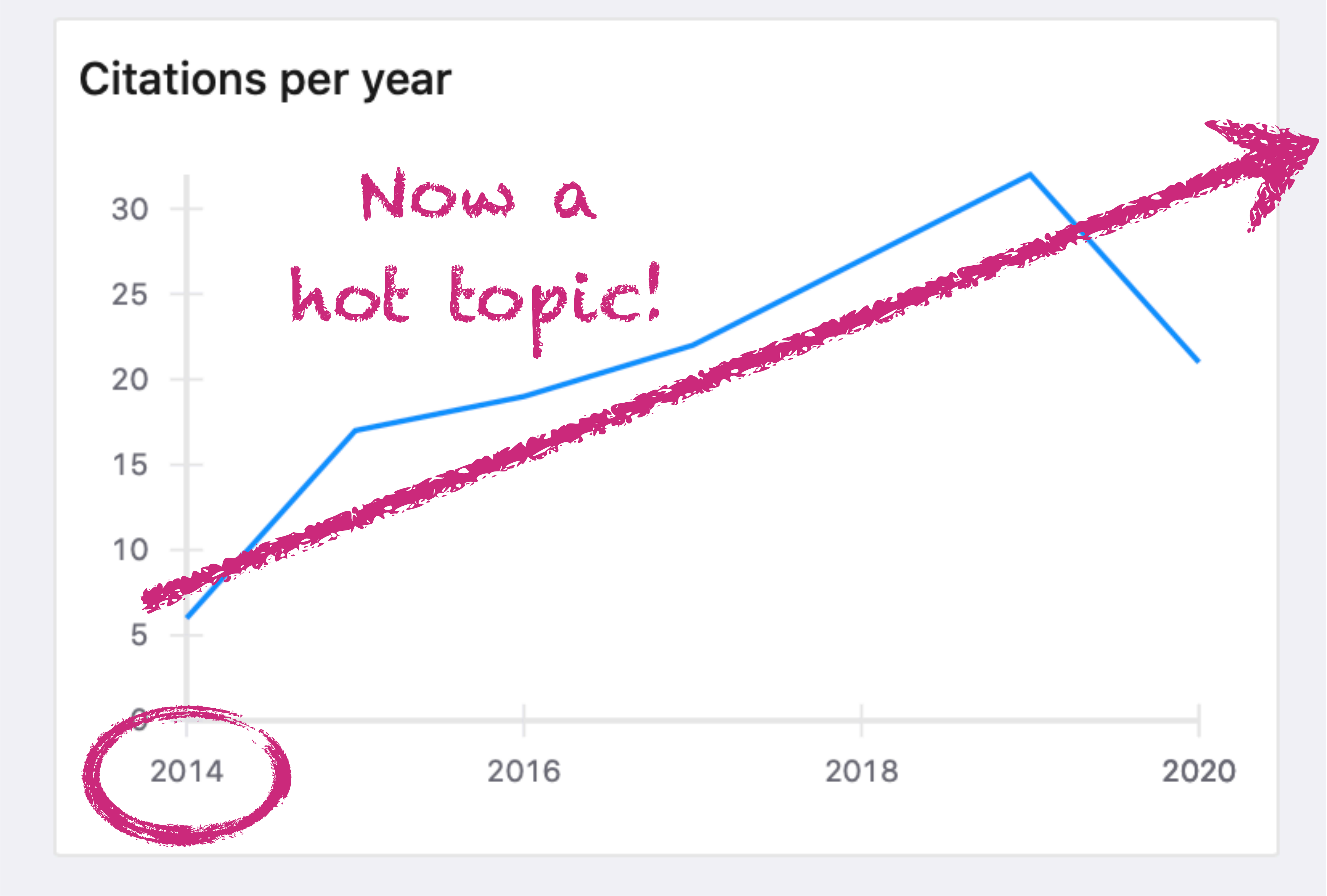

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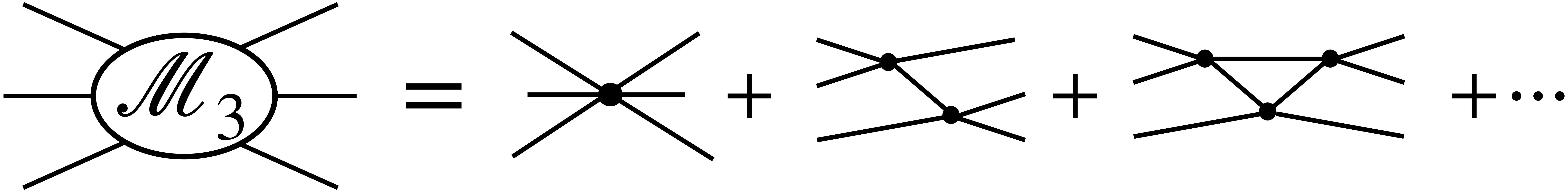
Three-particle amplitudes

Qualitatively more complicated than the two-particle case!



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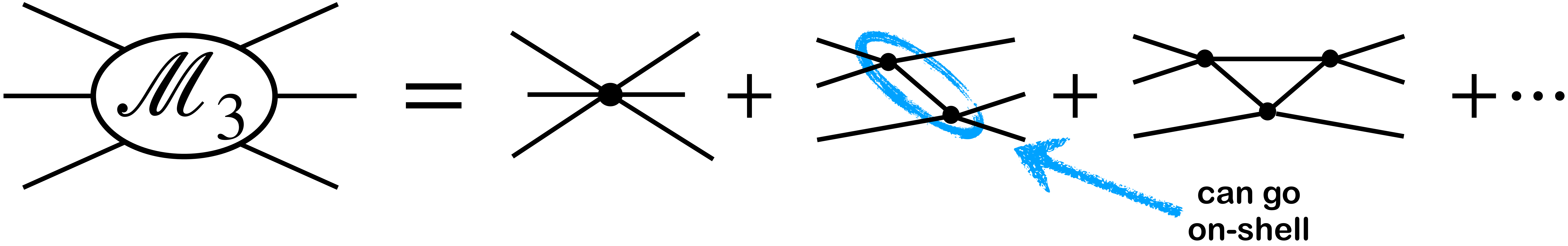

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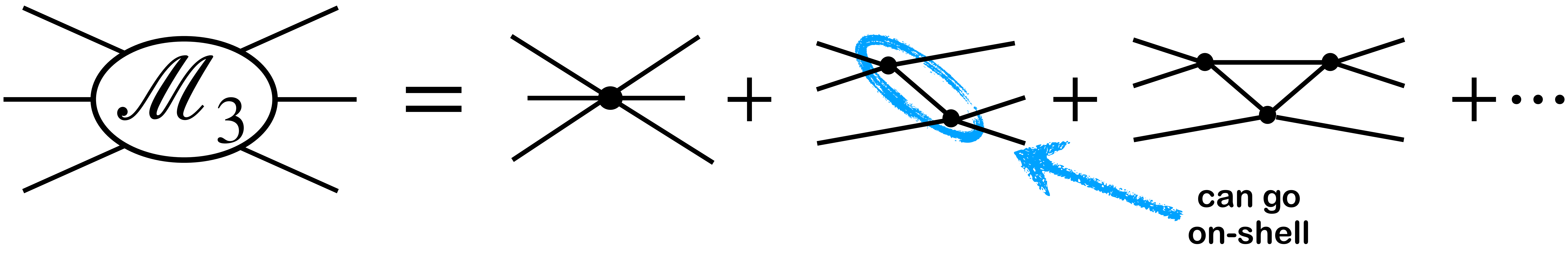
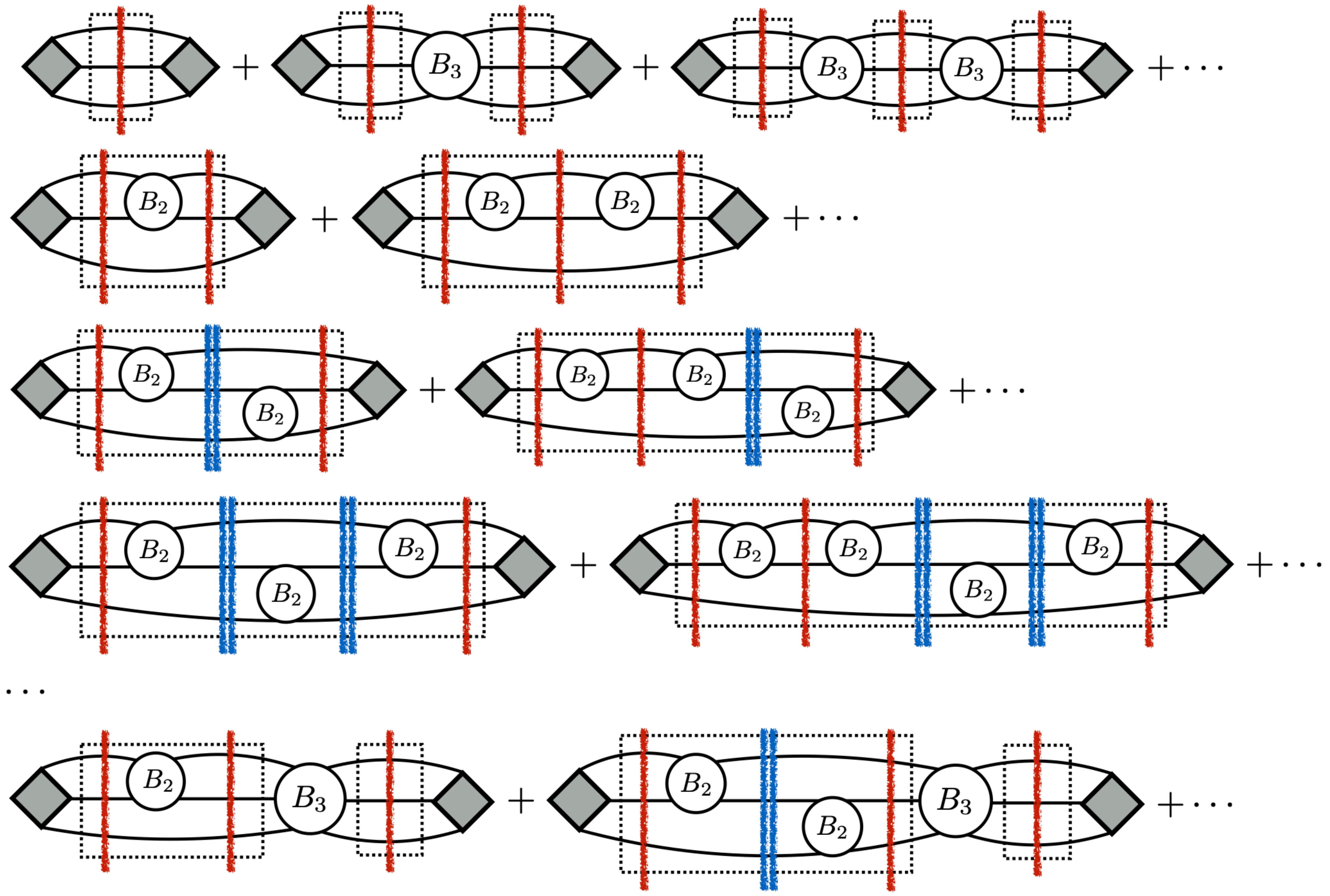


- Three-particle scattering amplitudes can be divergent for specific kinematics.
- They depend also on two-to-two interactions.
  - But any separation between “two-particle” and “three-particle” effects is not well-defined

Quantization Condition (I)

Skeleton expansion

$C_L =$



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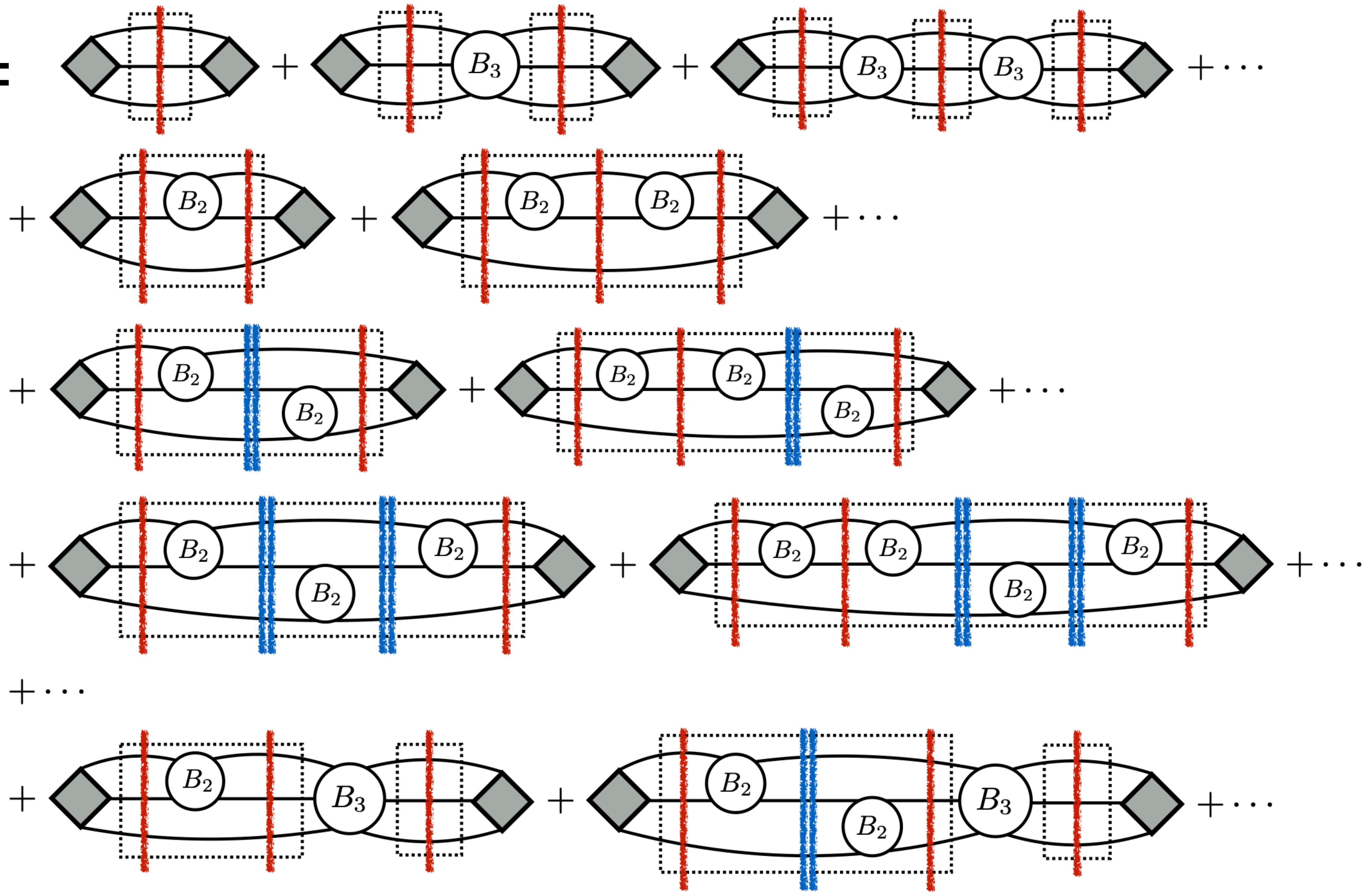

$$= C_\infty(P) + A_3 \frac{1}{\mathcal{K}_{df,3} + F_3^{-1}} A_3' + O(e^{-mL})$$

Easier derivation: Blanton, Sharpe [2007.16188]

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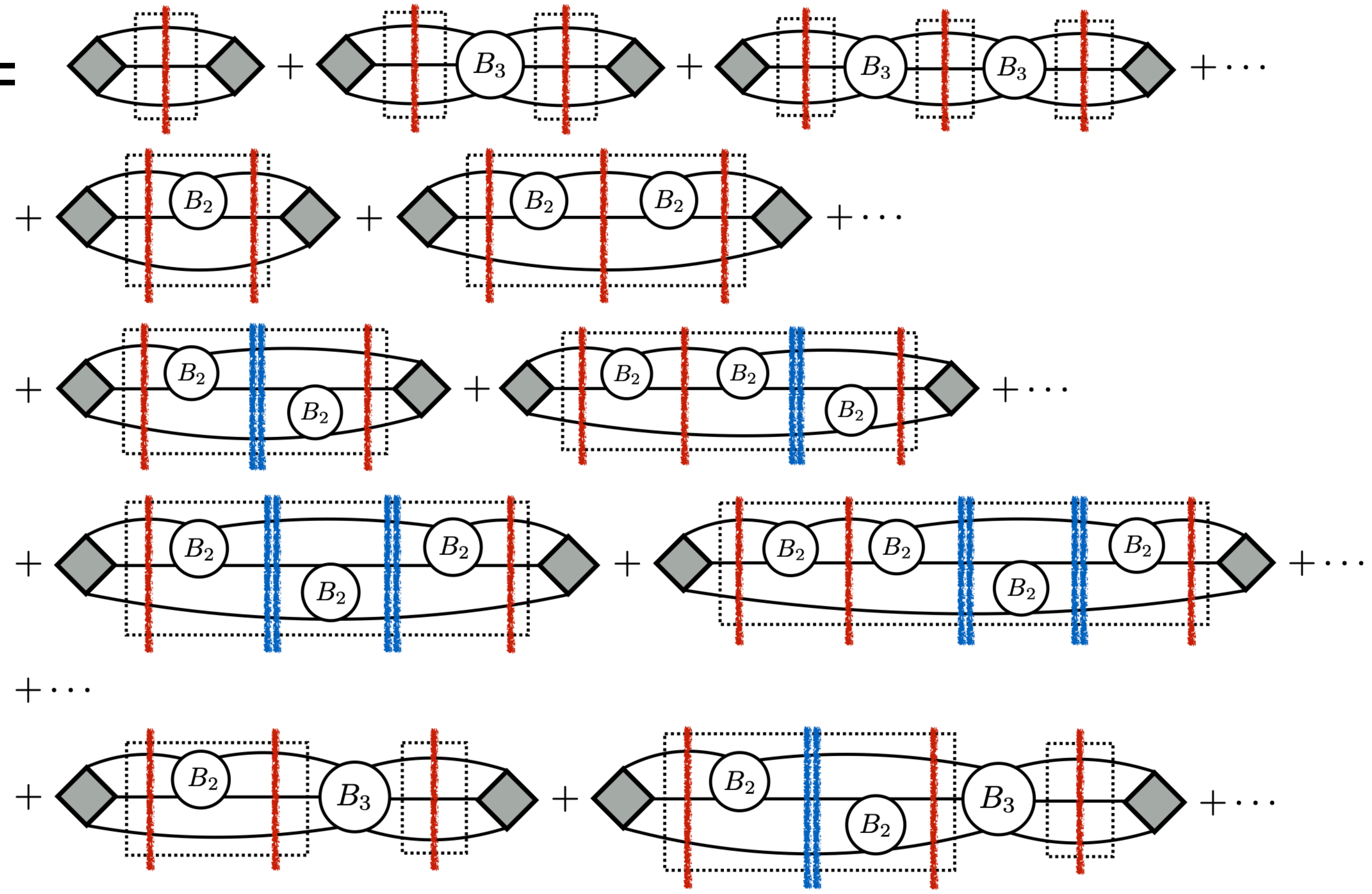

