Finite volume matrix elements of two-body states

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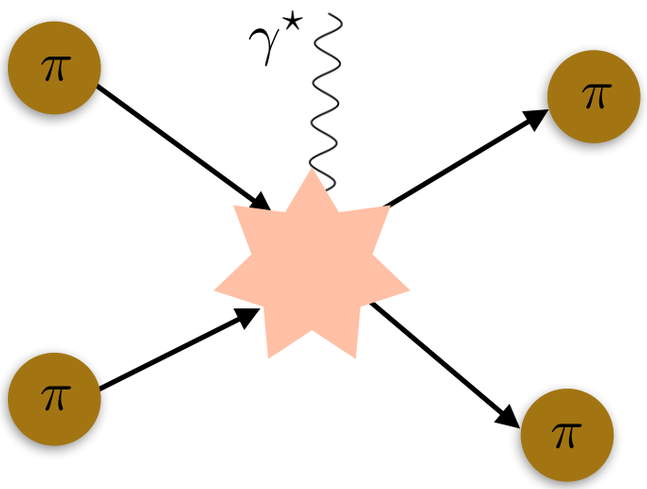
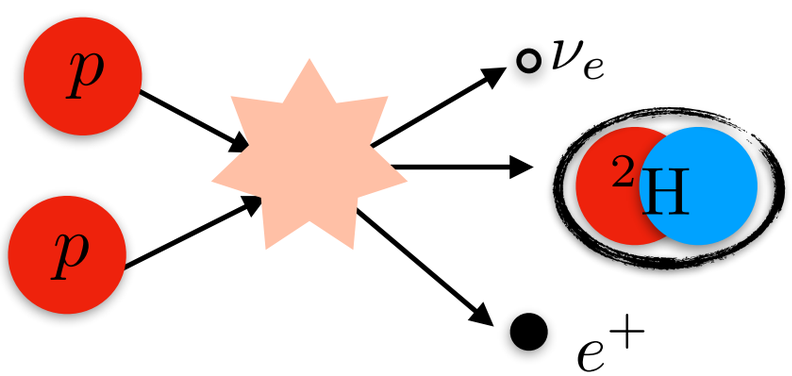
F. G. Ortega
William and Mary

Jefferson Lab 2019

Few-body electroweak processes

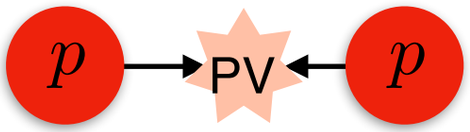


- Can be used to access resonance
  - Obtain structural information about resonances (form factors)



- Start of the main chain reaction that powers the Sun

Recently in LQCD (Savage et al. 2015)



- Not enough experimental data, theory poorly constrained

Kurth et al. (2015)

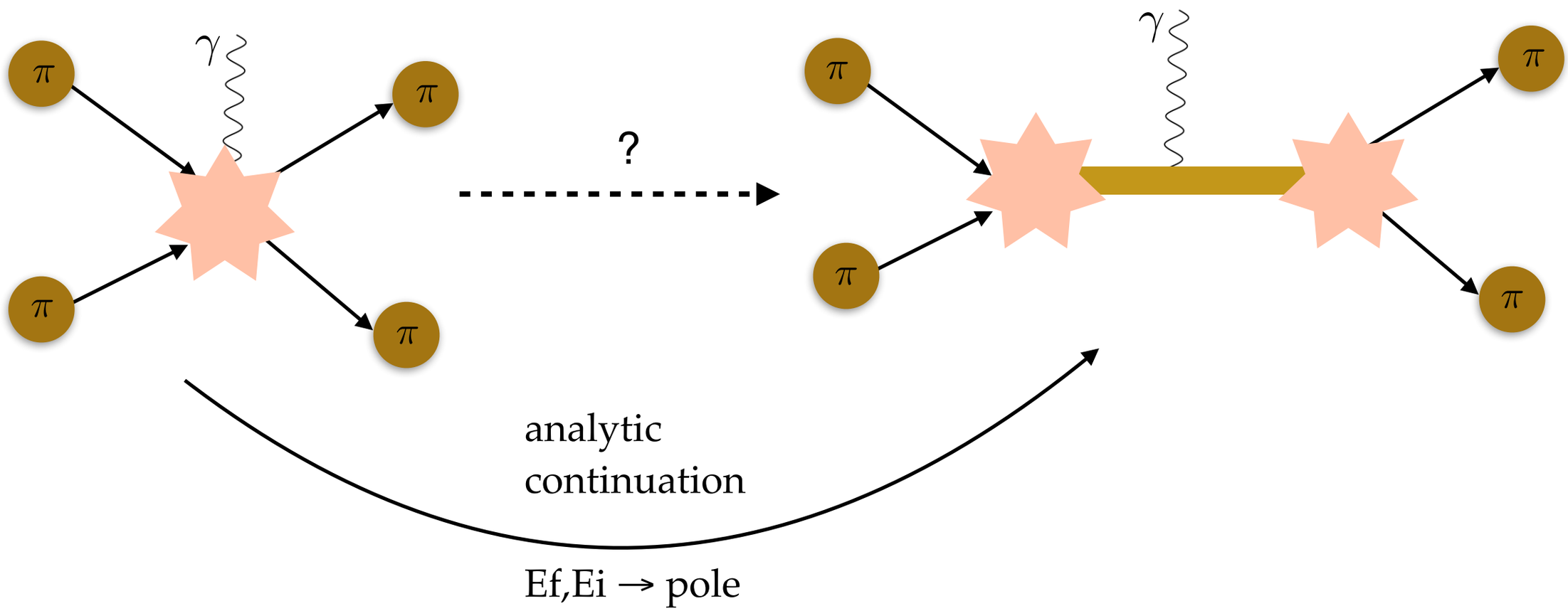
Plenty of other reactions relevant for nuclear physics:

- $n + p \to d + \gamma$
- Neutrino-deuteron low energy scattering

. . .

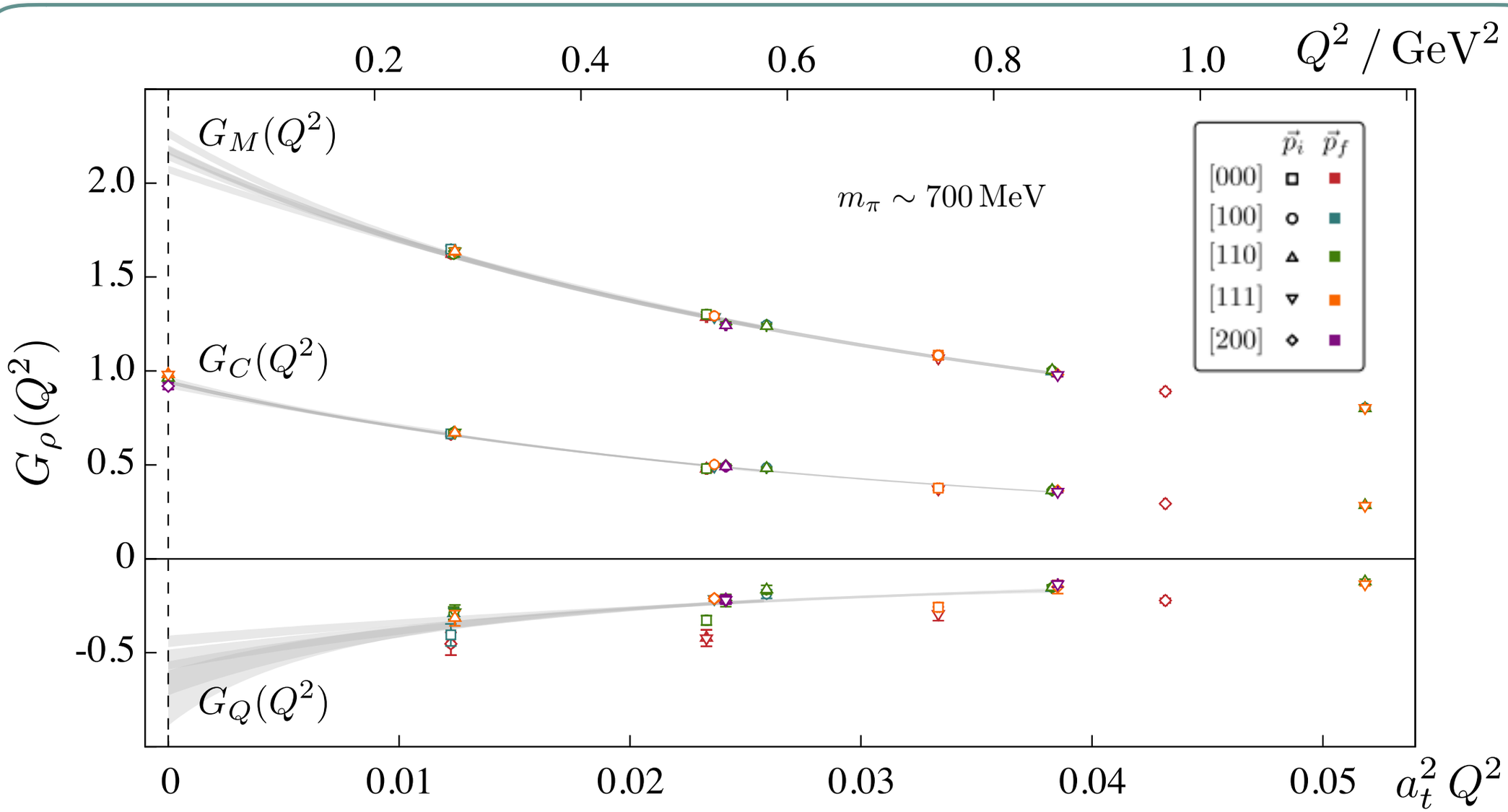
Form factor of a resonance

- Resonance are not asymptotic states → Form factor?
  (Standard LSZ theorem cannot be used)