Separation of finite and infinite volume terms:

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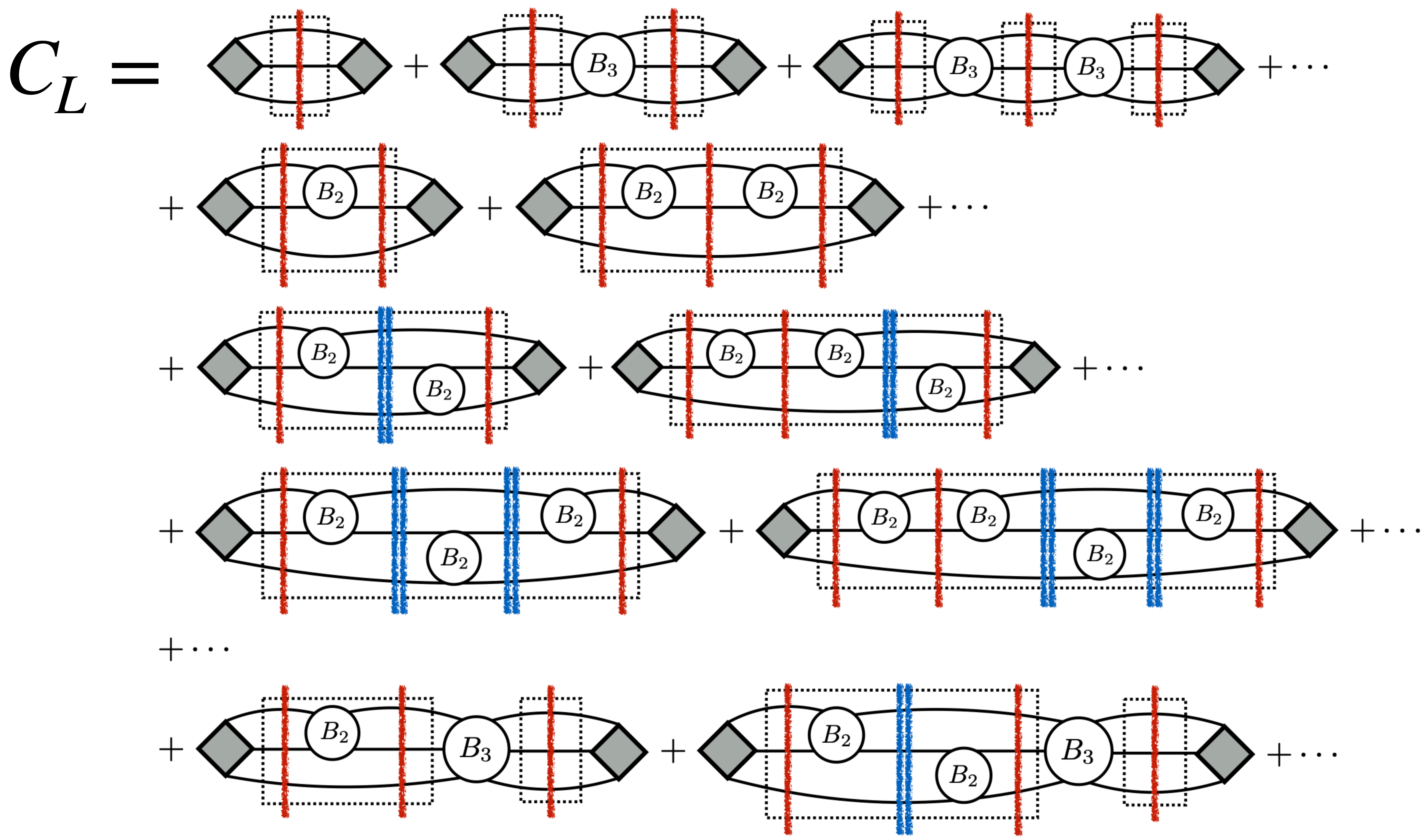

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$$\det\left[\mathcal{K}_{df,3}(E) + F_3^{-1}(E, \vec{P}, L)\right] = 0$$

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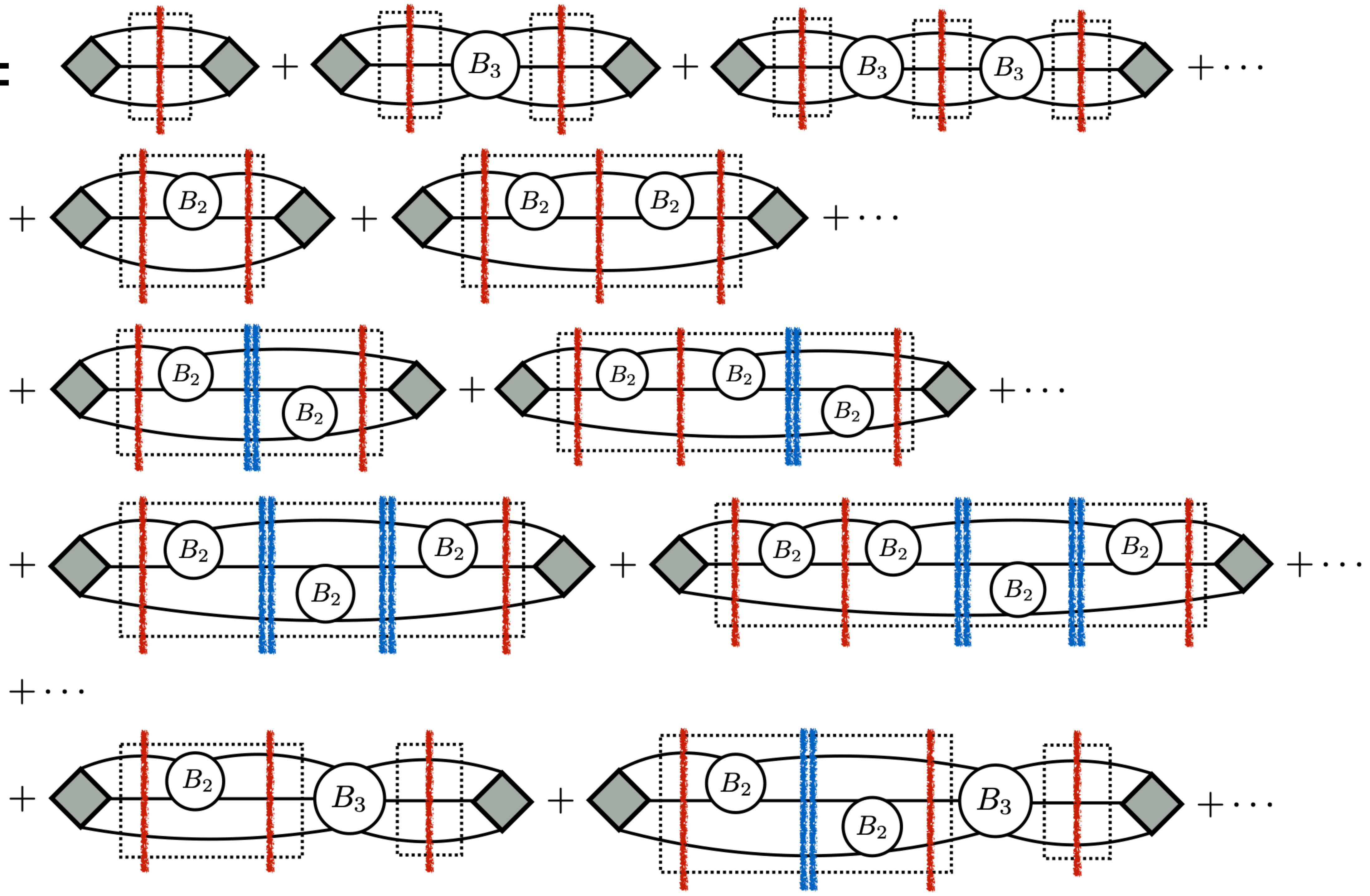

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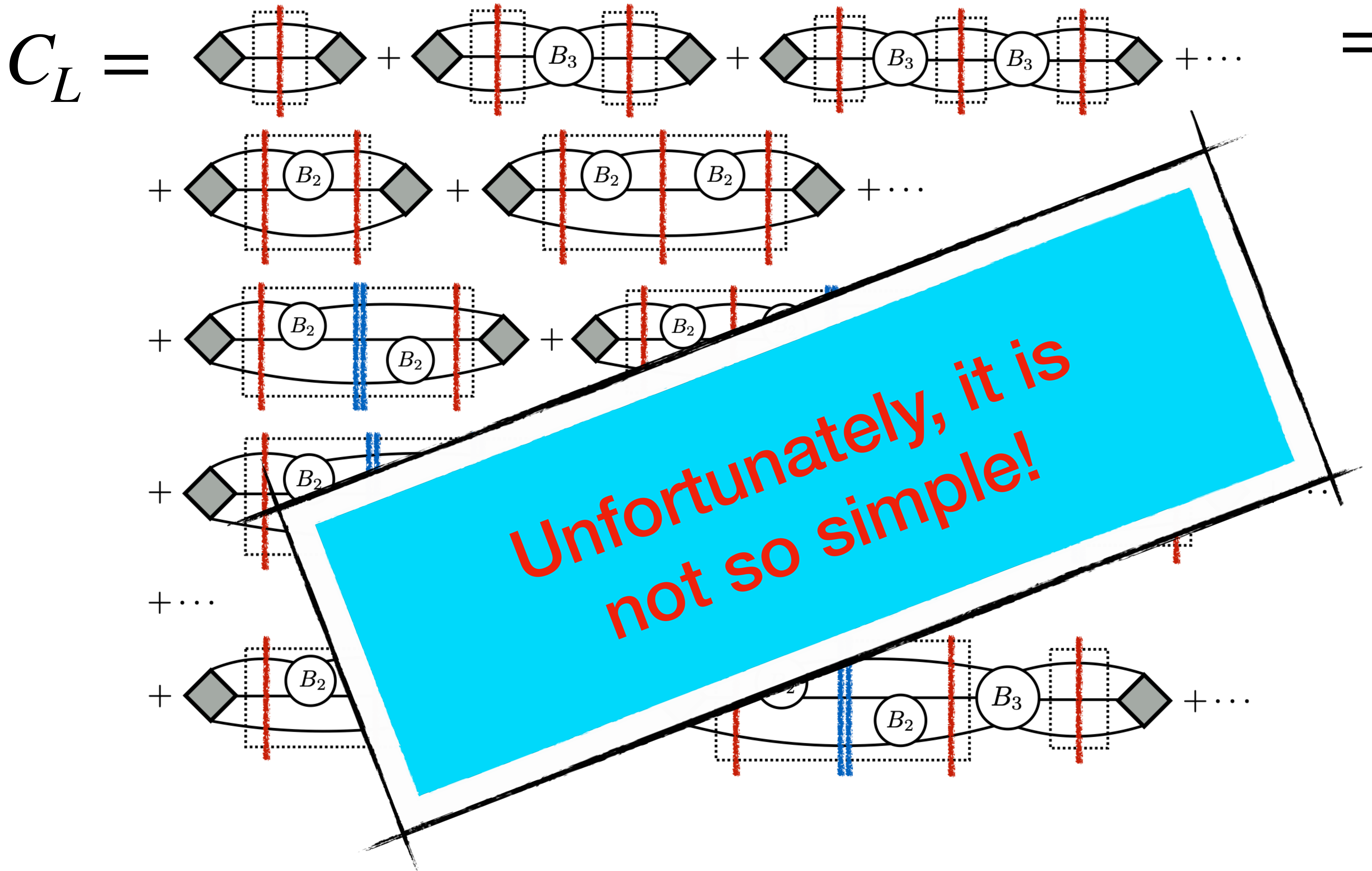

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Recovering the physical amplitude requires a further step

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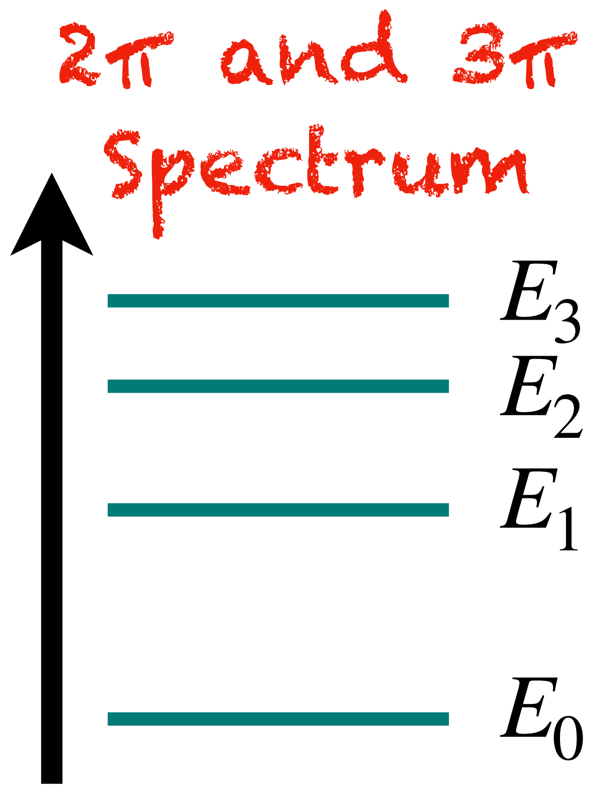
A two-step process

1. **Determine $\mathcal{K}_2$ and $\mathcal{K}_{df,3}$ from the two and three-pion spectrum**

Hansen, Sharpe [arXiv:1408.5933]

A two-step process



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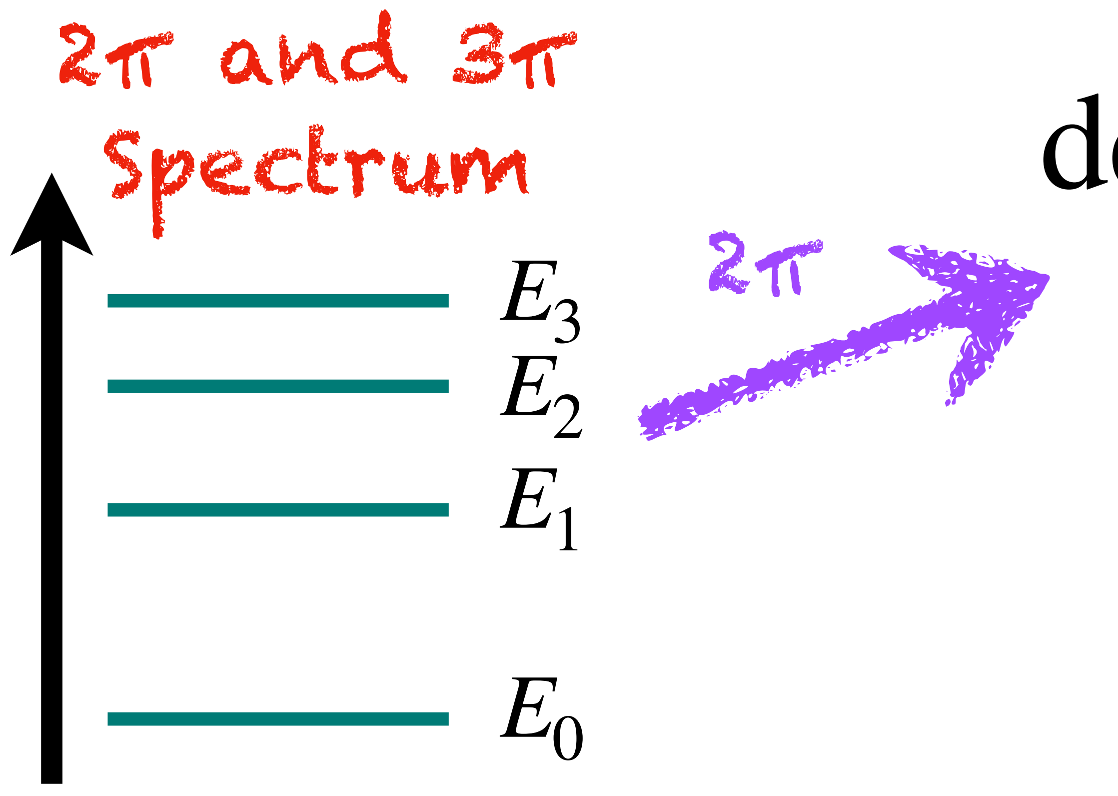

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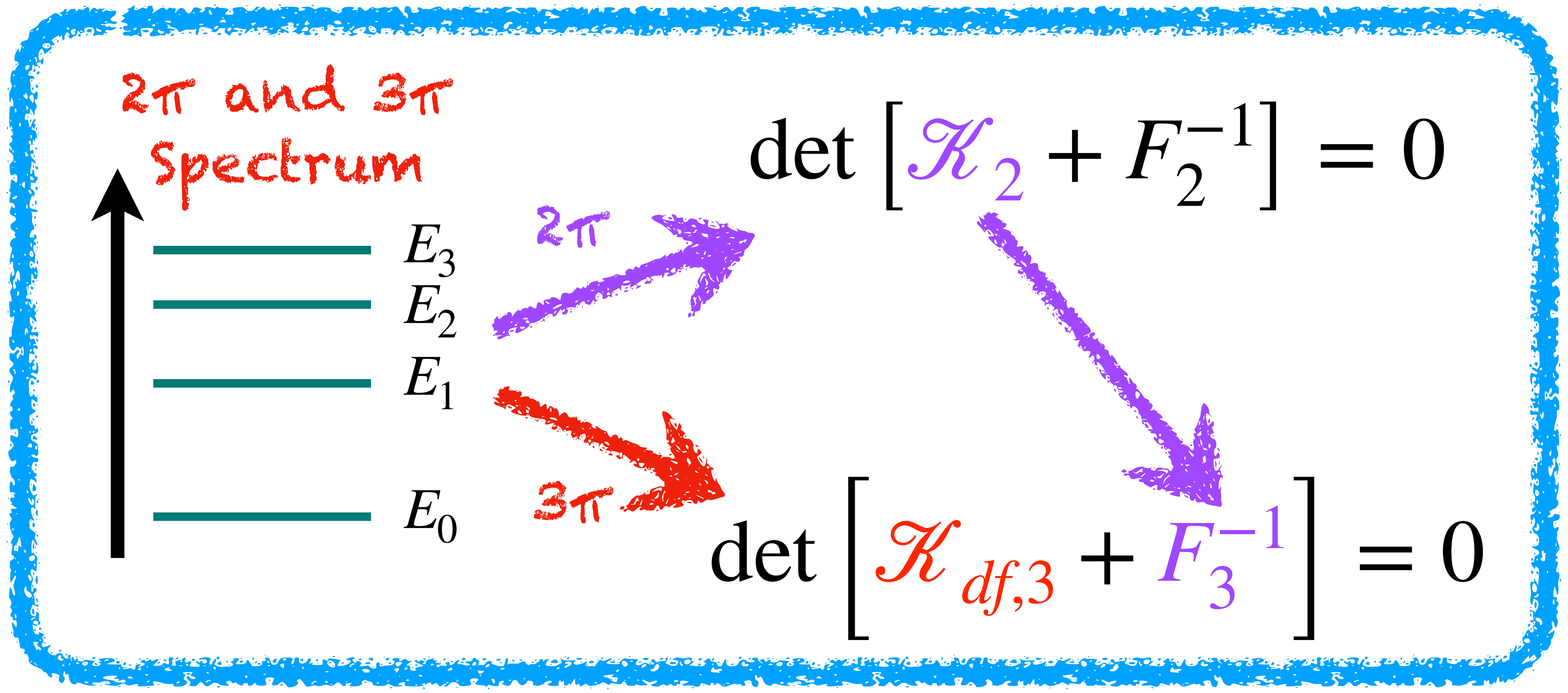

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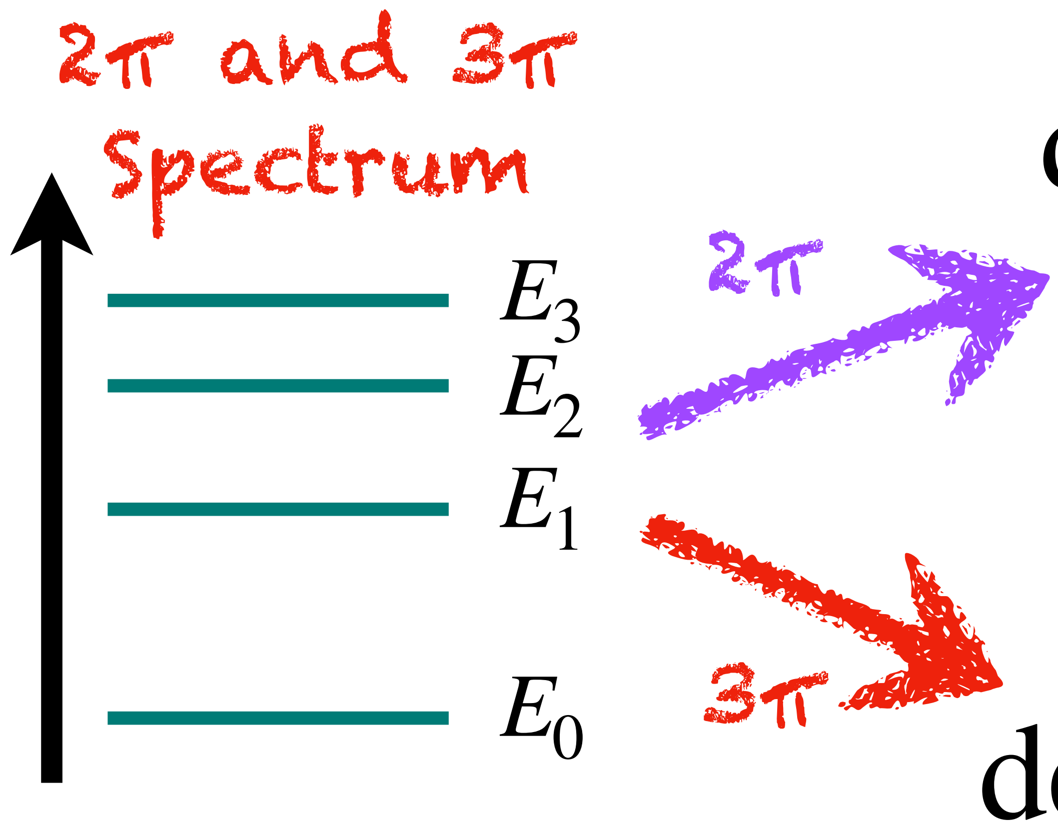

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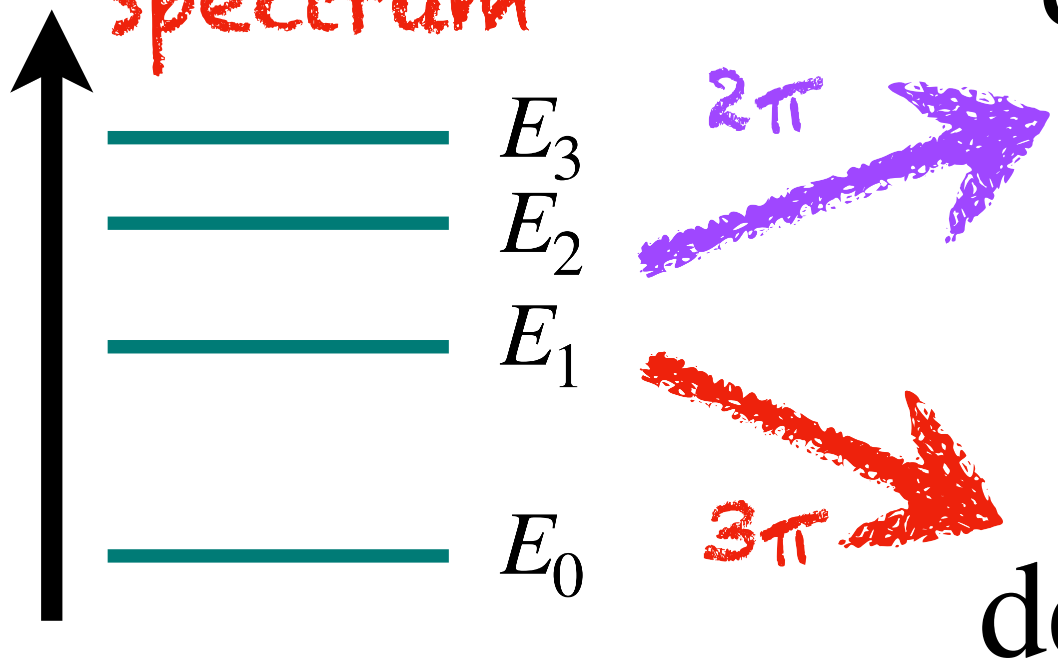
A two-step process

2π and 3π Spectrum



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This formalism provides a unitary parametrization of scattering amplitudes! (independent of lattice QCD)

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Integral equations

Three-pion scattering from the lattice

two & three-pion spectrum

Two- and three-pion finite-volume spectra at maximal isospin from lattice QCD

Ben Hörz*
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(Dated: October 8, 2019)

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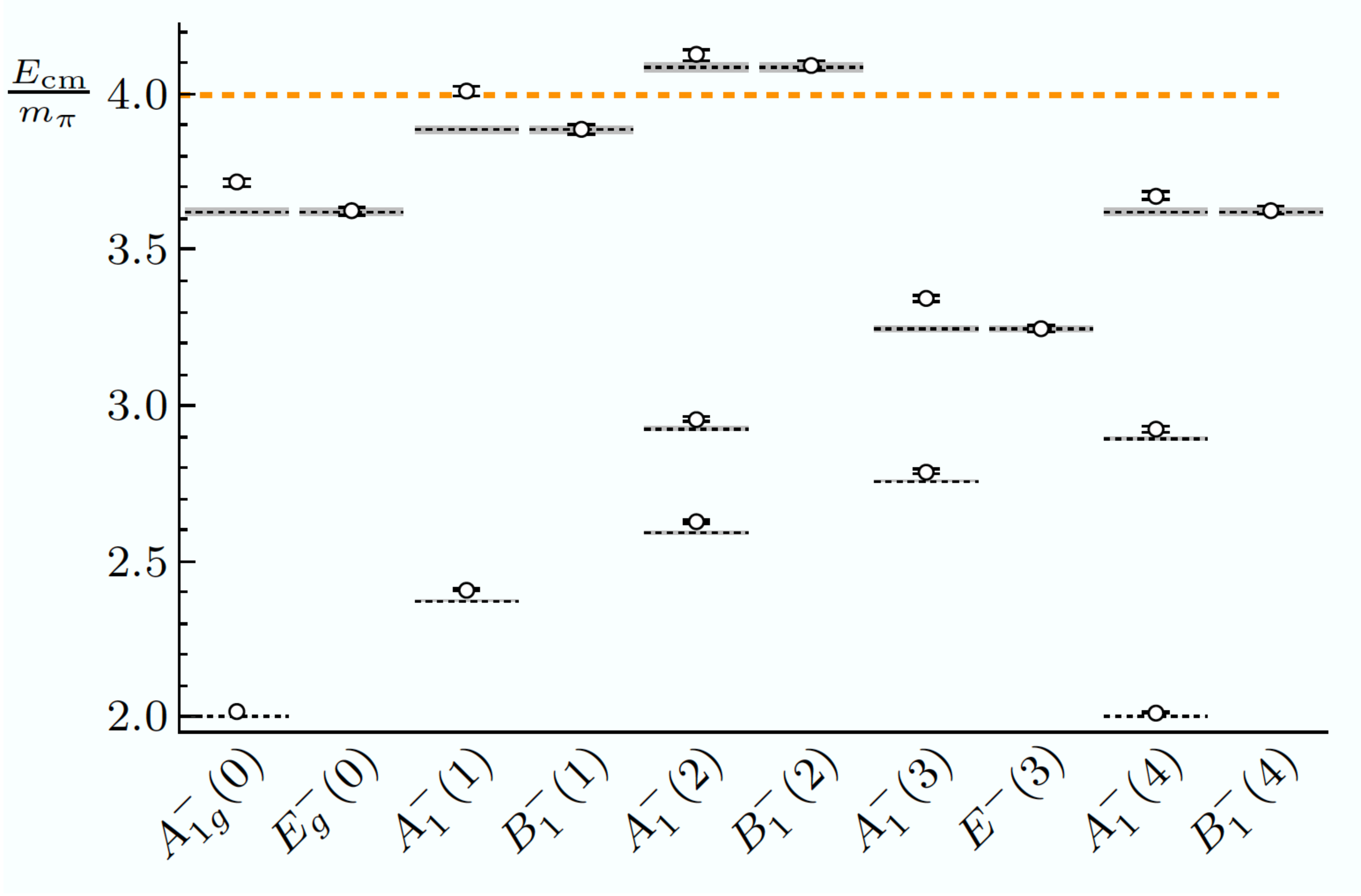

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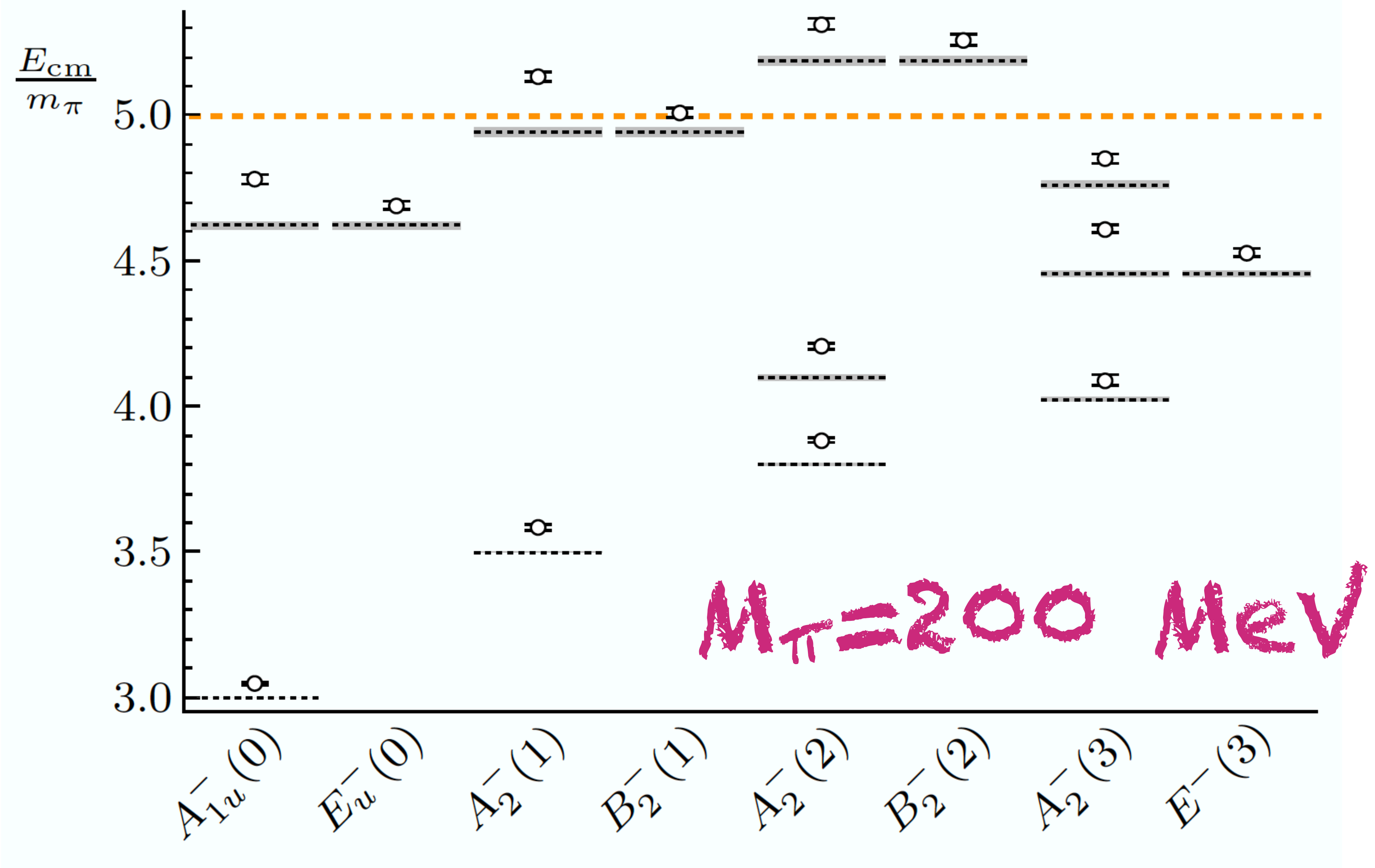

Analyzing the spectrum

$I = 3$ three-pion scattering amplitude from lattice QCD

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- **First analysis of the full finite-volume spectrum of $2\pi^+$ and $3\pi^+$**
- **Include only s-wave interactions**
- **Compare to Chiral Perturbation Theory**

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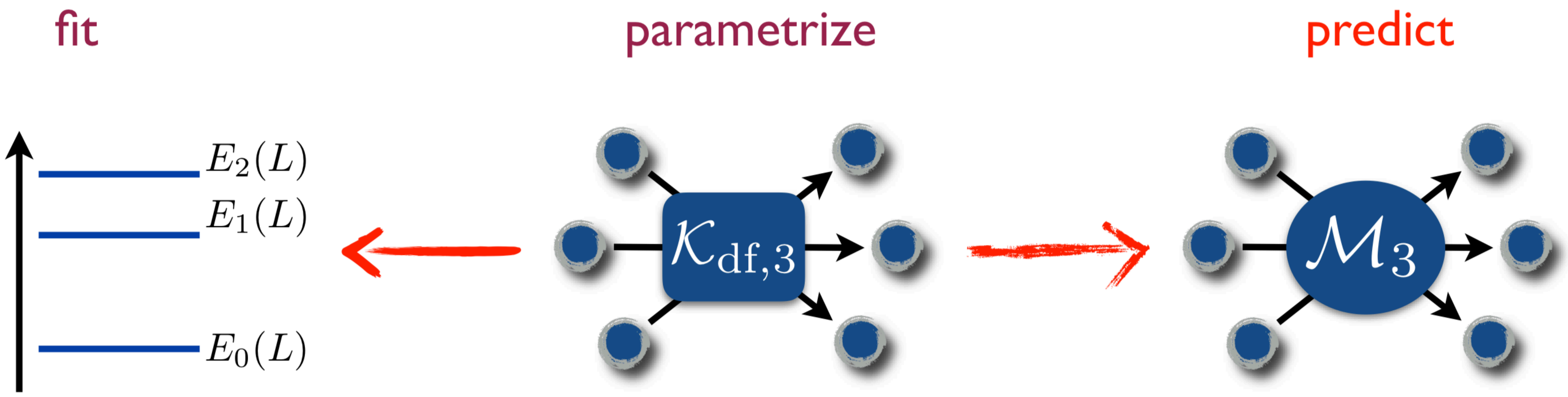

Fit results: two-pion sector

- **Model 1: standard Effective Range Expansion (ERE)**

$$\frac{q}{M}\cot\delta_0 = -\frac{1}{Ma_0} + \frac{1}{2}Mr_0\frac{q^2}{M^2} + \cdots$$

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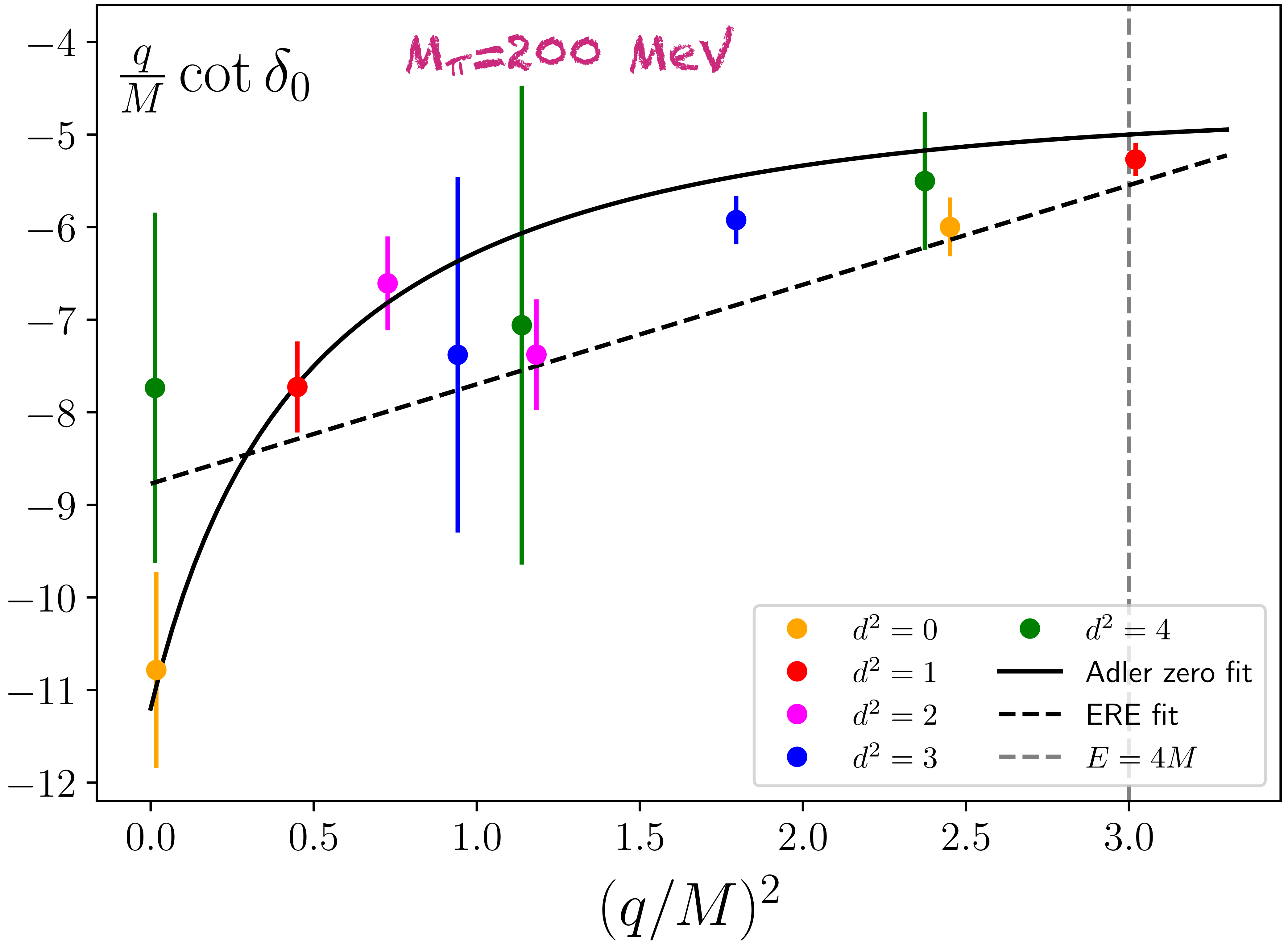