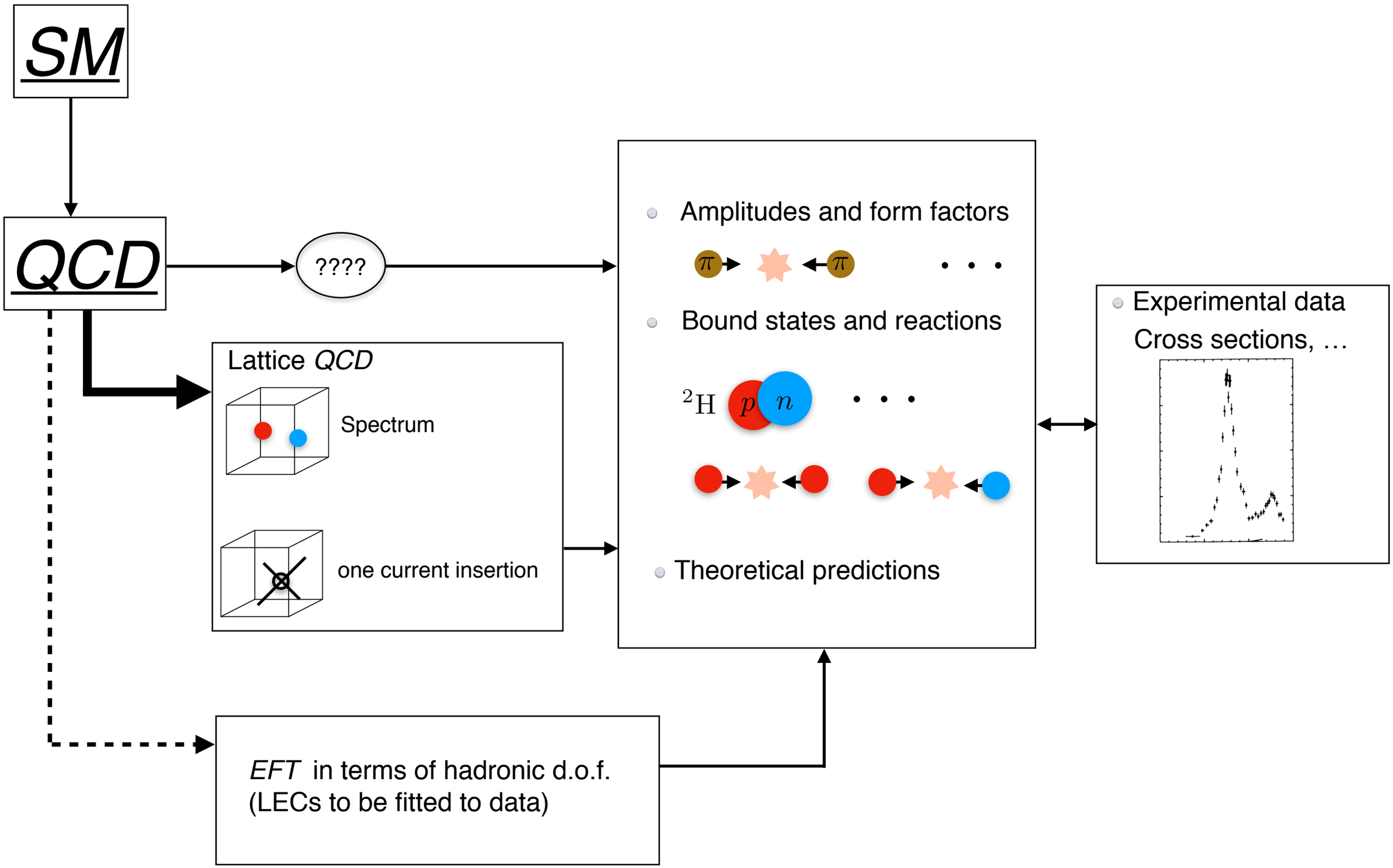
Looking at the scattering amplitude we can access

Form factor of a stable "resonance"

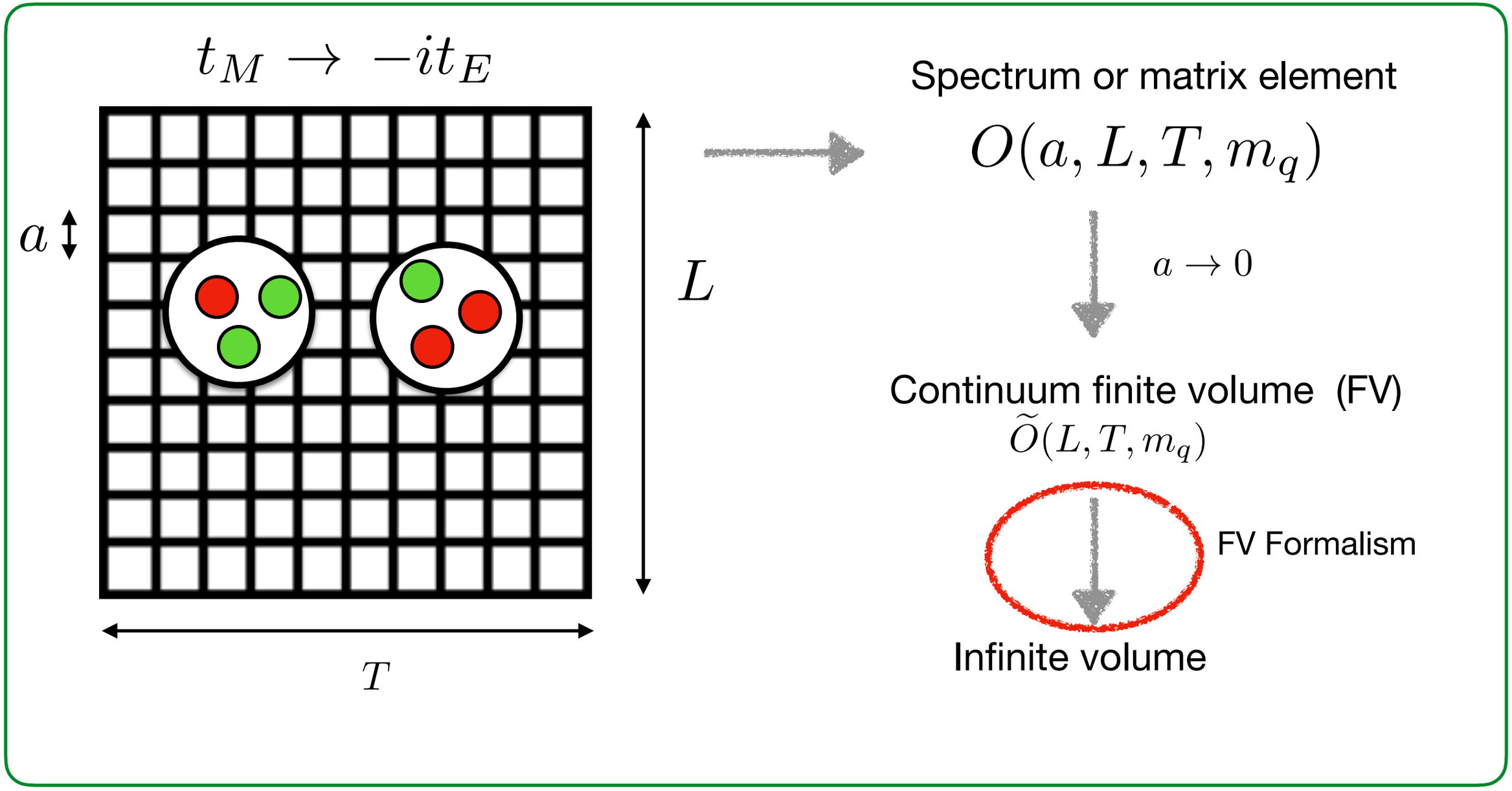


Shultz, Dudek, and Edwards PRD (2015)

Roadmap


SM
QCD
????
Lattice QCD
Spectrum
one current insertion
Amplitudes and form factors
π
π
. . .
Bound states and reactions
^{2}H
p
n
. . .
Theoretical predictions
Experimental data
Cross sections, …
EFT in terms of hadronic d.o.f.
(LECs to be fitted to data)

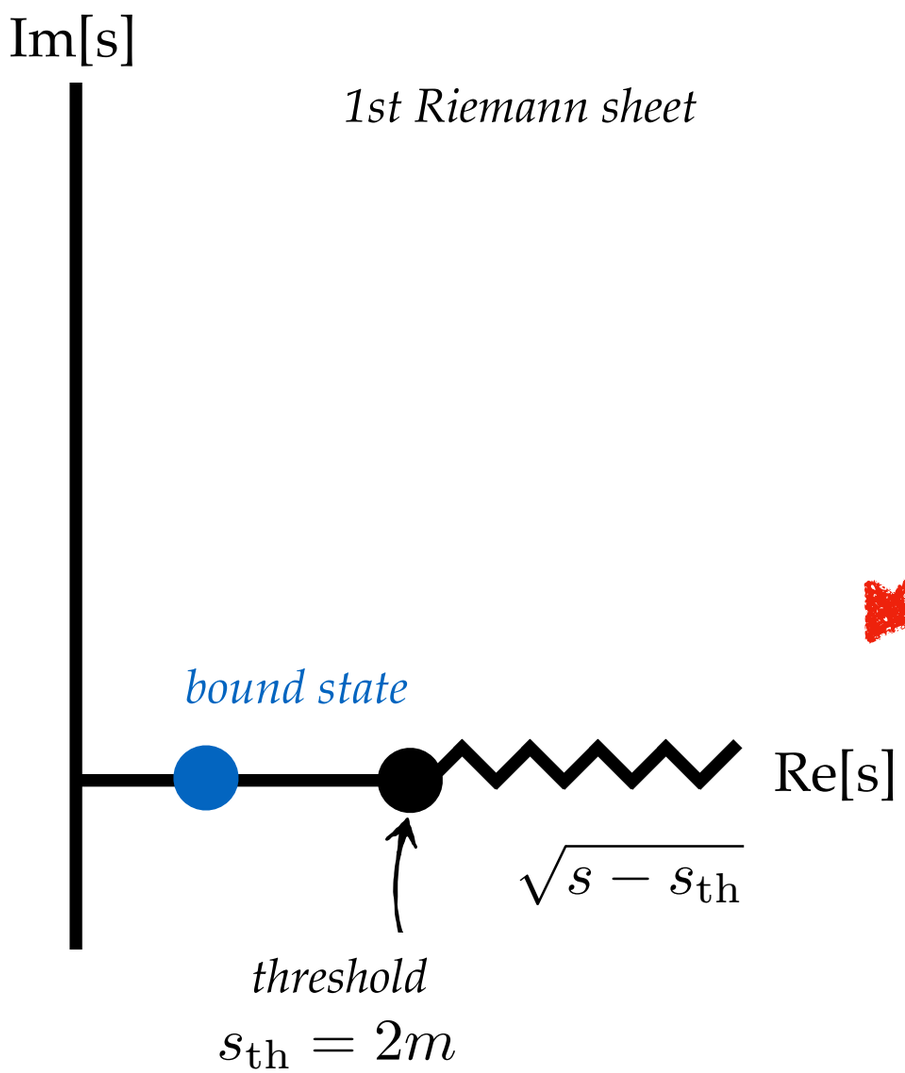
LQCD

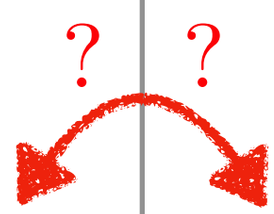

$t_M \to -it_E$
a
L
T
Spectrum or matrix element
$O(a, L, T, m_q)$
$a \to 0$
Continuum finite volume (FV)
$\widetilde{O}(L, T, m_q)$
FV Formalism
Infinite volume