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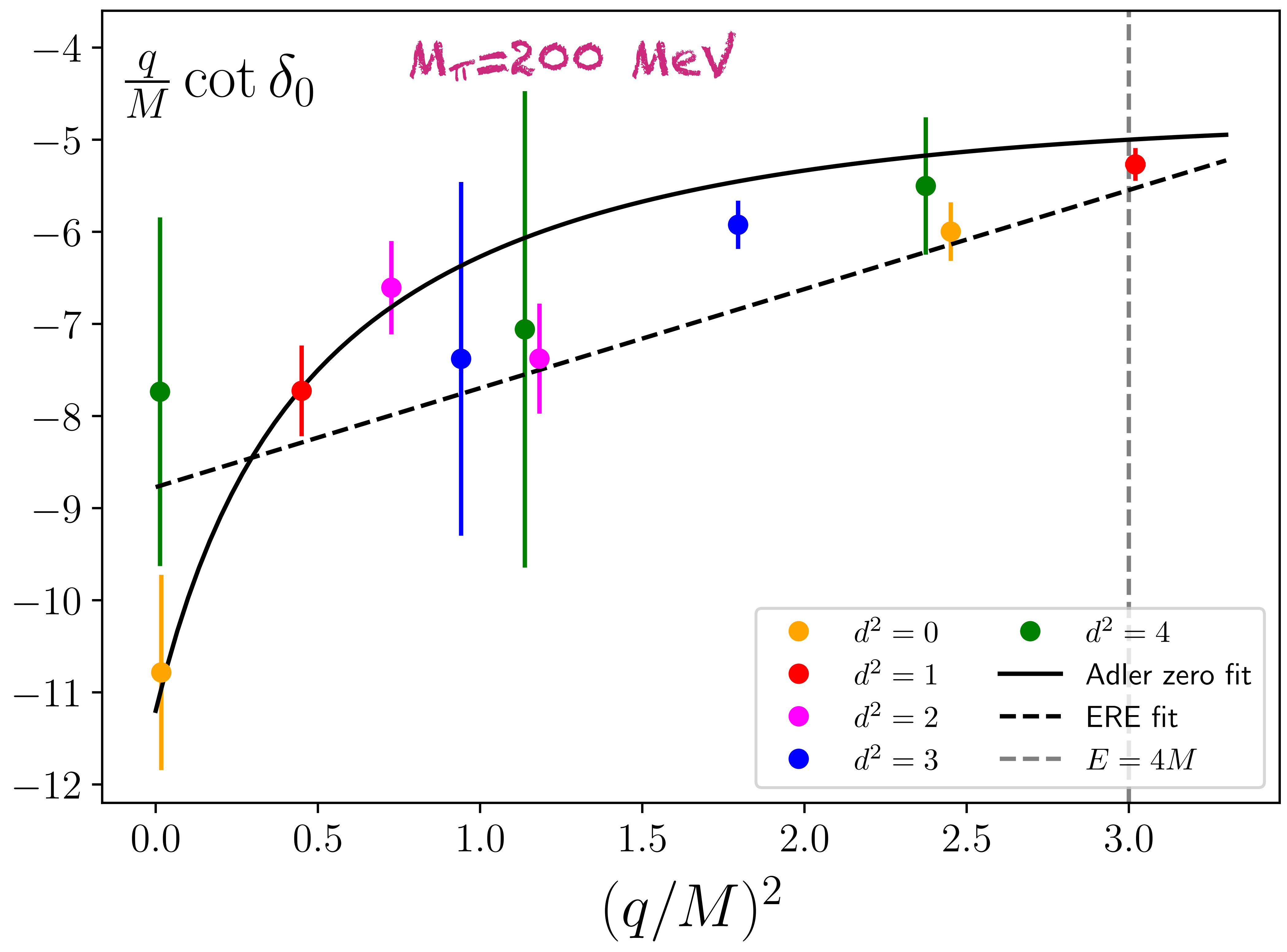

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$$\chi^2_{Adler}/dof = 1.3 \ll \chi^2_{ERE}/dof = 3.3$$

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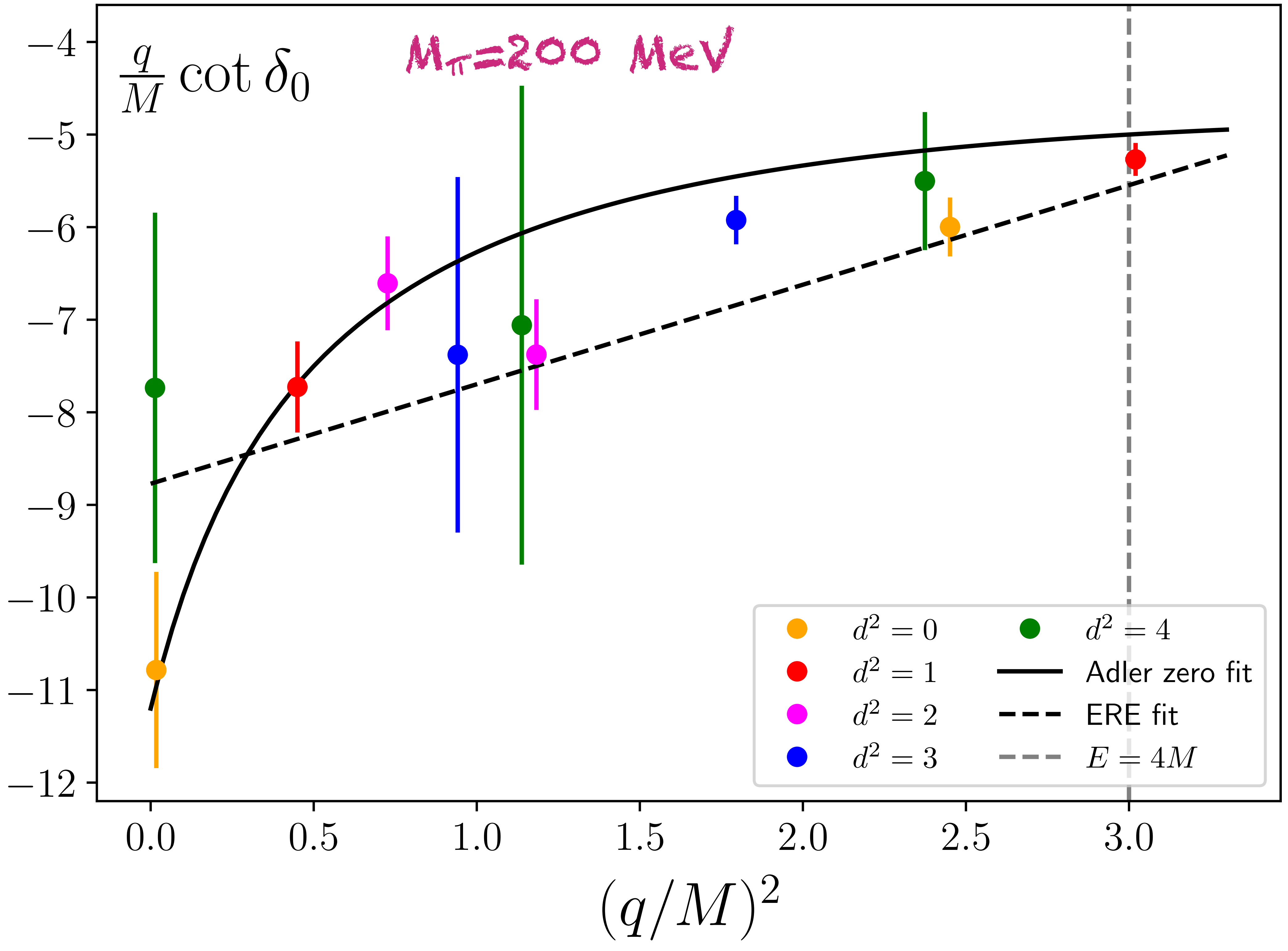

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$$Ma_0 = 0.0938(12)$$
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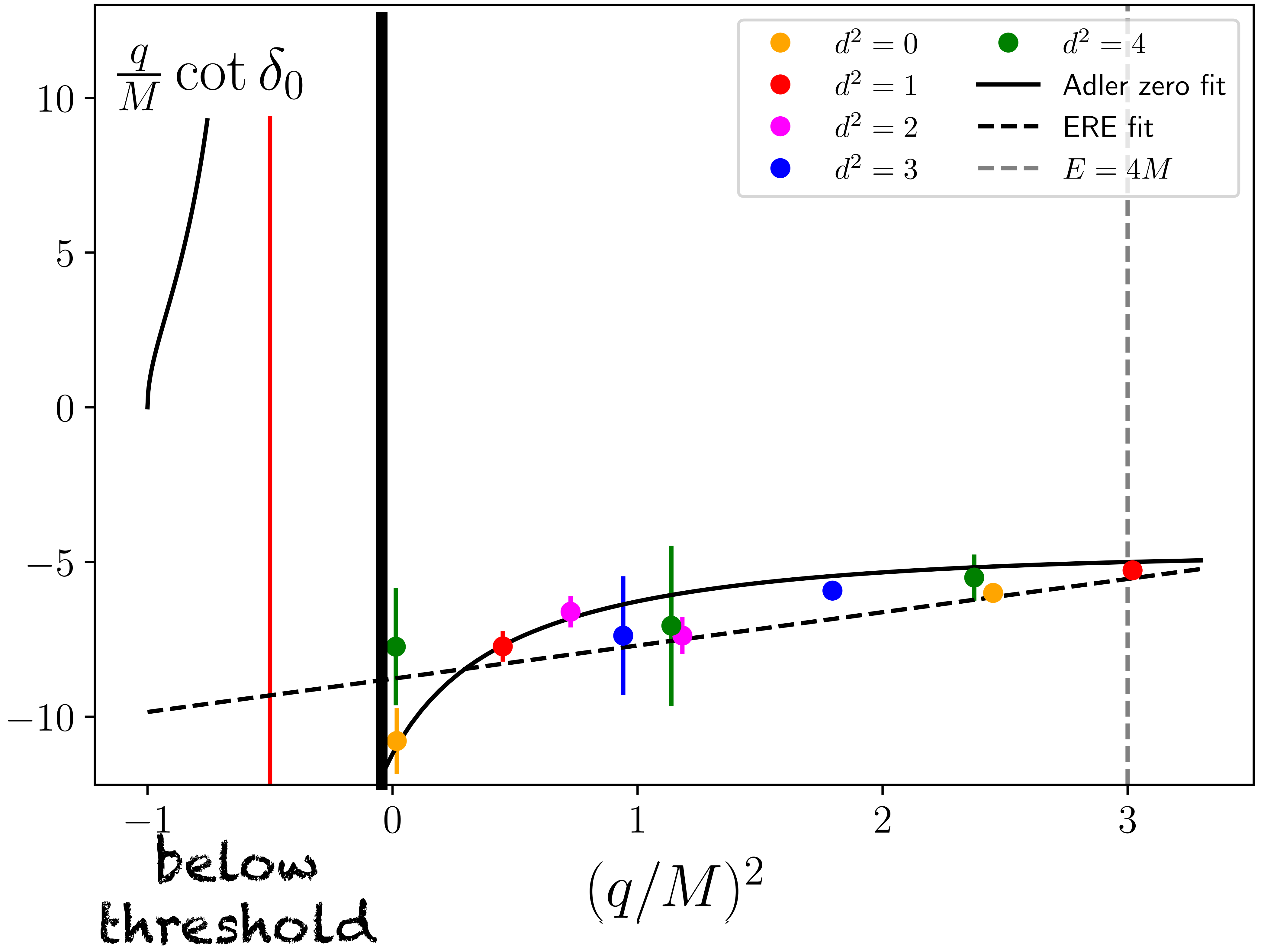

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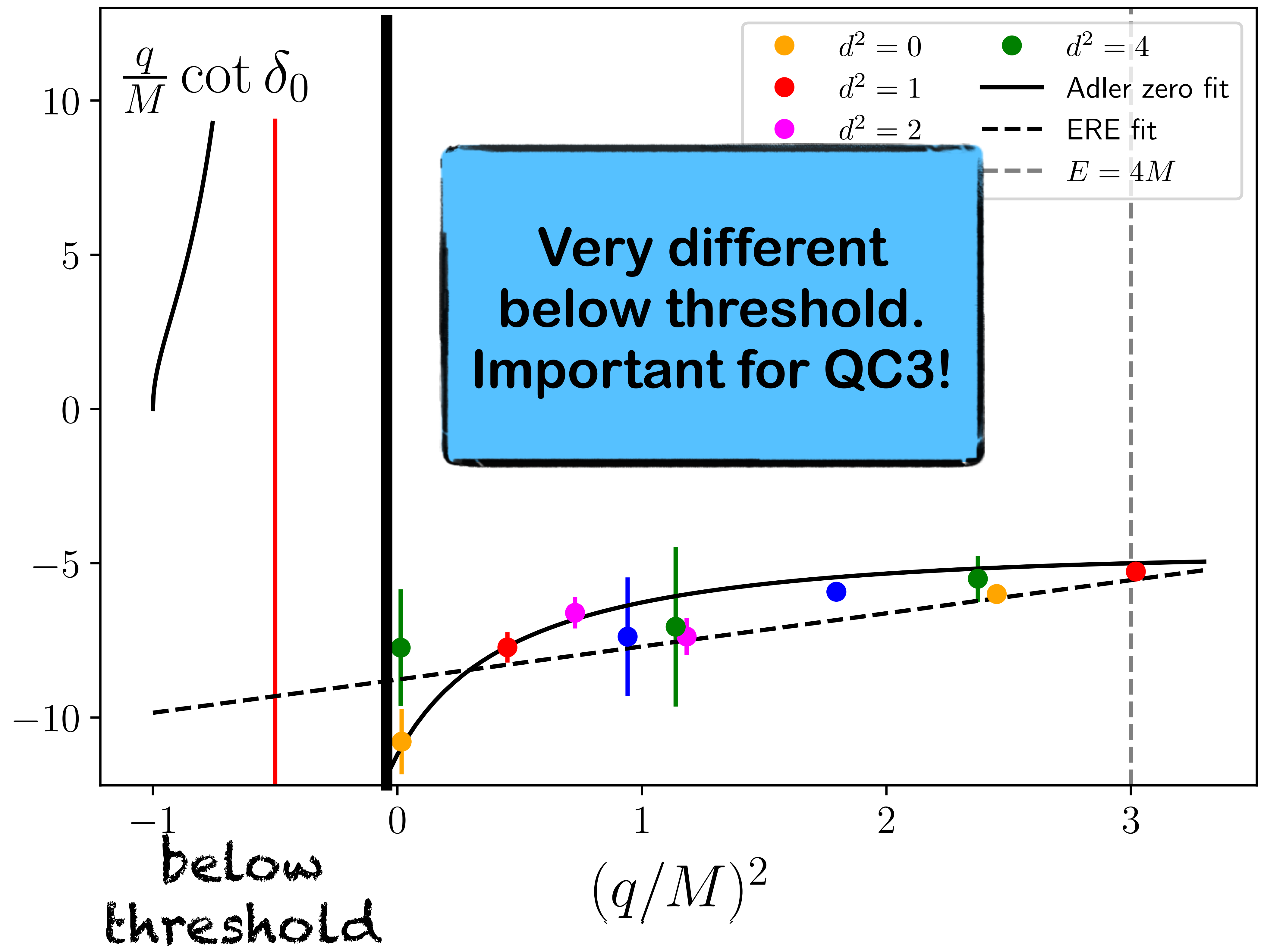

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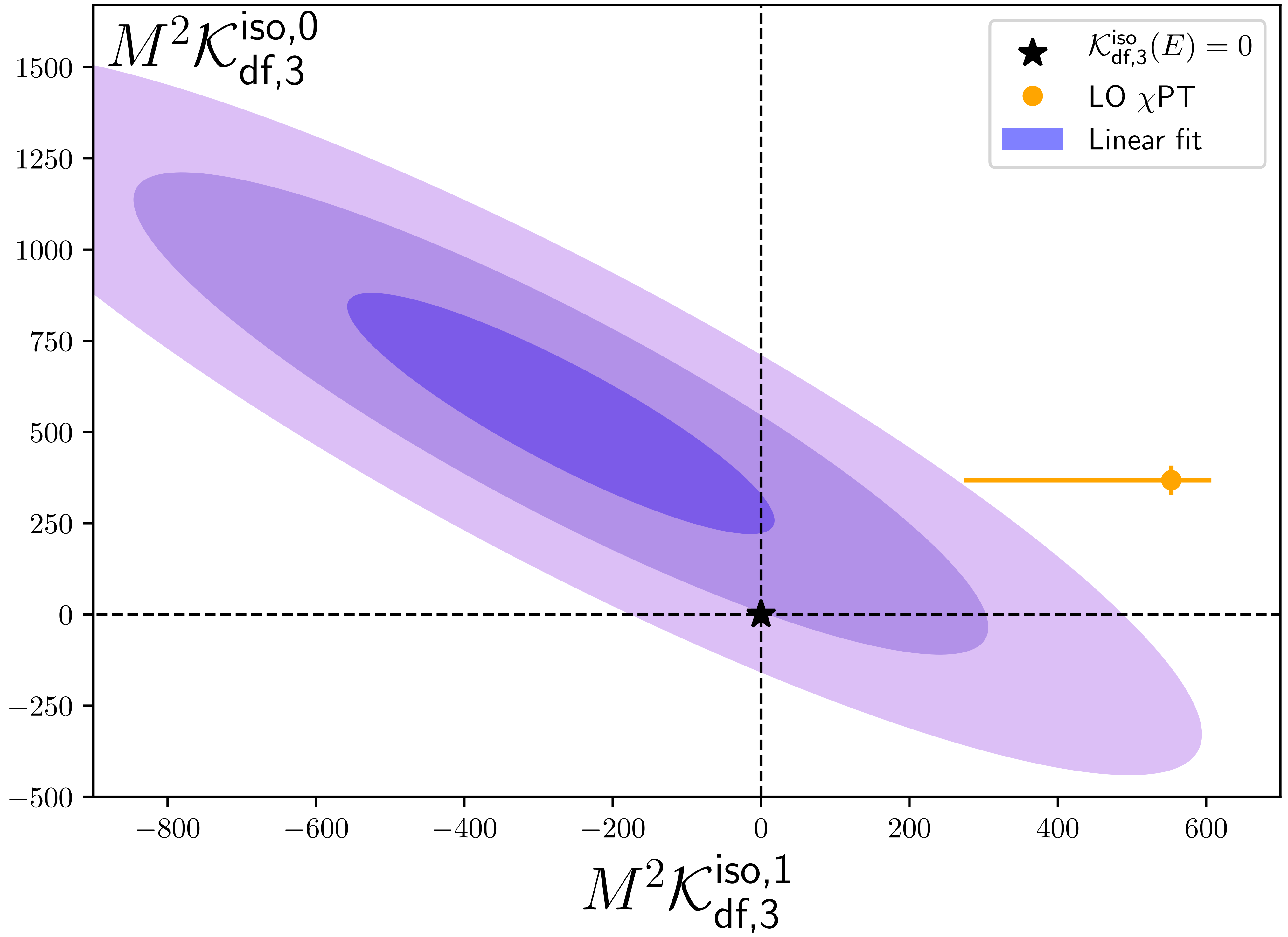

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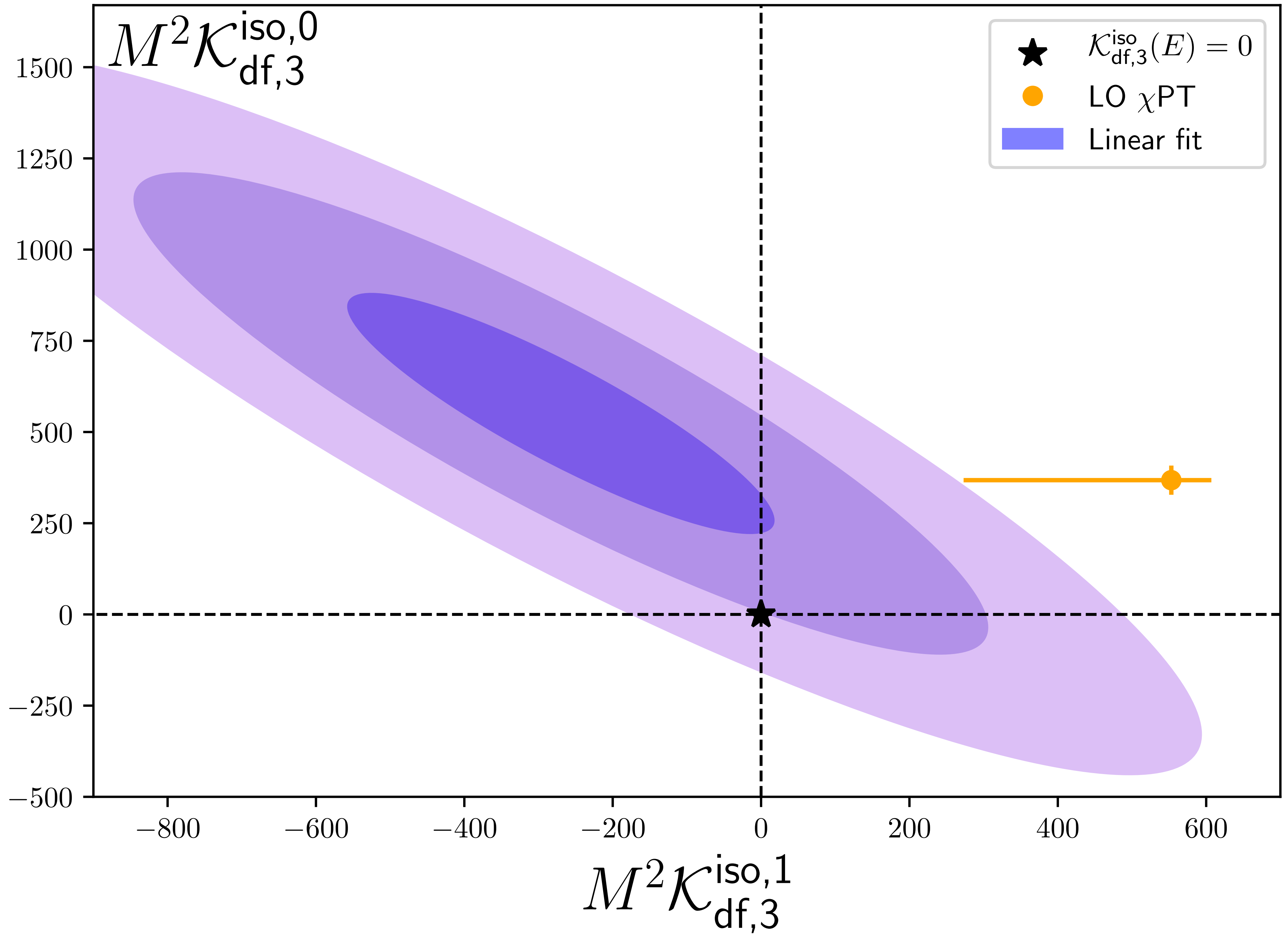

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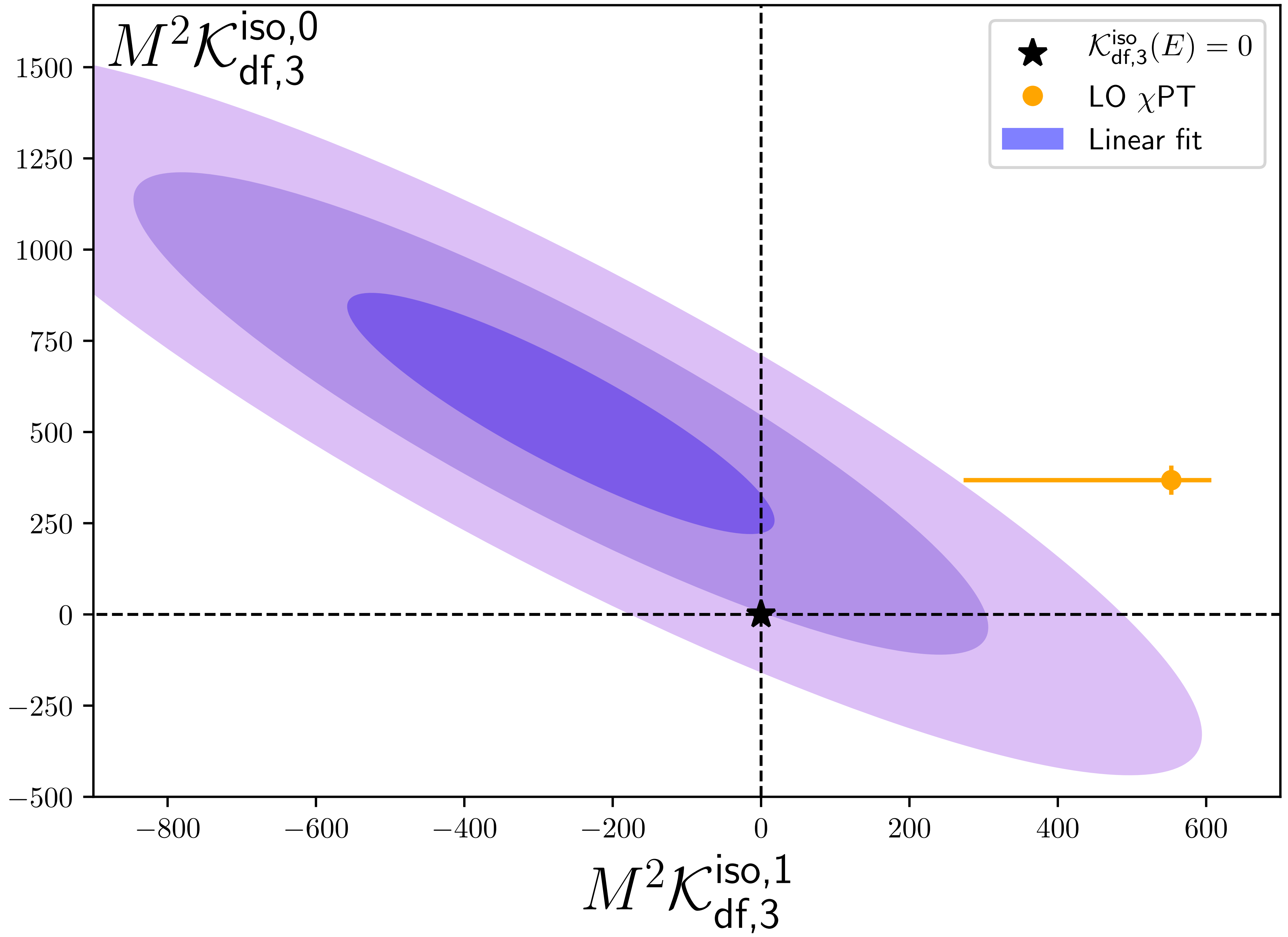

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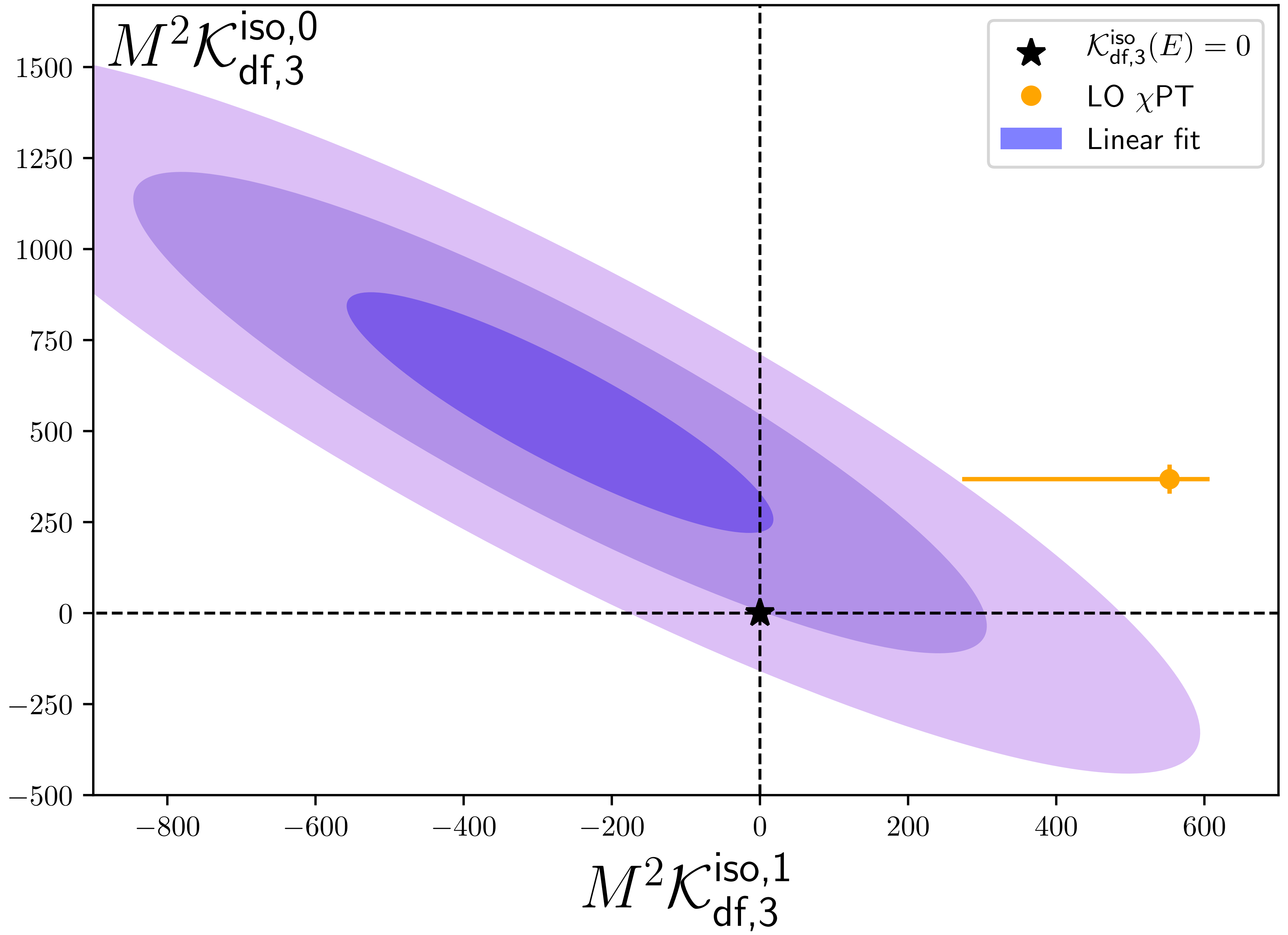

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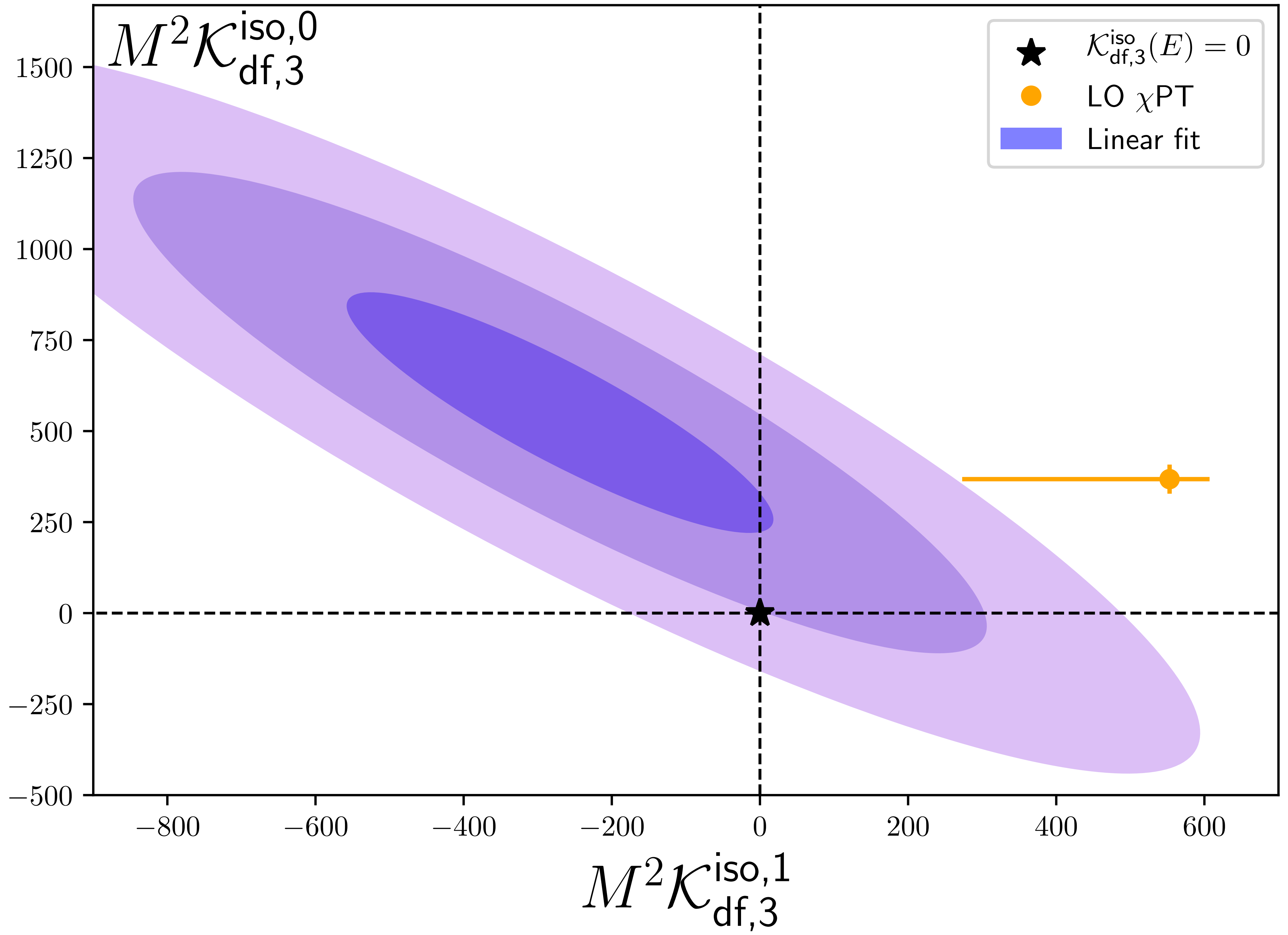

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$\mathcal{K}_{df,3} \longrightarrow \mathcal{M}_3$

Trivial only to leading order in ChPT

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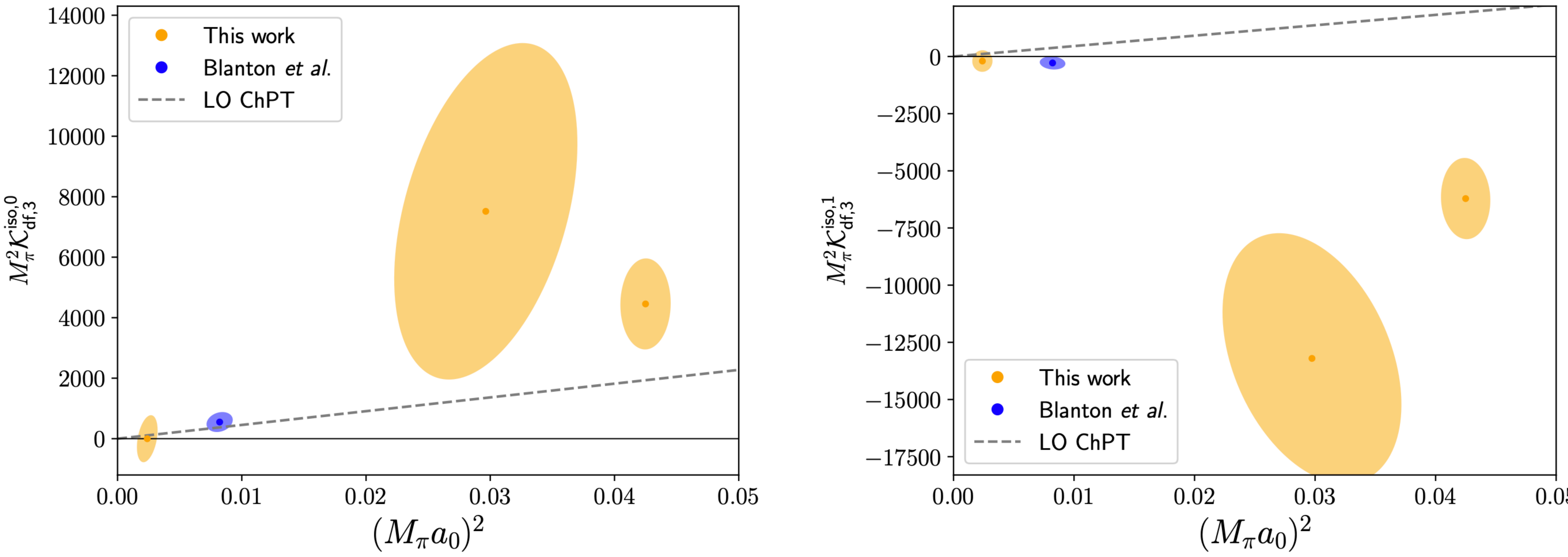