Scattering amplitude

Infinite volume

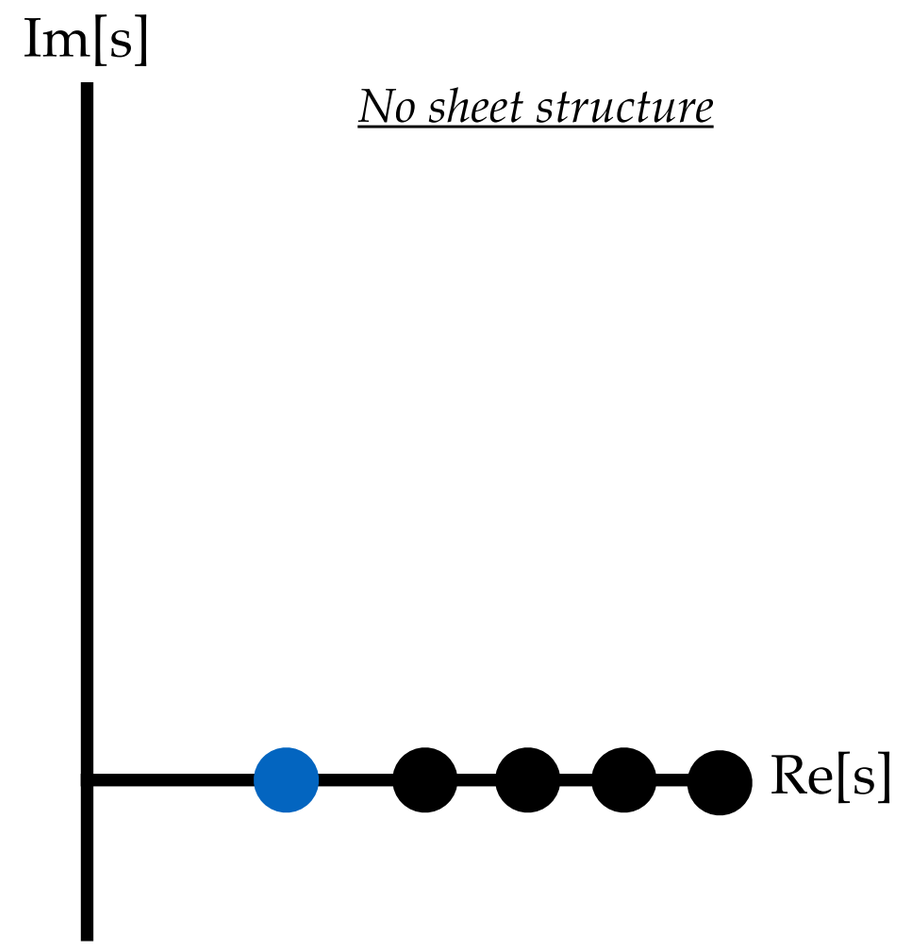
$$s = E_{\mathrm{cm}}^2$$





Spectrum

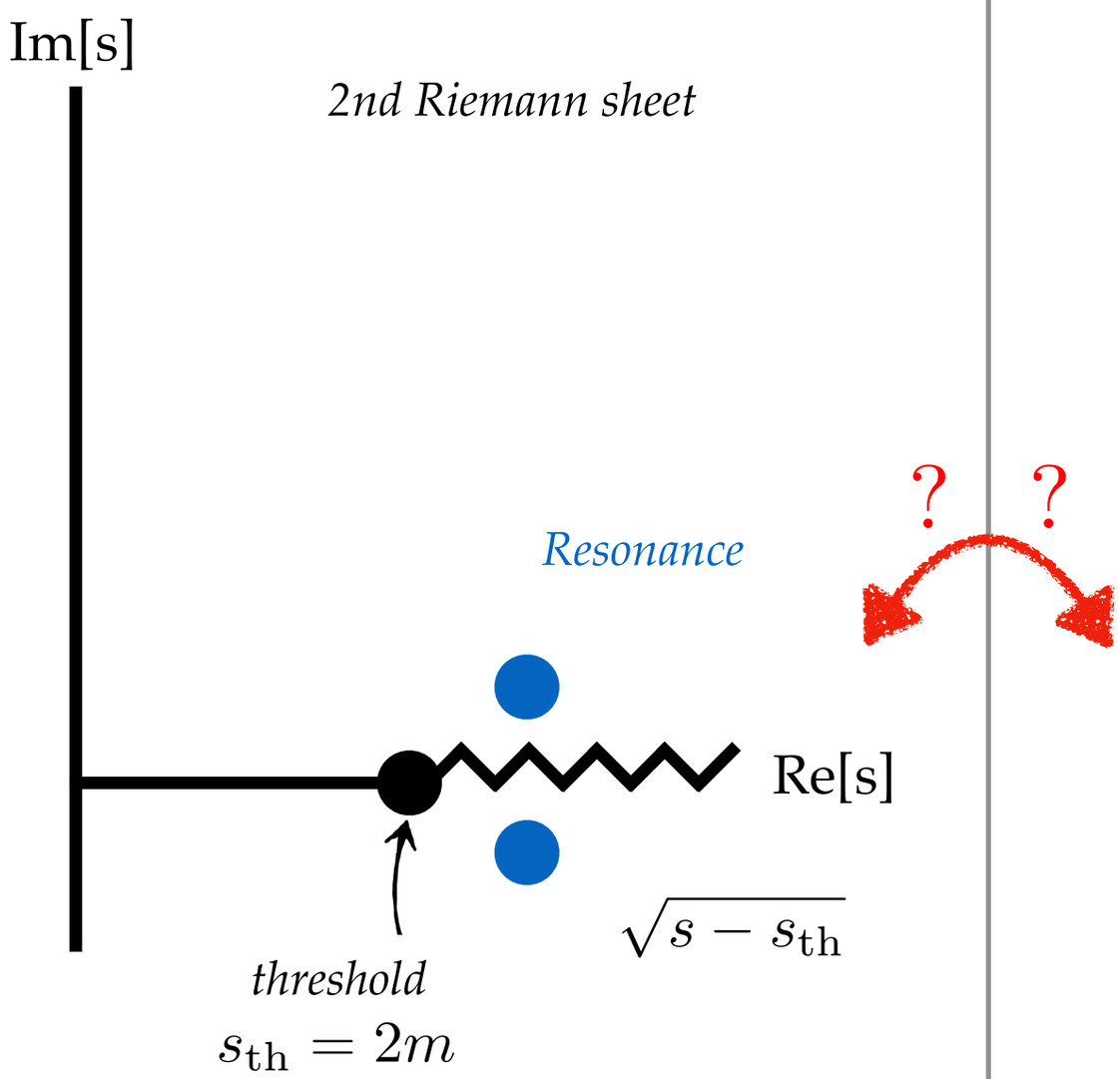
Finite volume



Scattering amplitude

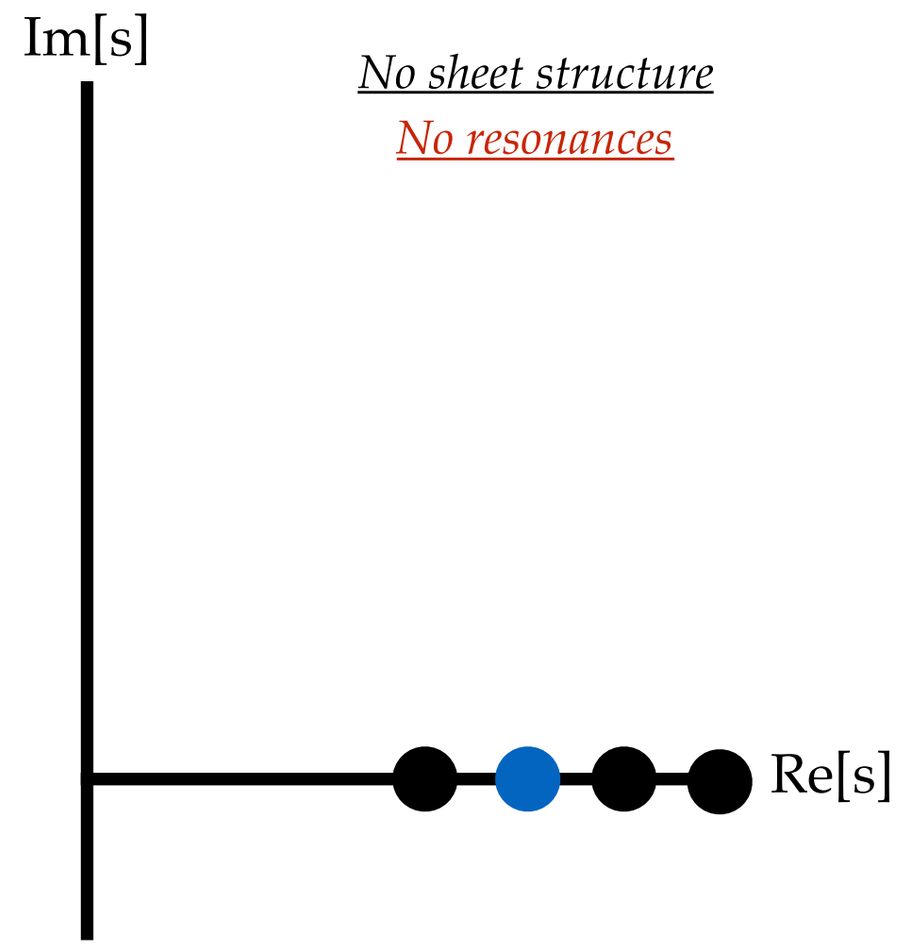
Infinite volume

$$s = E_{\mathrm{cm}}^2$$



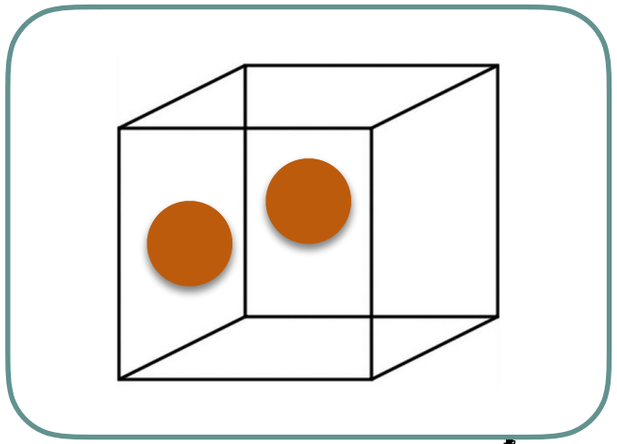
Spectrum

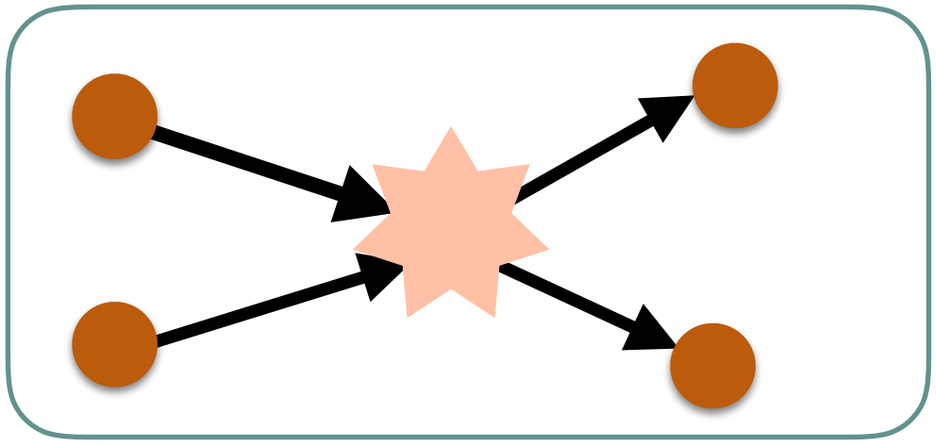
Finite volume



2→2

- FV spectra to infinite volume purely hadronic amplitudes
- Holds for a generic QFT with hadronic d.o.f, up to multi-particle thresholds