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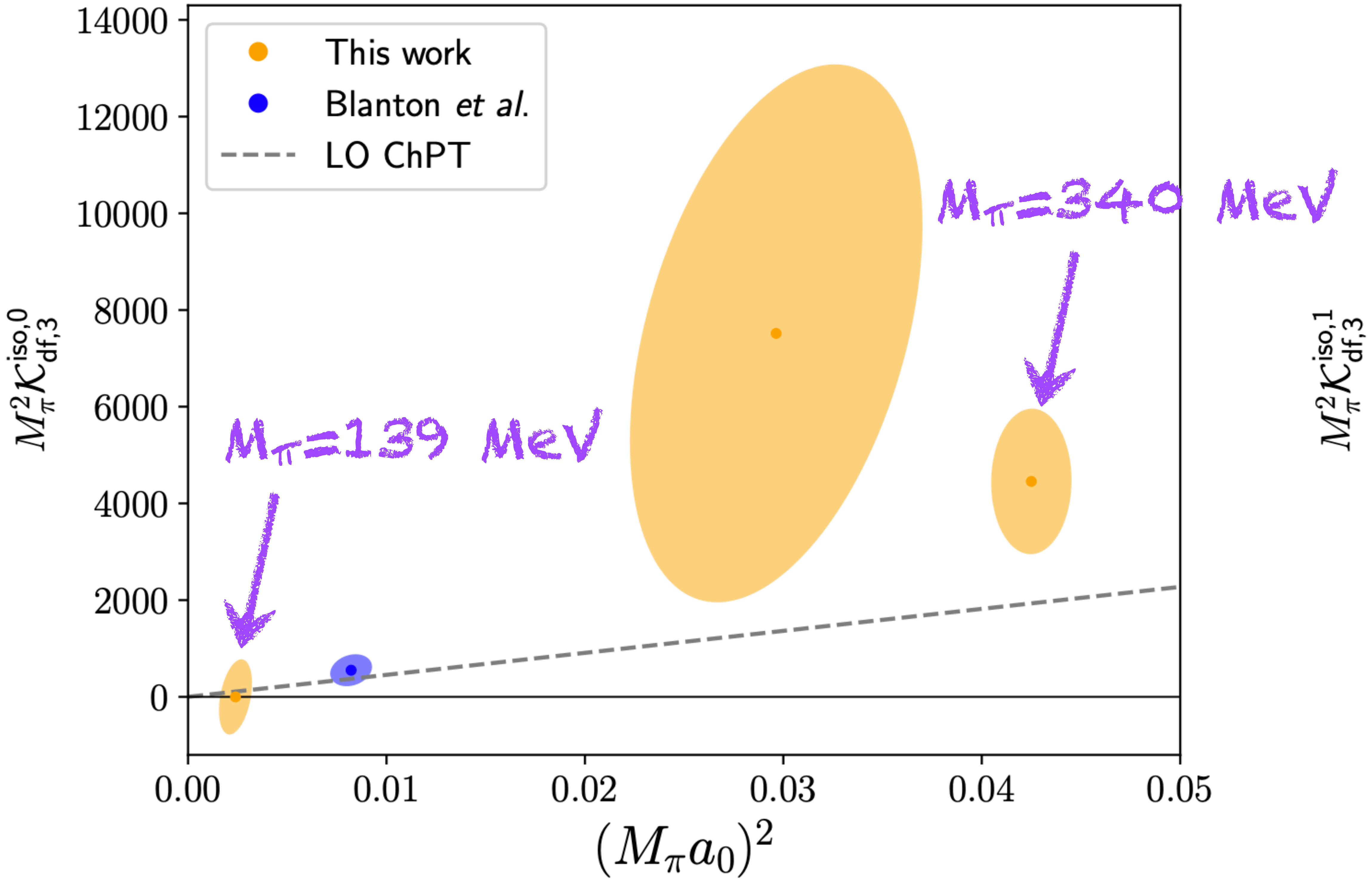

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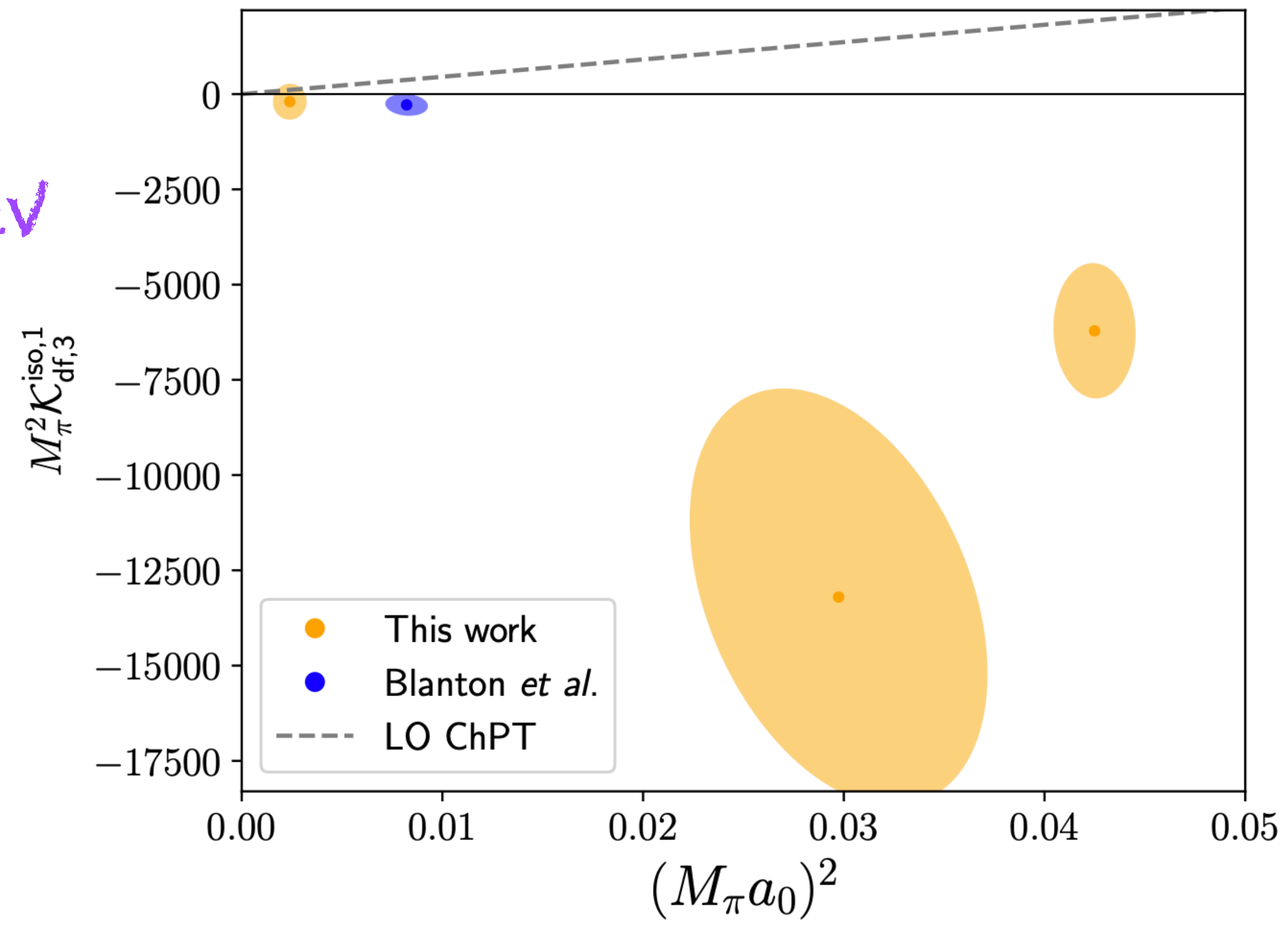

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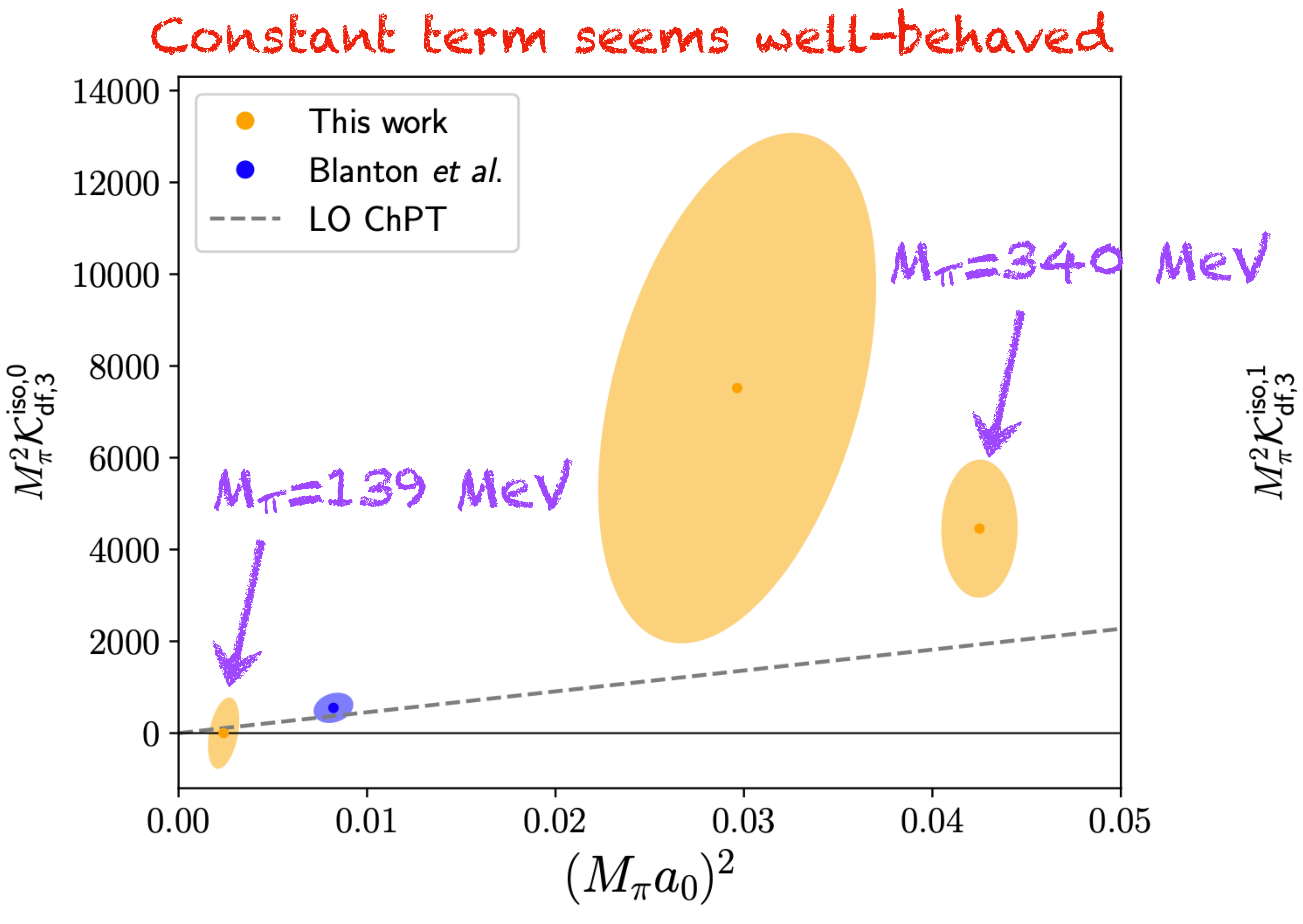

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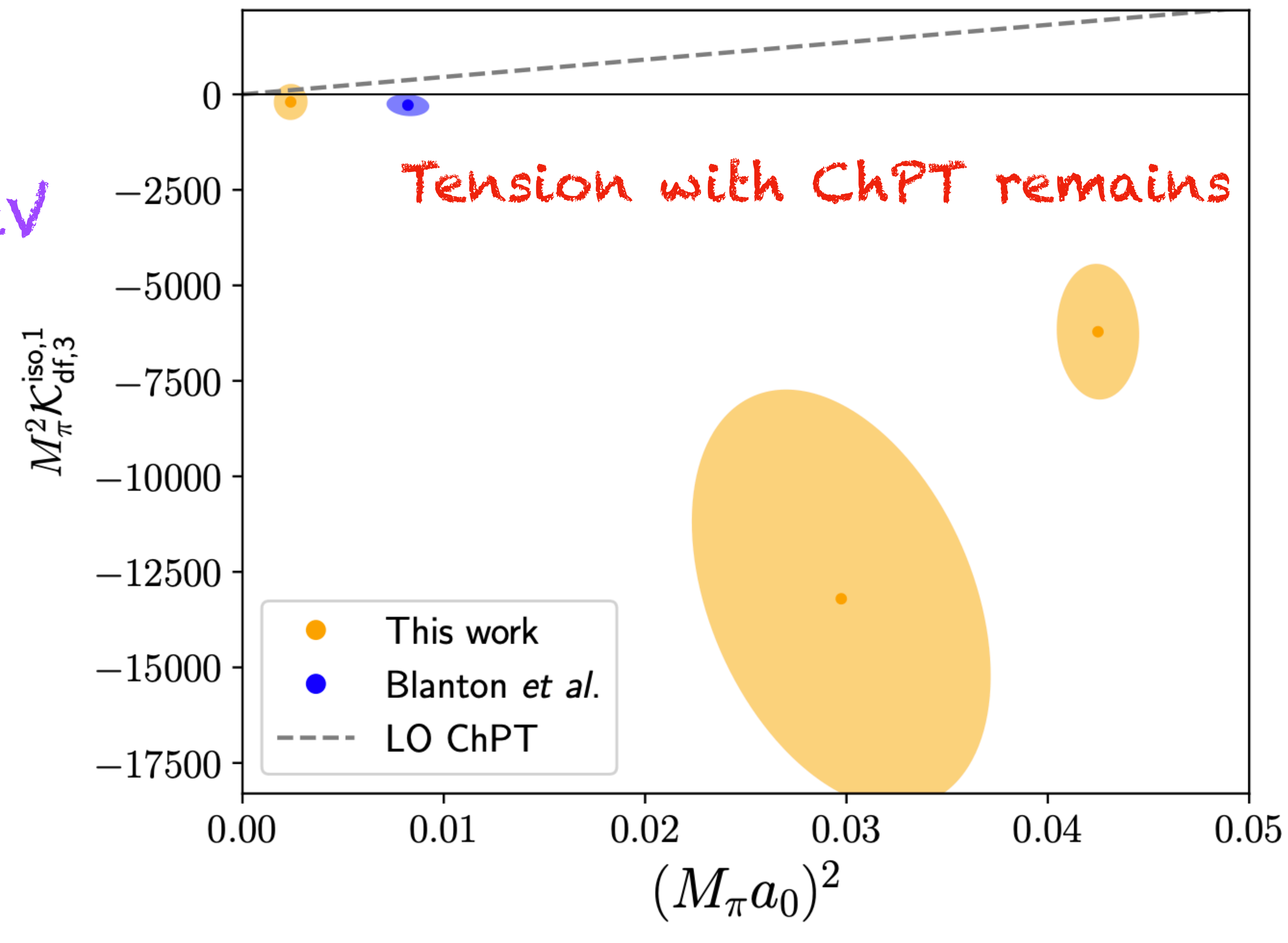

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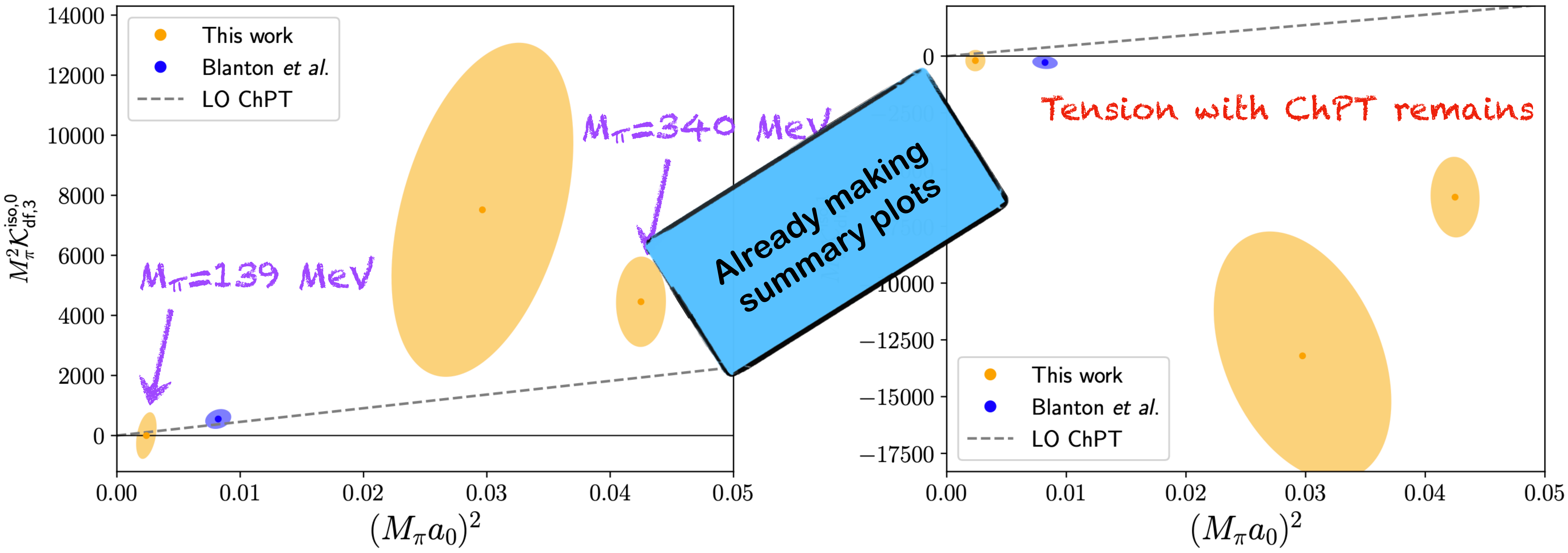

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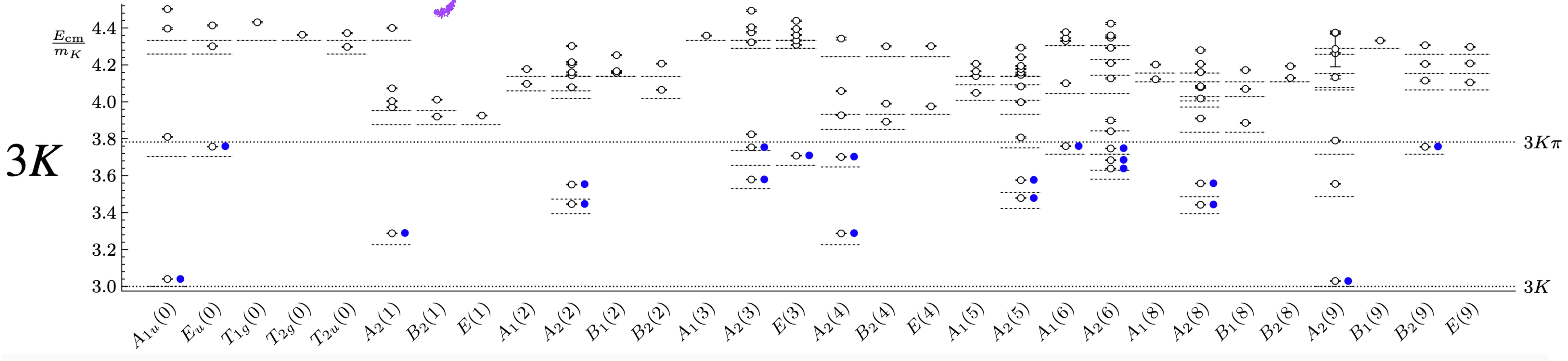
Two- and three-kaons

Study of 3K system
[Alexandru et al.]

- Other simple systems can also be studied: $2K^+$ & $3K^+$
- Many energy levels that allow for s- and d-wave interactions to be extracted!

Preliminary!



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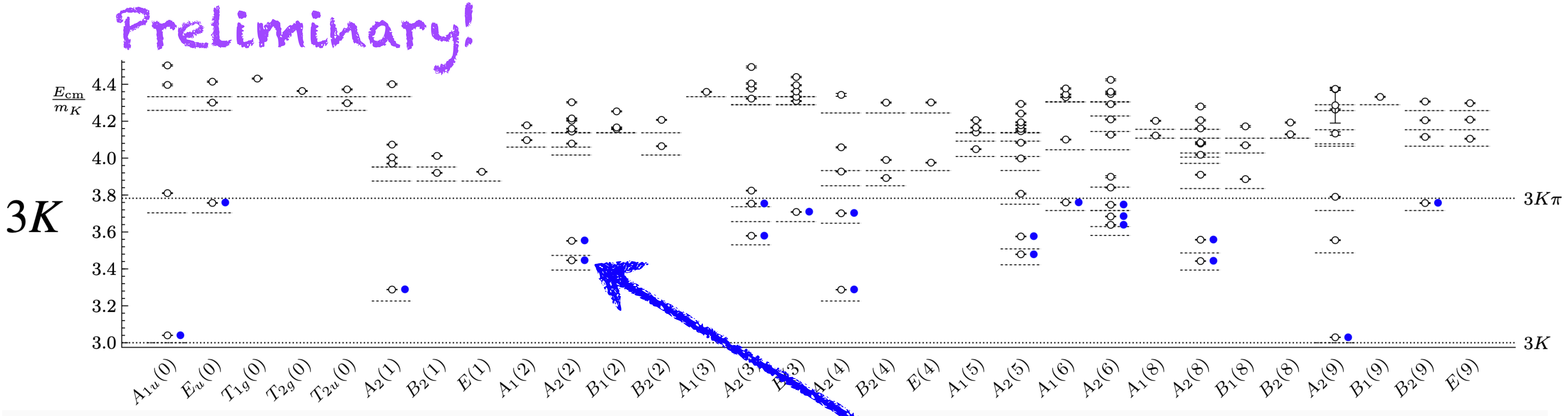

prediction from QC3

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Solving the integral equations

Final step

Physical 3->3 amplitude

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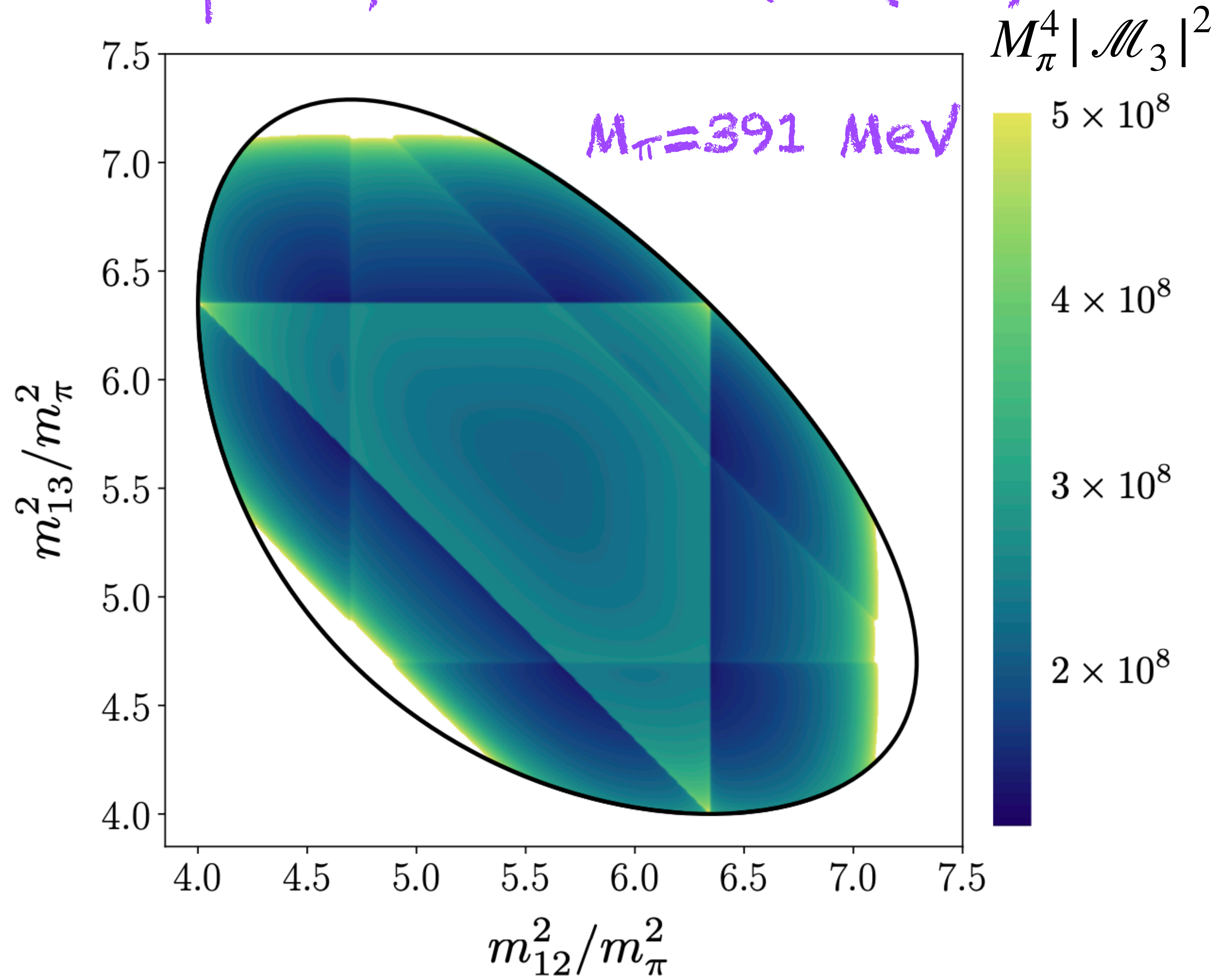

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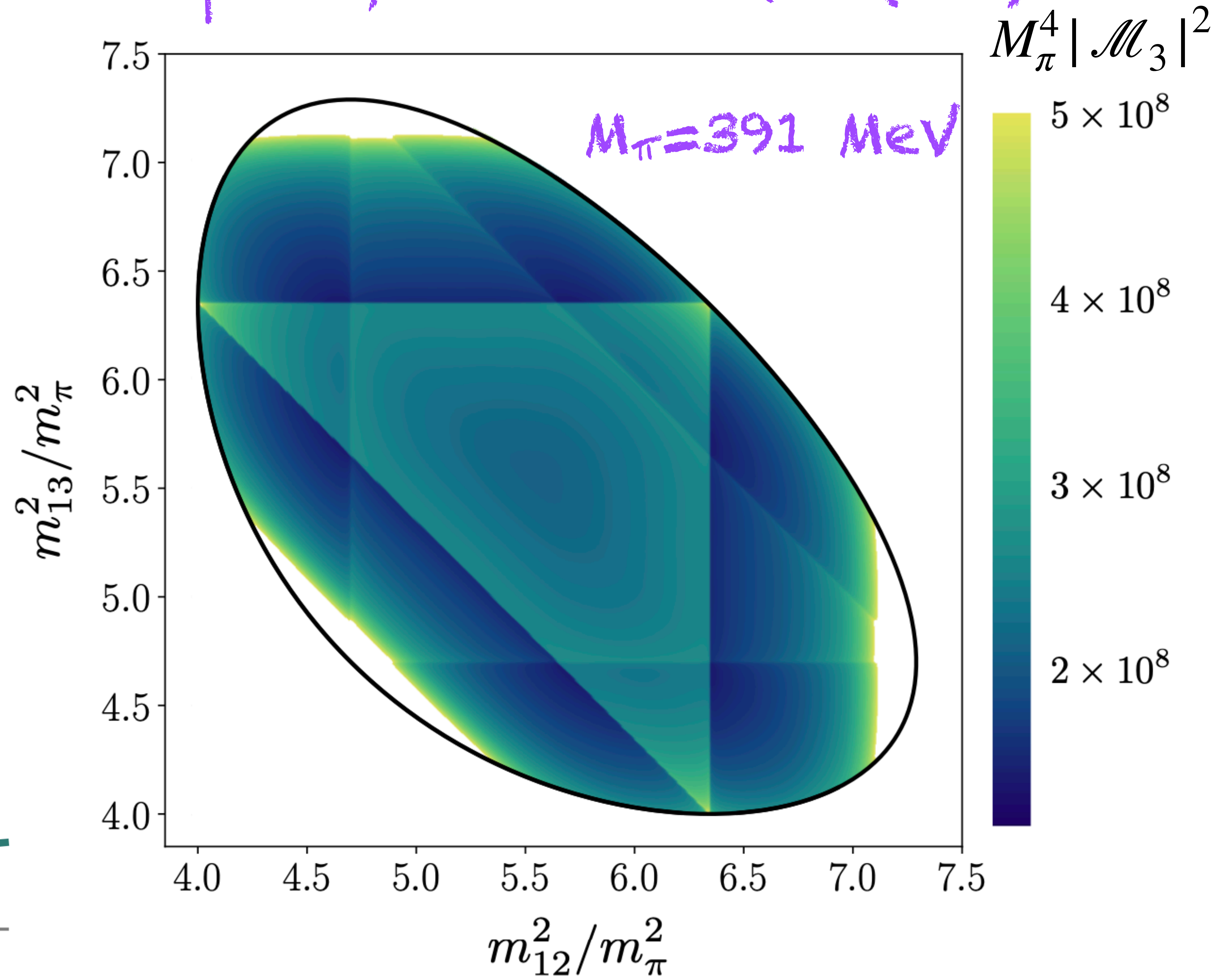

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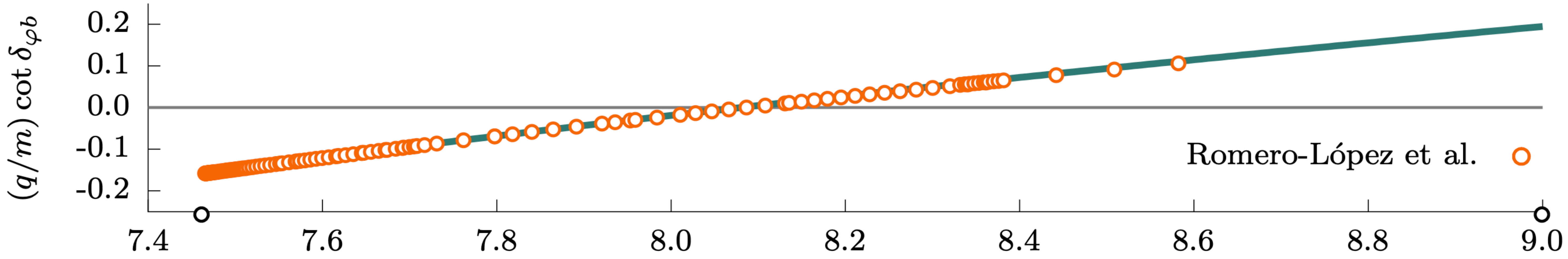
Particle-Dimer phase shift [Jackura et al.]



Beyond three identical particles

Generic 3π system

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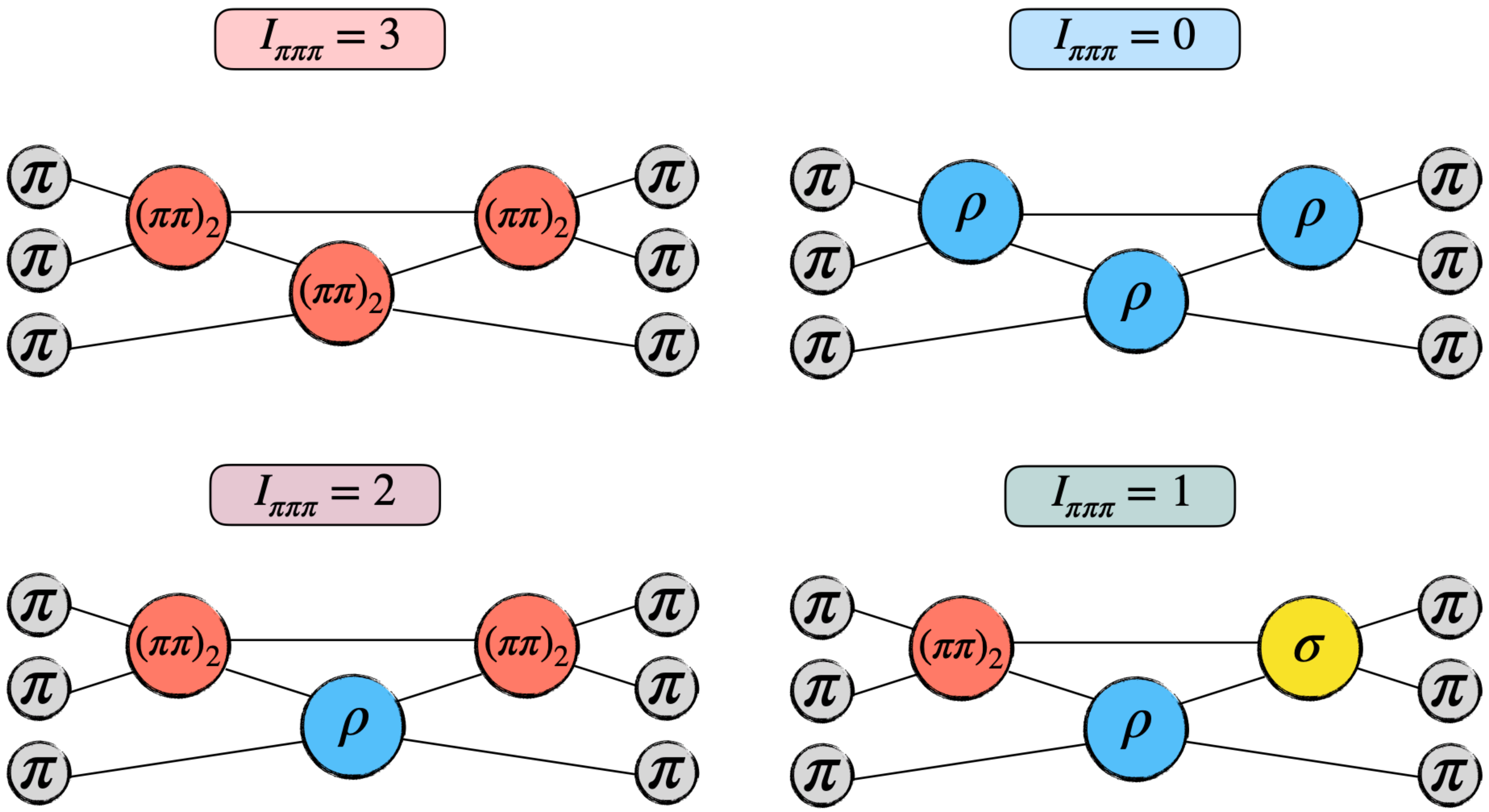

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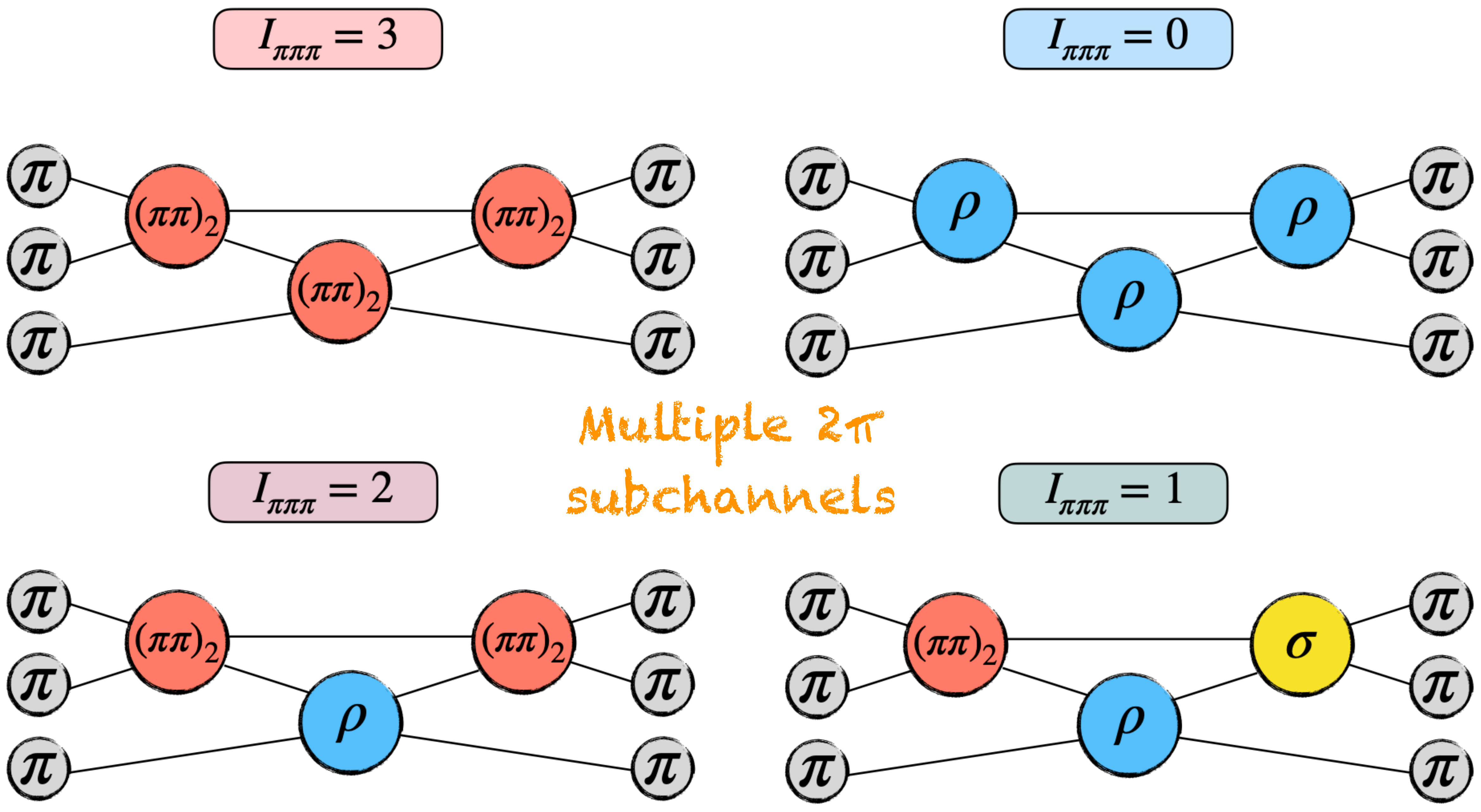