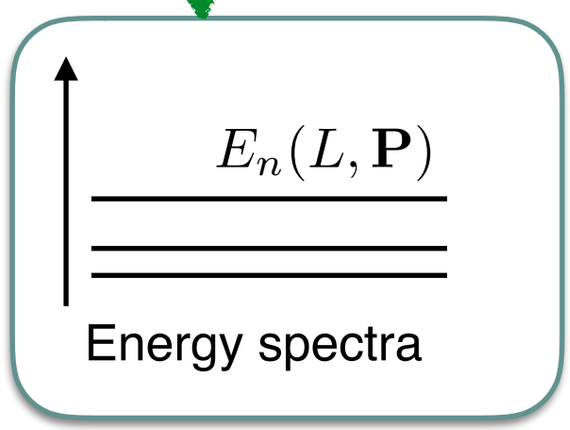


$$\det\left[F^{-1}(E_L, L) + M(E_L)\right] = 0$$

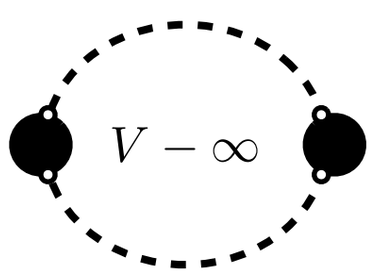
- Lüscher (1986, 1991) [elastic scalar bosons]
- Rummukainen & Gottlieb (1995) [moving elastic scalar bosons]
- Kim, Sachrajda, & Sharpe/Christ, Kim & Yamazaki (2005) [QFT derivation]
- Feng, Li, & Liu (2004) [inelastic scalar bosons]
- Hansen & Sharpe / Briceño & Davoudi (2012) [moving inelastic scalar bosons]
- Briceño (2014) [general 2-body result]

2→2

$$\det\left[F^{-1}(E_L, L) + M(E_L)\right] = 0$$

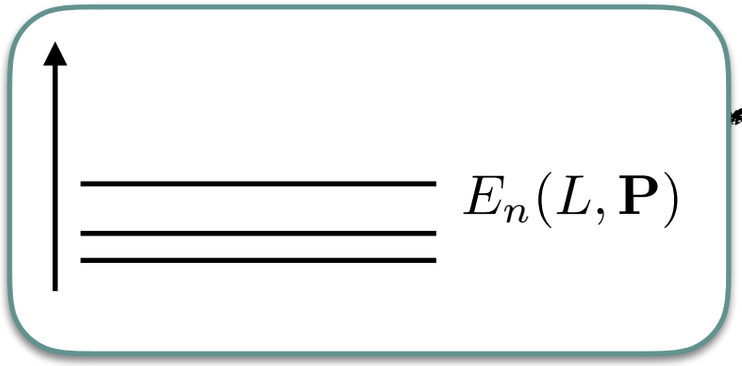


2→ 2 scatt. Amplitude
Calculated at FV energies

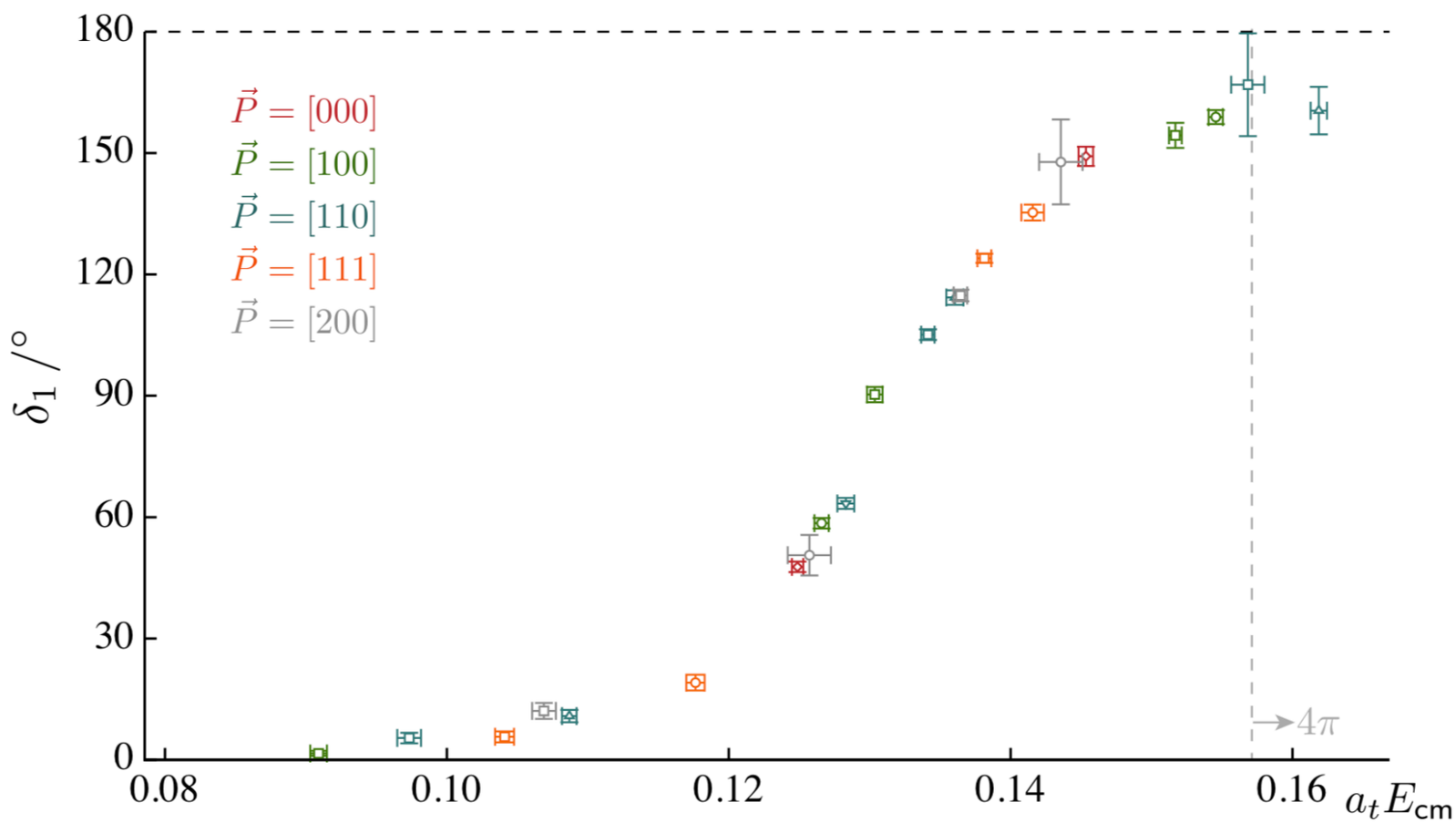
$F =$ 

$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right](\cdots)$

Application



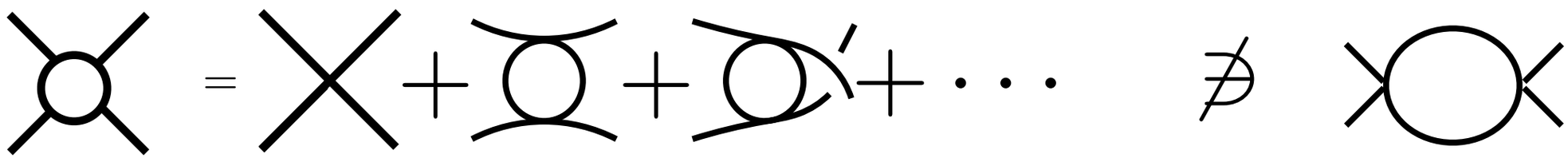
$$\det\left[F^{-1}(E_L, L) + M(E_L)\right] = 0$$



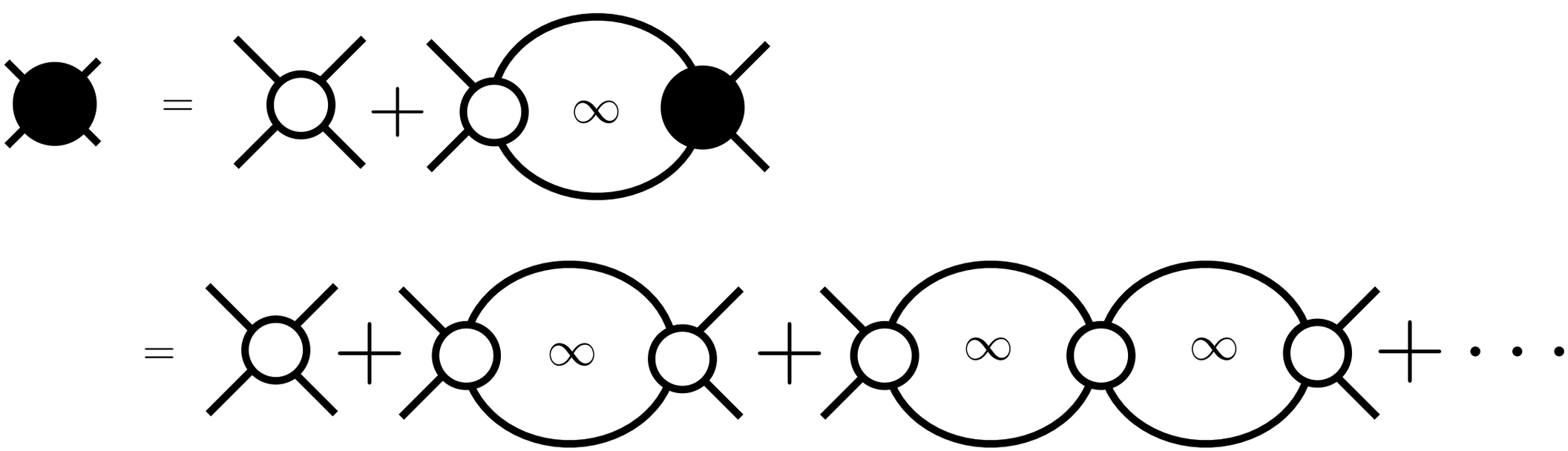
Wilson, Briceño, Dudek, Edwards, and Thomas (2015)

All orders "perturbation" theory

- Define the following kernel (all Feynman diagrams but s-channel)



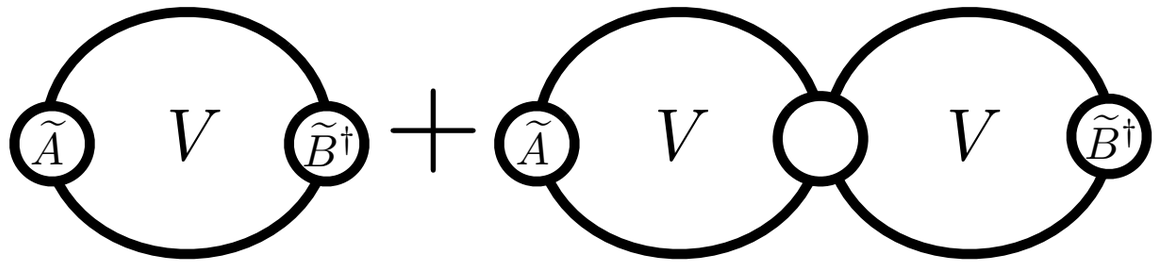
- With the above definition we can represent the scattering amplitude in the following way



Derivation

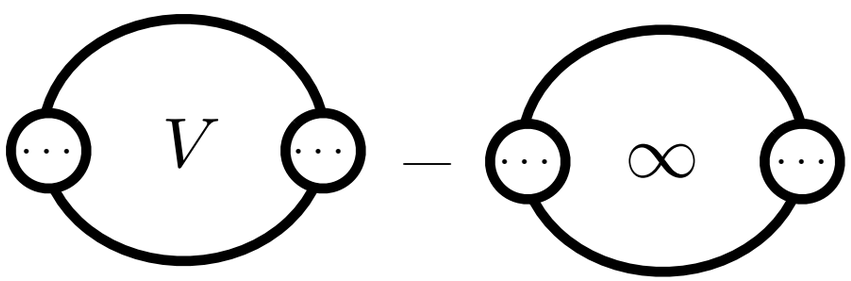
- Lets look now at the FV correlator

$$C_L^{2pt.}(P) =$$



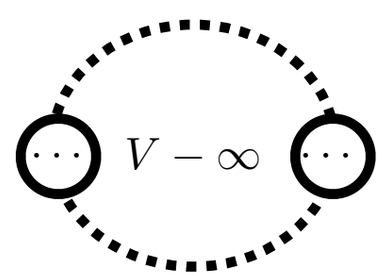
$$+ \cdots$$

- Lets look at



$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \int dk^0 \frac{(\cdots)\ (\cdots)}{(k^2 - m^2 + i\epsilon)((P-k)^2 - m^2 + i\epsilon)}$$

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \frac{(\cdots)(\cdots)|_{k^0=\omega_k}}{2\omega_k(E - \omega_k + i\epsilon)} =$$



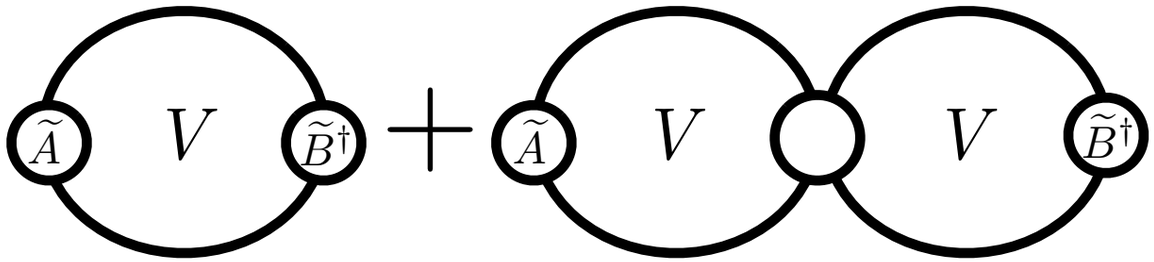
$$= (\cdots)|_{\text{on}}\, iF\, (\cdots)|_{\text{on}}$$

Plugging this in the FV correlator above

Derivation

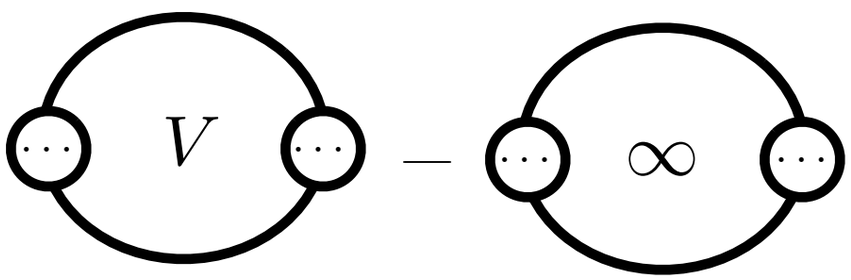
- Lets look now at the FV correlator

$$C_L^{2pt.}(P) =$$



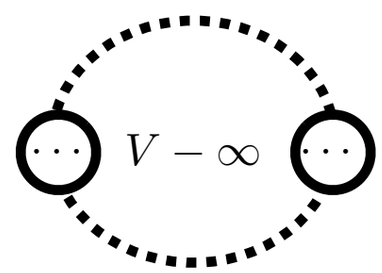
$$+ \cdots$$

- Lets look at



$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \int dk^0 \frac{(\cdots)\ (\cdots)}{(k^2 - m^2 + i\epsilon)((P-k)^2 - m^2 + i\epsilon)}$$

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \frac{(\cdots)(\cdots)|_{k^0=\omega_k}}{2\omega_k(E - \omega_k + i\epsilon)} =$$



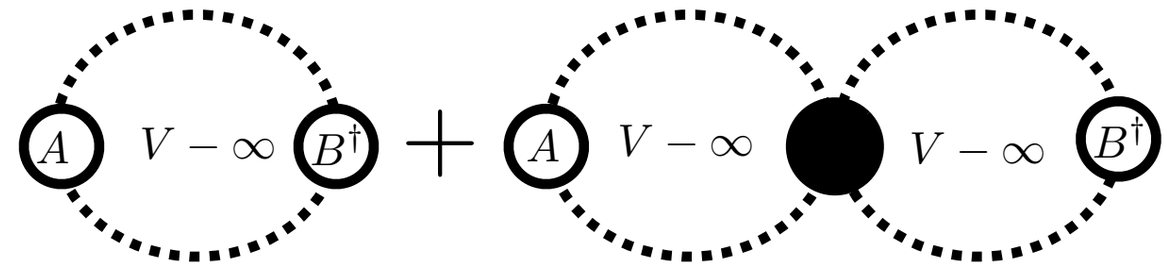
$$= (\cdots)|_{\text{on}}\, iF\, (\cdots)|_{\text{on}}$$

Plugging this in the FV correlator above

Derivation

- After properly rearranging the series we arrive at

$$C_L^{2pt.}(P) = C_\infty(P) +$$



$$+ \cdots$$

- But this is “only” a geometric series!

$$C_L^{2pt.}(P) = \quad A\frac{1}{F^{-1}+M}B^\dagger + \cdots$$

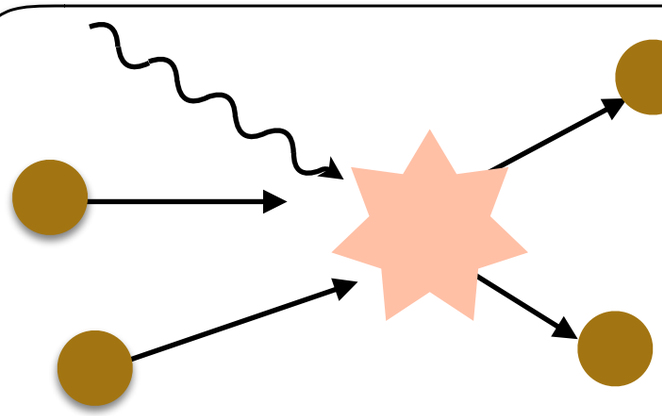
Smooth terms

- From this we obtain the wanted equation (the correlator has singularities at FV energies)

$$\boxed{\det\left[F^{-1}(E_L, L) + M(E_L)\right] = 0}$$

Two-to-two scattering with one current insertion

- Scalar particles



$= W = \langle$; out| $\mathcal{J}(0)$ | ; in$\rangle$

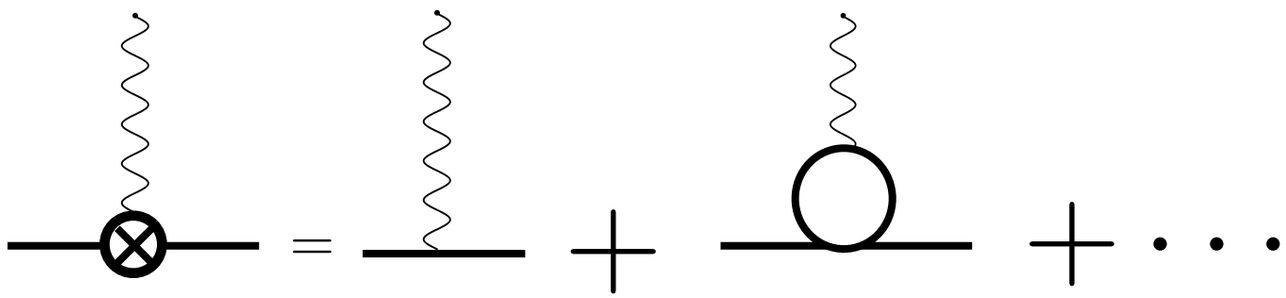
Local current

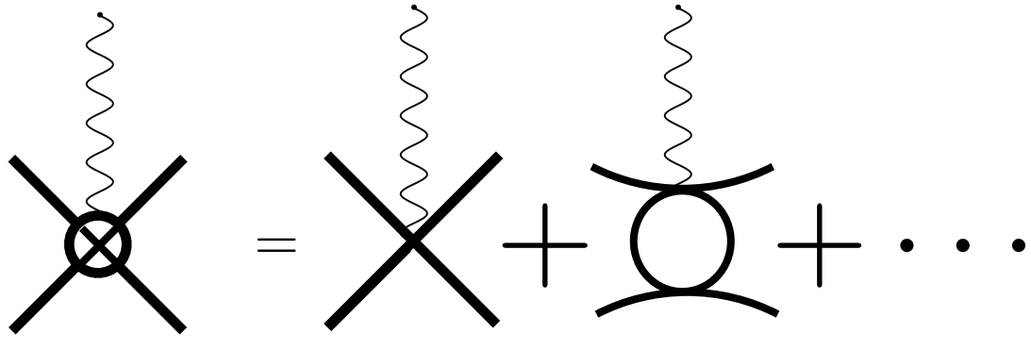
- How can we use all orders “perturbation” theory to represent this quantity?