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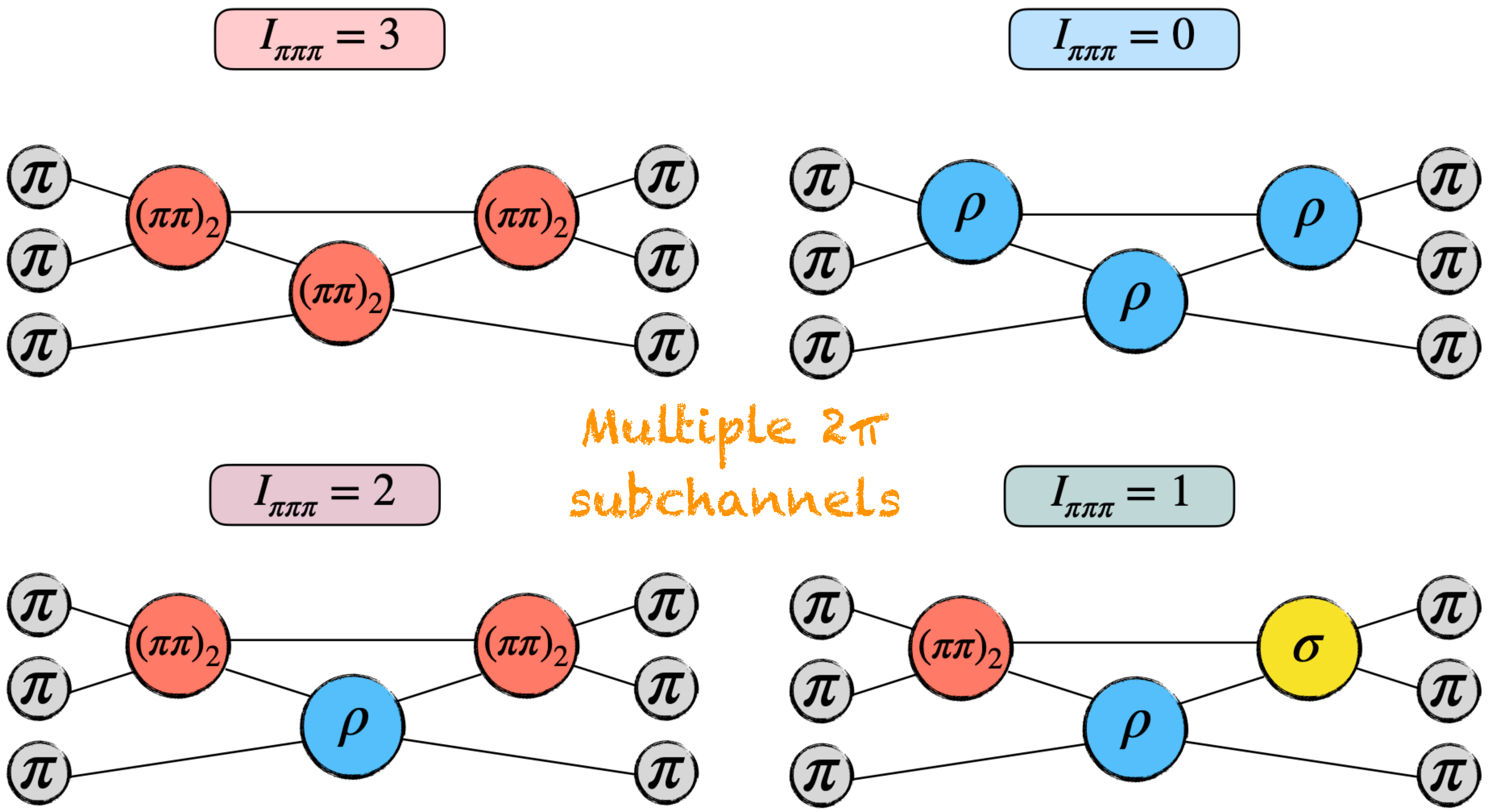
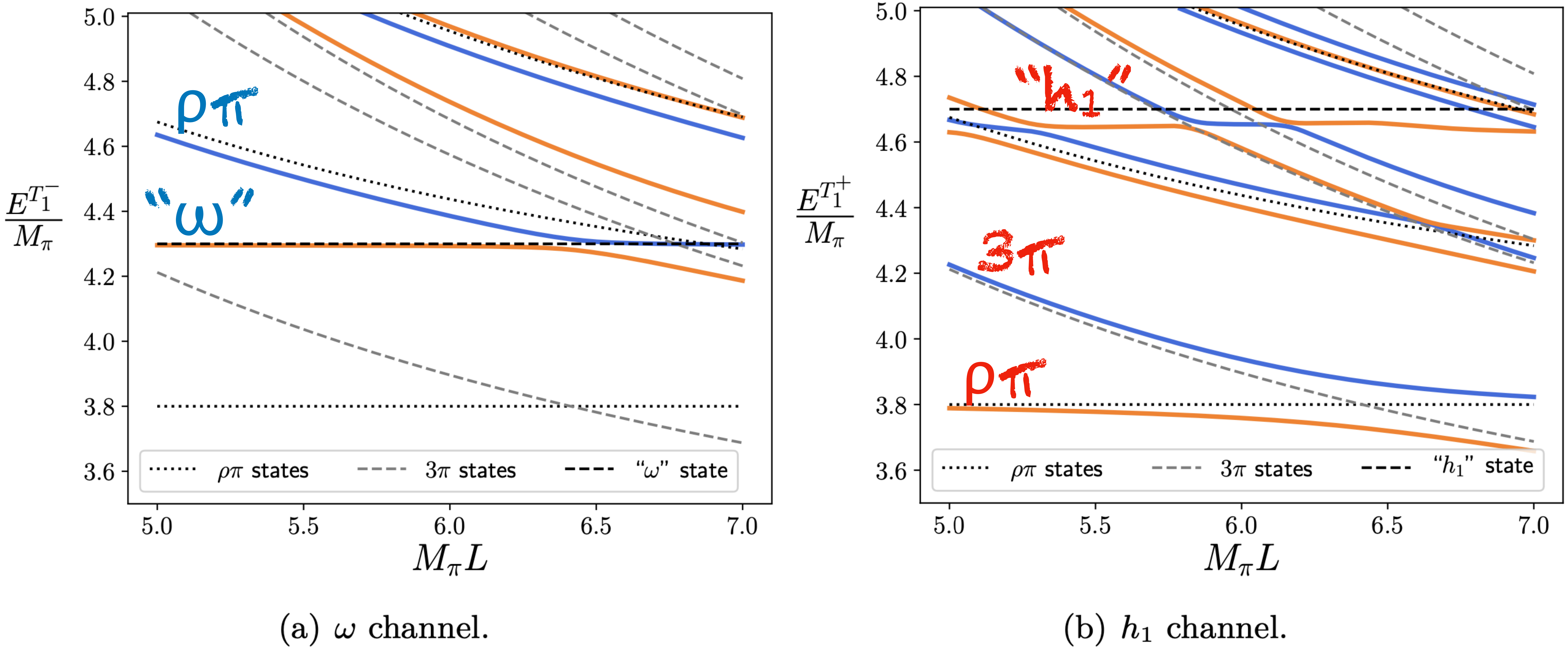


"All ingredients are now available for lattice studies of resonances with three-particle decay channels, such as the ω(782) and the $h_1(1170)$"

Toy example: "h_1" and "ω"



I=0, $J^P=1^-$ [Hansen, FRL, Sharpe] I=0, $J^P=1^+$

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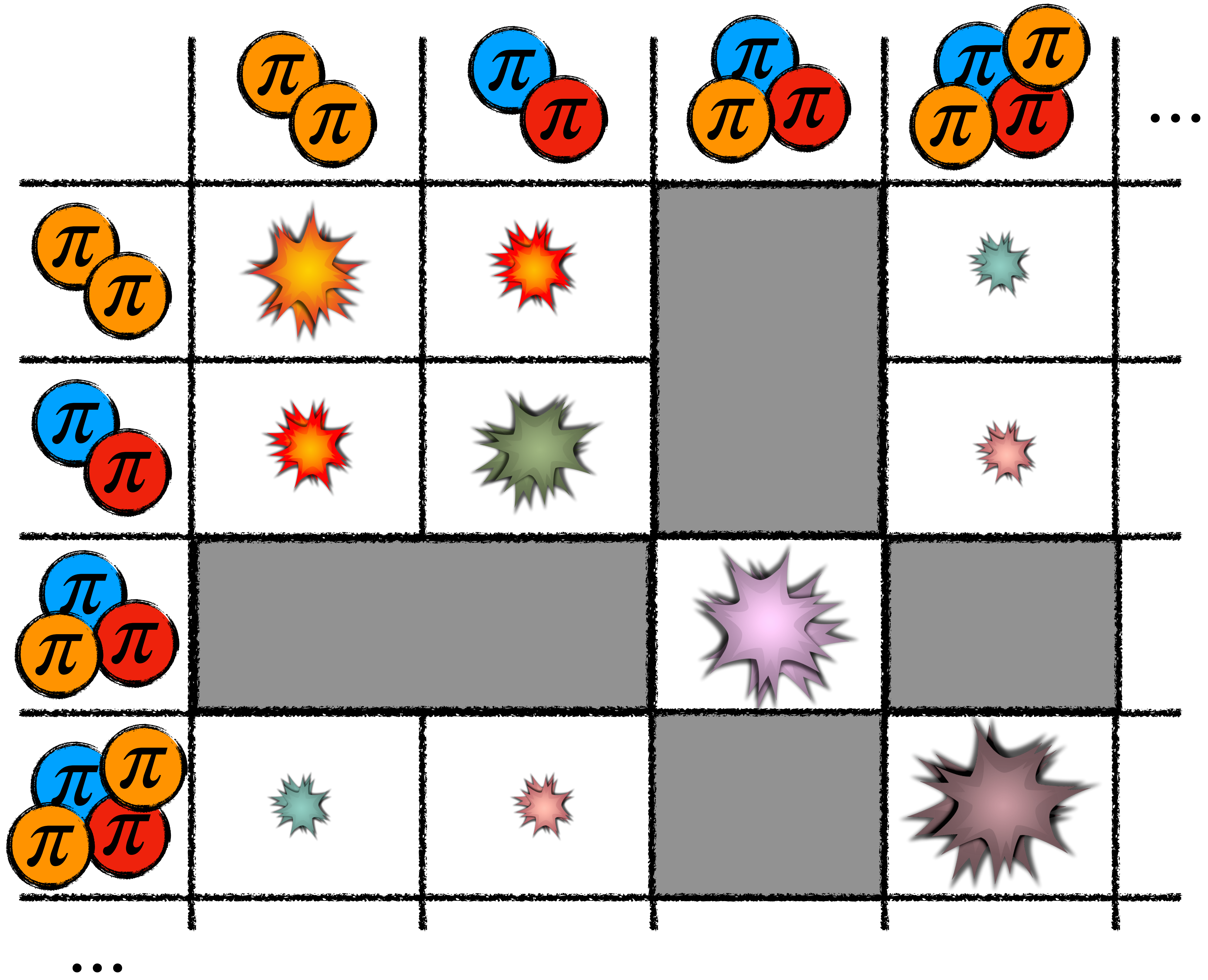
- Towards the Roper resonance! $N(1440) \to \Delta\pi \to N\pi\pi$

Conclusion



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- We are entering a new era of three-particle scattering studies.
  - Finite-volume formalism for identical particles.
  - A tool for solving relativistic integral equations for three body systems
  - Some lattice studies of three charged pions (and kaons)
  - The formalism for generic three-pion resonances is ready!
  - Progress in QC3 for nondegenerate scalar particles.

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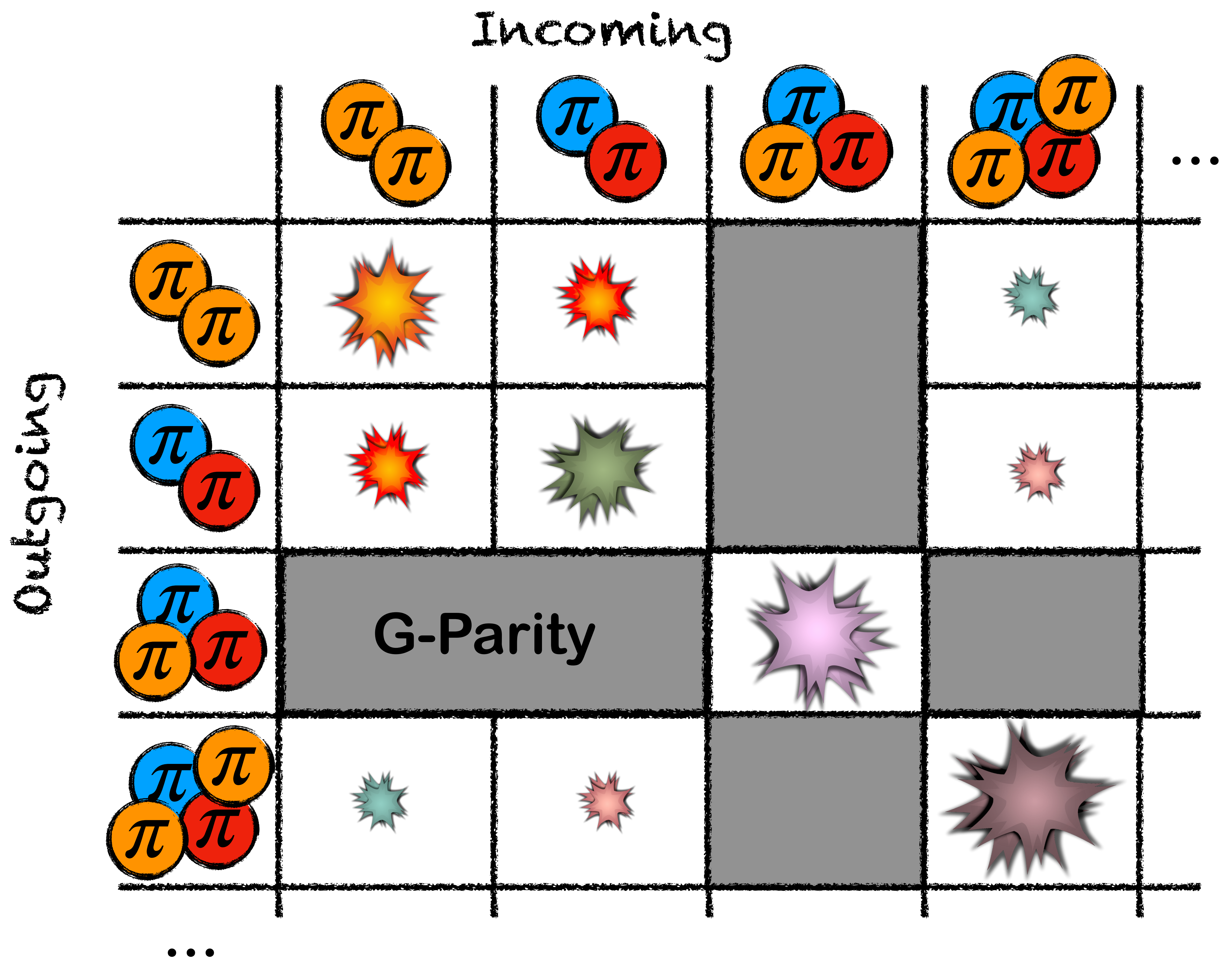

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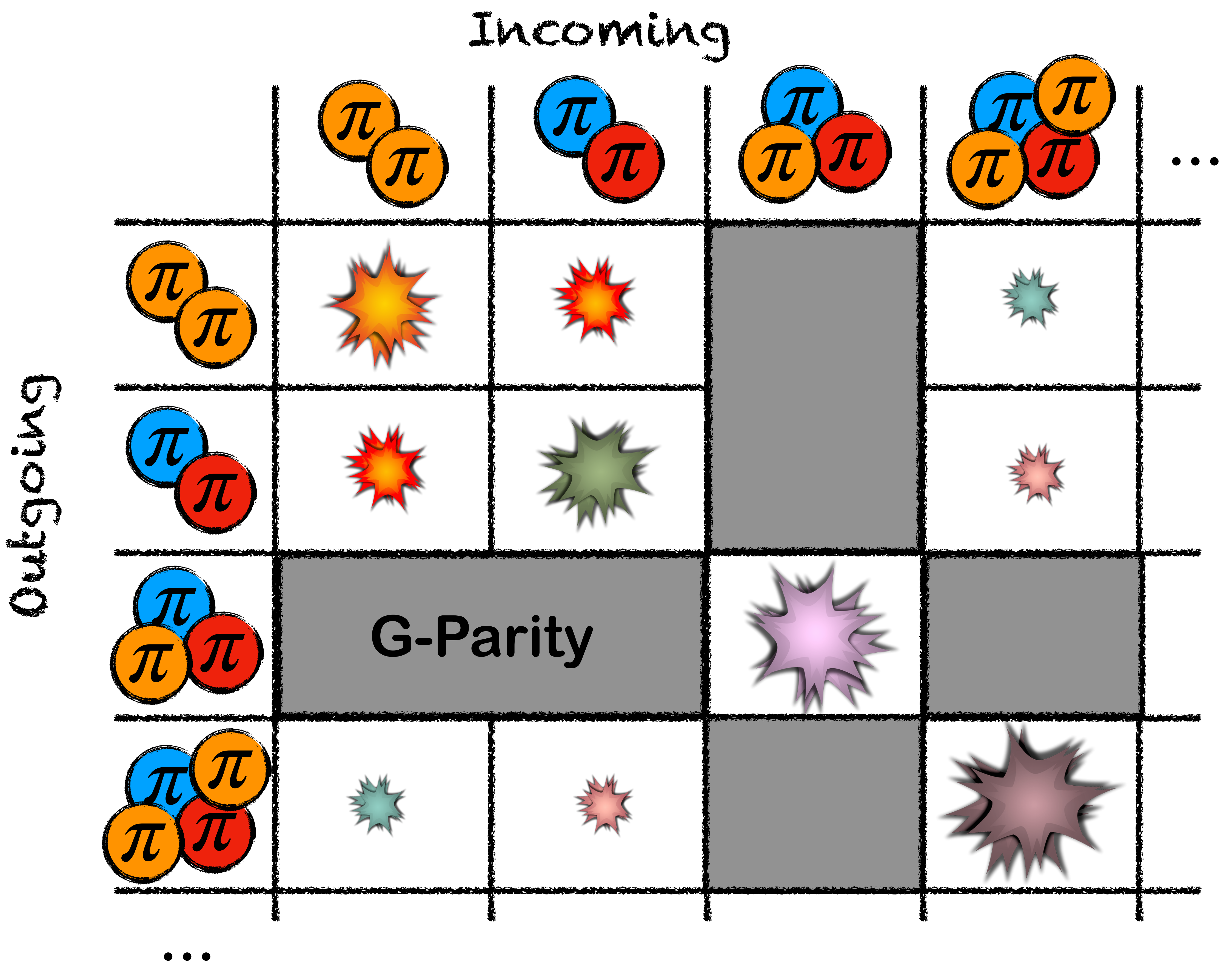

Thanks!