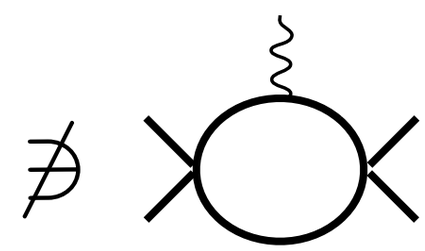
Briceño & Hansen (2016)

All orders "perturbation" theory

- Define the following kernels (all Feynman diagrams but s-channel)





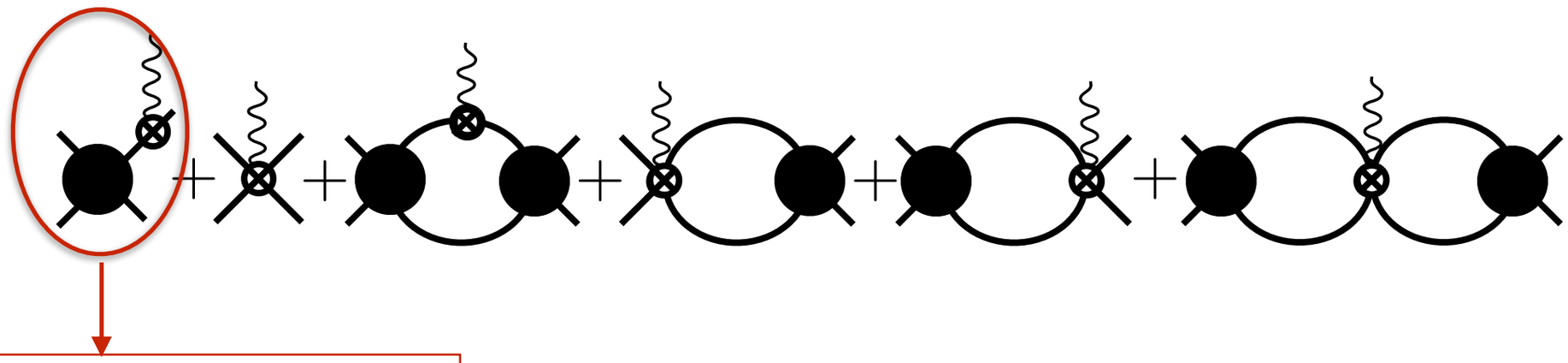


- **In the kinematic window of interest (below multi particle thresholds) there are no singularities at finite values of the intermediate momenta**

All orders "perturbation" theory

- Using the kernels of the previous slide we can define a skeleton expansion for the matrix element

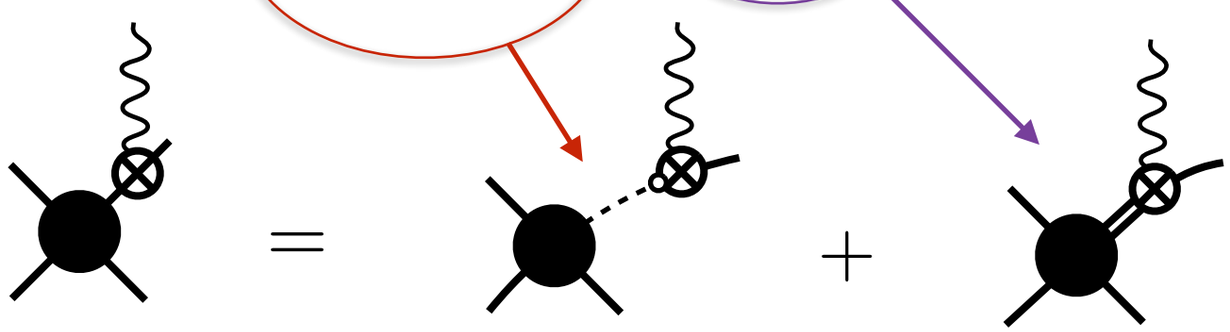
$iW =$



Can have kinematic singularities

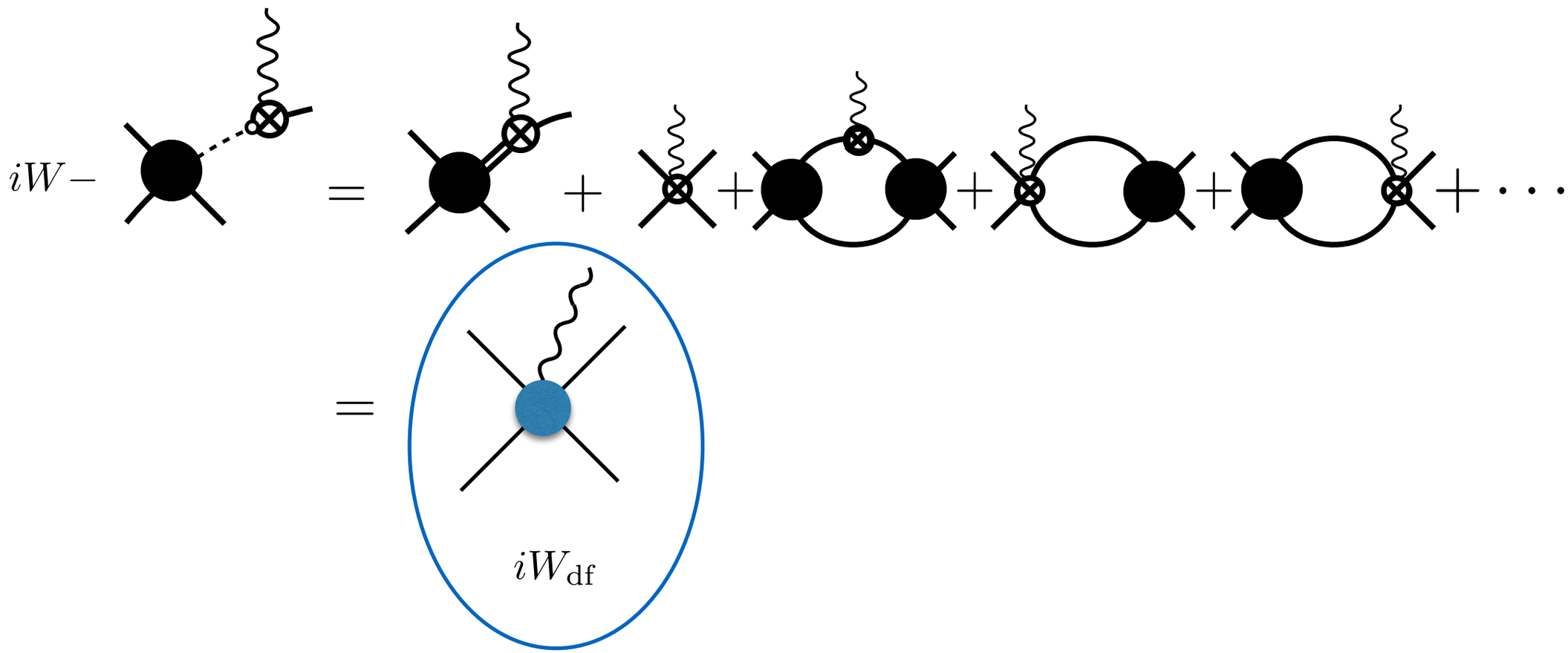
- For fully dressed propagators

$$i\Delta = \frac{i}{k^2 - m^2 + i\epsilon} + \text{smooth}$$



- Briceño & Hansen (2016)
- AB, Briceño, Hansen, Ortega (2018)

Two-to-two scattering with current

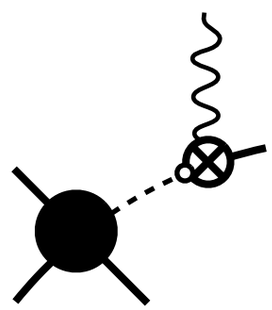


- The object related to the FV matrix element is not the full scattering amplitude
- We can recover the full scattering amplitude adding back diagrams with kinematic singularities

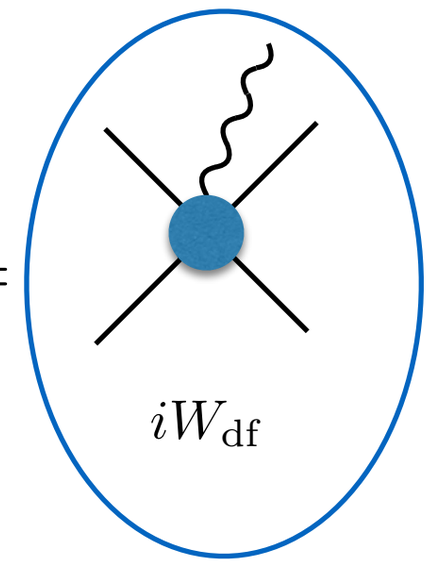
Two-to-two scattering with current

For resonances and bound states

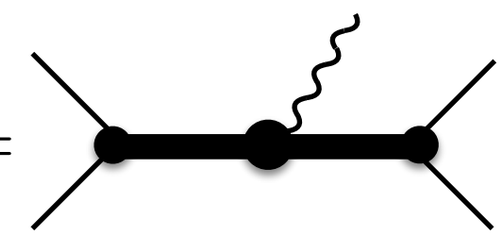
$$\lim_{E_i^{\rm cm}, E_f^{\rm cm} \to E_R \ (E_B)} iW -$$



$$- \cdots =$$



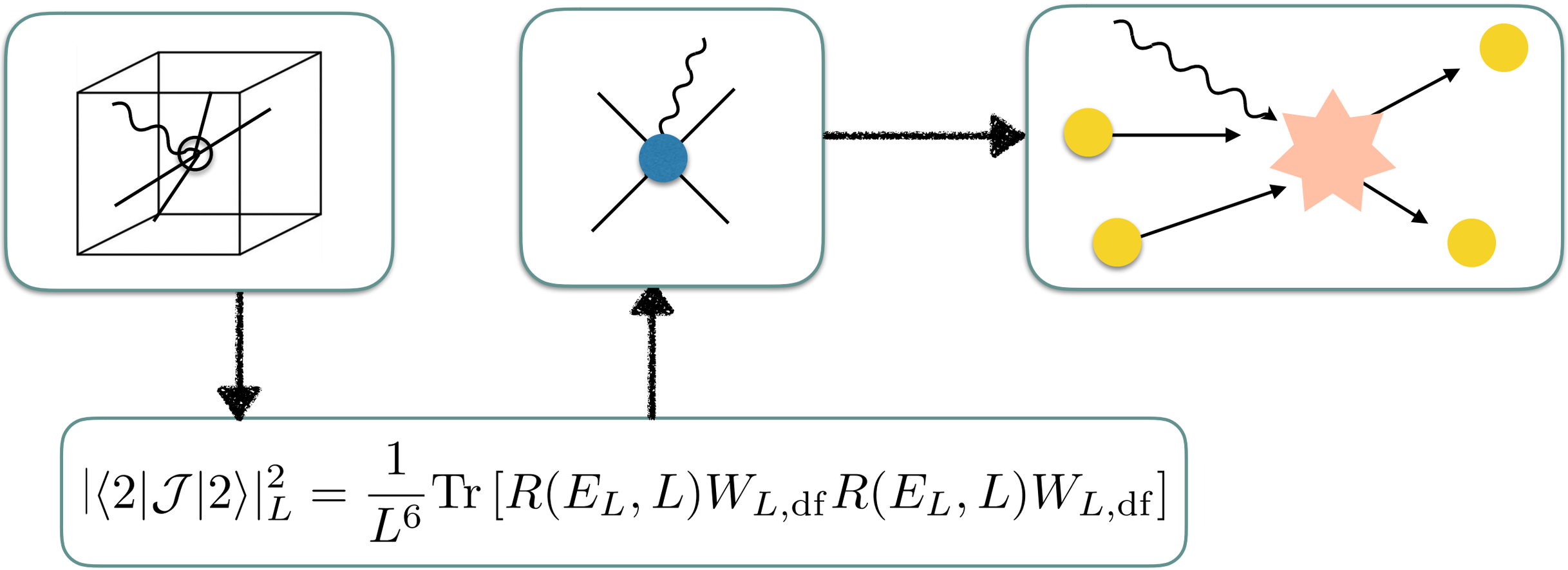
$$=$$



Form factors directly related to $W_{\rm df}$

$2+\mathcal{J} \to 2$

FV matrix elements to infinite volume electroweak amplitudes



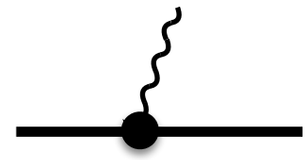
- Briceño & Hansen (2016)
- AB, Briceño, Hansen, Ortega (2018)

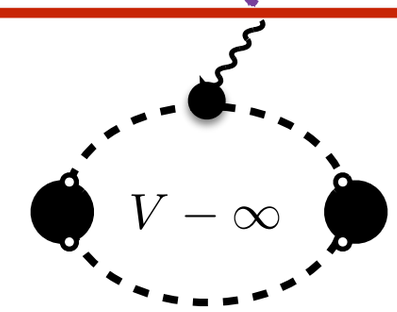
$2+\mathcal{J}\rightarrow 2$

$$\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L,L)W_{L,\mathrm{df}}R(E_L,L)W_{L,\mathrm{df}}\right]$$

$$W_{L,\mathrm{df}} = W_{\mathrm{df}} + MG(L,w)M$$

$M =$ 2→ 2 scatt. Amplitude Calculated at FV energies

$w =$ 



$$G = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right](\cdots)$$

Crucial ingredient that we studied

$2+\mathcal{J}\to 2$

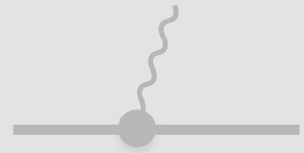
Special case

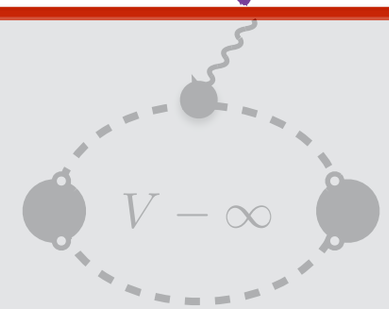
$$\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L,L)W_{L,\mathrm{df}}R(E_L,L)W_{L,\mathrm{df}}\right]$$

$$W_{L,\mathrm{df}} = W_{\mathrm{df}} + MG(L,w)M$$

$M =$

2→ 2 scatt. Amplitude
Calculated at FV energies

$w =$ 



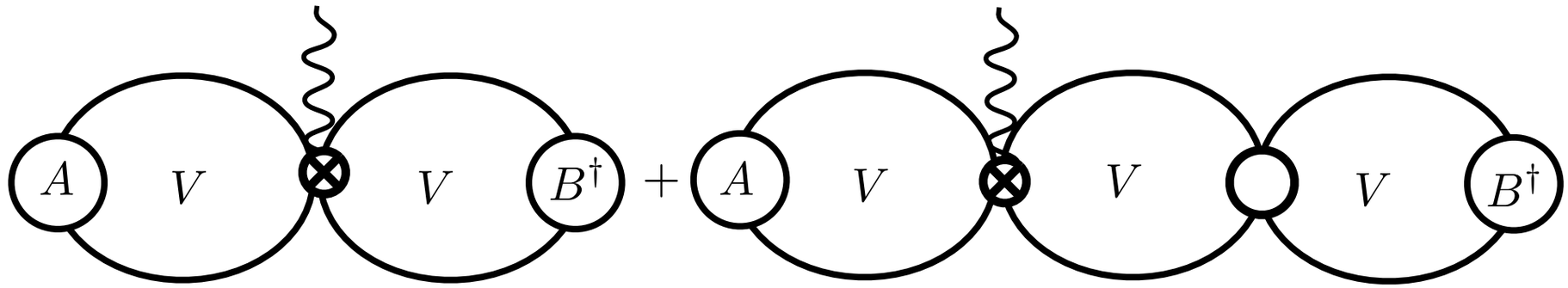
$$G = \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right](\cdots)$$

Crucial ingredient that we studied

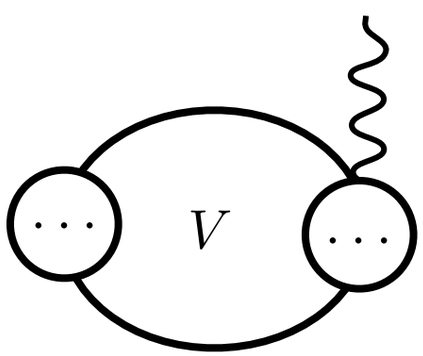
Finite volume correlator

Special case

$C_L^{3\,pt.}(P_i, P_f) =$



$+ \cdots$



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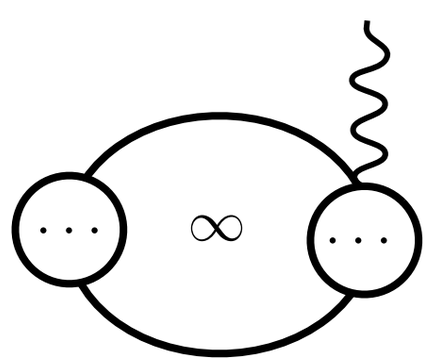
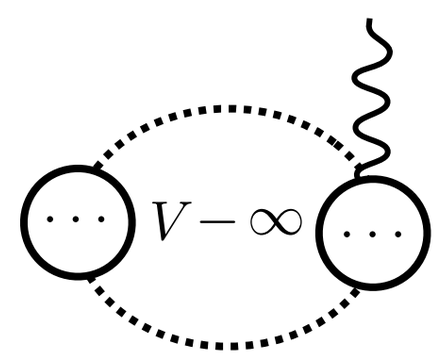


$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \int dk^0 \frac{(\cdots)\,(\cdots)}{(k^2 - m^2 + i\epsilon)[(P-k)^2 - m^2 + i\epsilon]}$$

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \frac{(\cdots)\,(\cdots)|_{k^0=\omega_k}}{(2\omega_k)^2(E - 2\omega_k + i\epsilon)} =$$



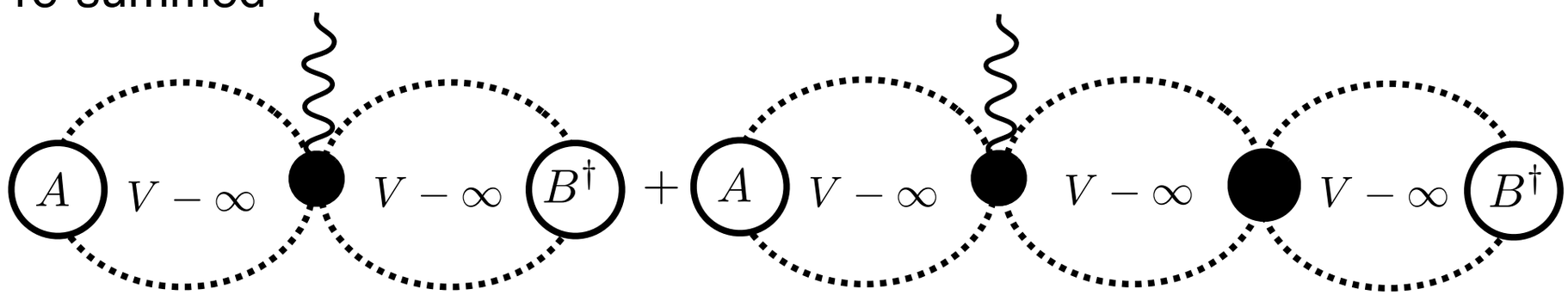
$$= (\cdots)|_{\text{on}}\, iF\, (\cdots)|_{\text{on}}$$

Finite volume correlator

Special case

- Series below can be re-summed

$C_L^{3\,pt.}(P_i, P_f) =$



$$+ \cdots$$

$$= A(P_i)\frac{1}{F^{-1}(P_i, L) + M(P_i)} W(P_i, P_f) \frac{1}{F^{-1}(P_f, L) + M(P_f)} B^\dagger(P_f)$$

$$+ \cdots$$

- Fourier transform of the correlator

$$C_L^{3\,pt.}(t_i, t_f, \mathbf{P}_i, \mathbf{P}_f) = L^3 \int \frac{dP_i^0}{2\pi} \int \frac{dP_f^0}{(2\pi)} e^{iP_i^0 t_i + iP_f^0 t_f} C_L^{3\,pt.}(P_i, P_f)$$

$$= L^3 \sum_{n_i, n_f} e^{-(E_{n_i} t_i + E_{n_f} t_f)} A(E_{n_i}, \mathbf{P}_i) R(E_{n_i}, \mathbf{P}_i) W(P_i, P_f) R(E_{n_f}, \mathbf{P}_f) B^\dagger(E_{n_f}, \mathbf{P}_f)$$

Finite volume correlator

Special case

- From previous slide

$$C_L^{3\,pt.}(t_i, t_f, \mathbf{P}_i, \mathbf{P}_f) = L^3 \sum_{n_i, n_f} e^{-(E_{n_i} t_i + E_{n_f} t_f)} A(E_{n_i}, \mathbf{P}_i) R(E_{n_i}, \mathbf{P}_i) W(P_i, P_f) R(E_{n_f}, \mathbf{P}_f) B^\dagger(E_{n_f}, \mathbf{P}_f)$$

- From standard spectral decomposition of the matrix element

$$C_L^{3\,pt.}(t_i, t_f, \mathbf{P}_i, \mathbf{P}_f) = L^9 \sum_{n_i, n_f} e^{-(E_{n_i} t_i + E_{n_f} t_f)} \left[\langle 0|A(0)|E_{n_i}, \mathbf{P}_i, L\rangle\right]_L \left[\langle E_{n_i}, \mathbf{P}_i, L|\mathcal{J}(0)|E_{n_f}, \mathbf{P}_f, L\rangle\right]_L \left[\langle E_{n_f}, \mathbf{P}_f, L|B^\dagger(0)|0\rangle\right]_L$$

- Matching the two expressions we obtain the wanted relation (and few more passages..)

$$\left[\langle E_{n_i}, \mathbf{P}_i, L|\mathcal{J}(0)|E_{n_f}, \mathbf{P}_f, L\rangle\right]_L^2 = \frac{1}{L^6} \mathrm{Tr}\left[W(P_i, P_f) R(E_{n_f}, \mathbf{P}_f) W(P_f, P_i) R(E_{n_i}, \mathbf{P}_i)\right]$$

Finite volume correlator

Special case

- From previous slide

$$C_L^{3\,pt.}(t_i,t_f,\mathbf{P}_i,\mathbf{P}_f) = L^3 \sum_{n_i,n_f} e^{-(E_{n_i}t_i+E_{n_f}t_f)} A(E_{n_i},\mathbf{P}_i)R(E_{n_i},\mathbf{P}_i)W(P_i,P_f)R(E_{n_f},\mathbf{P}_f)B^\dagger(E_{n_f},\mathbf{P}_f)$$

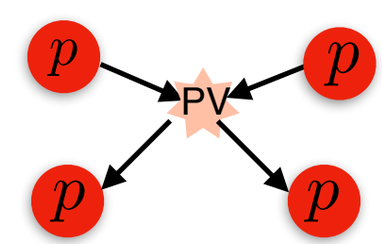
- From standard spectral decomposition of the matrix element

$$C_L^{3\,pt.}(t_i,t_f,\mathbf{P}_i,\mathbf{P}_f) = L^9 \sum_{n_i,n_f} e^{-(E_{n_i}t_i+E_{n_f}t_f)} \left[\langle 0|A(0)|E_{n_i},\mathbf{P}_i,L\rangle\right]_L \left[\langle E_{n_i},\mathbf{P}_i,L|\mathcal{J}(0)|E_{n_f},\mathbf{P}_f,L\rangle\right]_L \left[\langle E_{n_f},\mathbf{P}_f,L|B^\dagger(0)|0\rangle\right]_L$$

- Matching the two expressions we obtain the wanted relation (and few more passages..)

$$\left[\langle E_{n_i},\mathbf{P}_i,L|\mathcal{J}(0)|E_{n_f},\mathbf{P}_f,L\rangle\right]_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[W(P_i,P_f)R(E_{n_f},\mathbf{P}_f)W(P_f,P_i)R(E_{n_i},\mathbf{P}_i)\right]$$

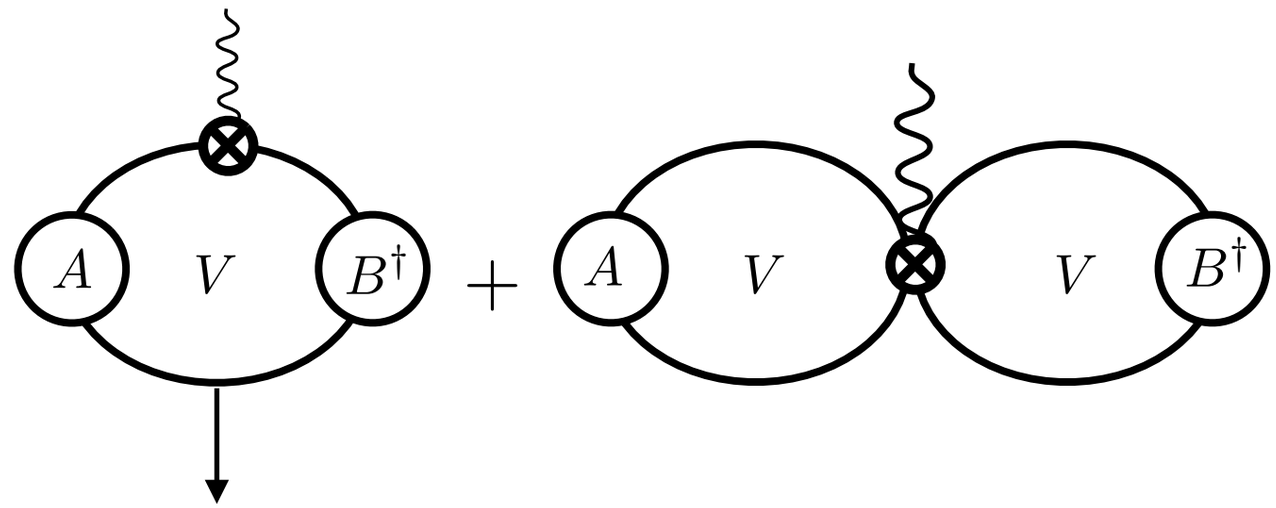
Theories without $1+\mathcal{J}\to 1$ processes



Kurth et al. (2015)

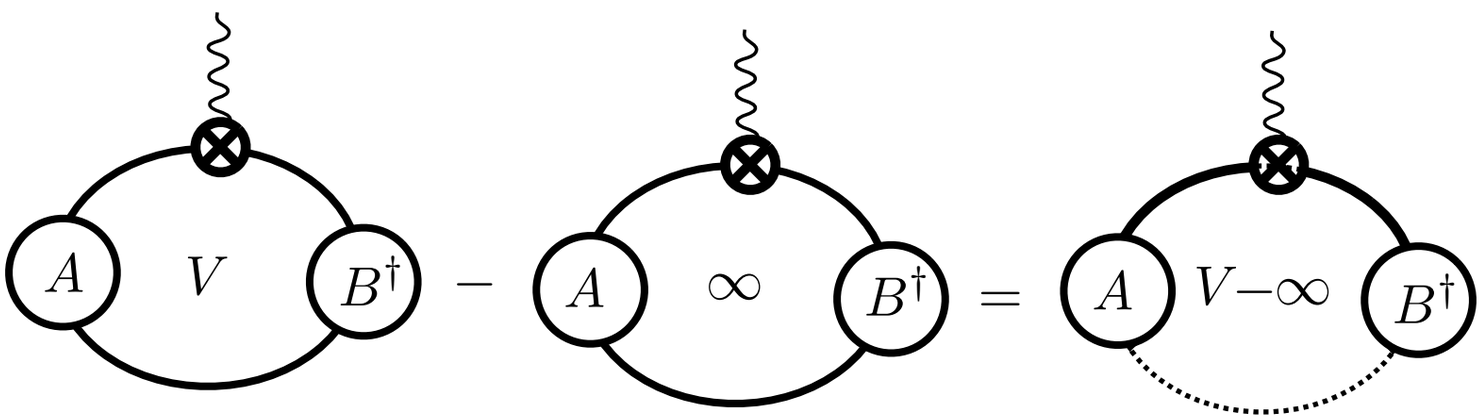
Finite volume correlator

$$C_L^{3\,pt.}(P_i, P_f) =$$



$+\cdots$

New topology not present before

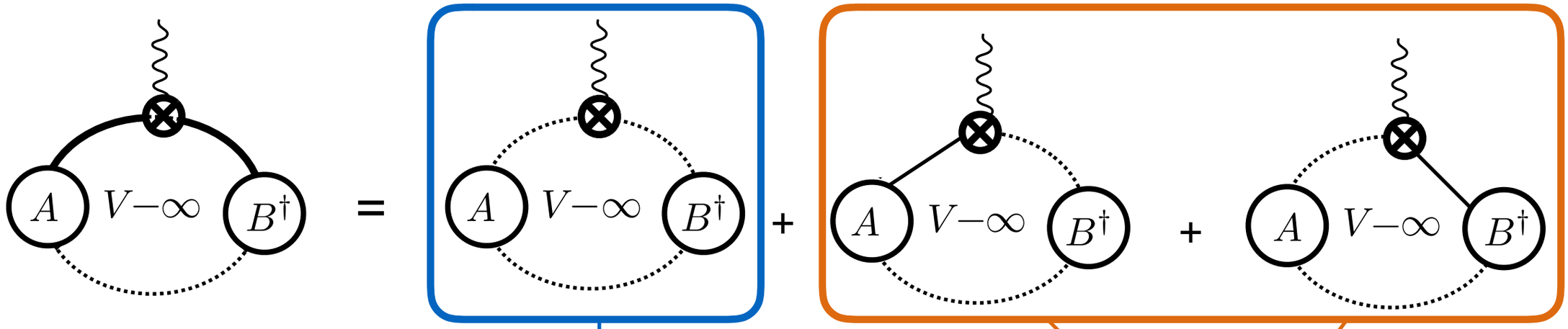


$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \int dk^0 \frac{A(P_i)\, w(P_i, Pf, k)\ B^\dagger(P_f)}{(k^2 - m^2 + i\epsilon)[(P_i - k)^2 - m^2 + i\epsilon][(P_f - k)^2 - m^2 + i\epsilon]}$$

$$= \left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int d\mathbf{k}\right] \frac{A(P_i)\, w(P_i, Pf, k)\ B^\dagger(P_f)}{2\omega_k[(P_i - k)^2 - m^2 + i\epsilon][(P_f - k)^2 - m^2 + i\epsilon]}\Bigg|_{k_0=\omega_k}$$

+exp suppressed terms

Finite volume correlator



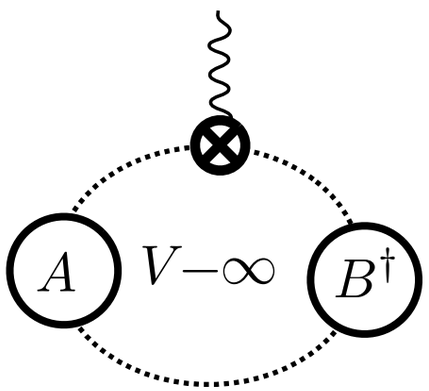
They will give the F function

$$G(P_i, P_f, L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right) f(P_i, P_f, \mathbf{k})$$

Inserting the above contributions in the previously showed skeleton expansion we obtain the final equation

$$|\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6} \mathrm{Tr}\left[R(E_L, L) W_{L,\mathrm{df}} R(E_L, L) W_{L,\mathrm{df}}\right]$$

New kinematic function



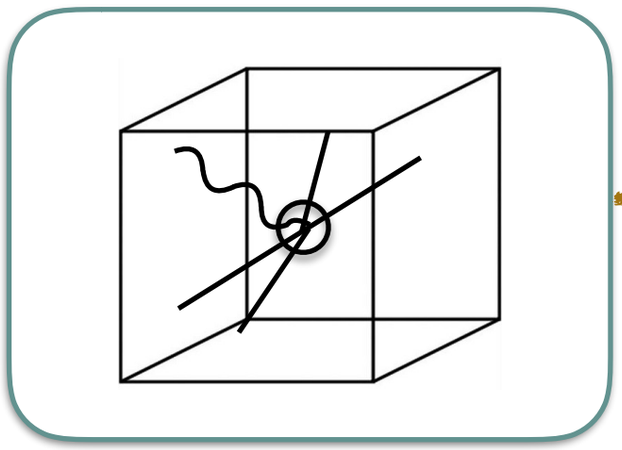
$$= \quad G(P_i, P_f, L) = \left(\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right) f(P_i, P_f, \mathbf{k})$$

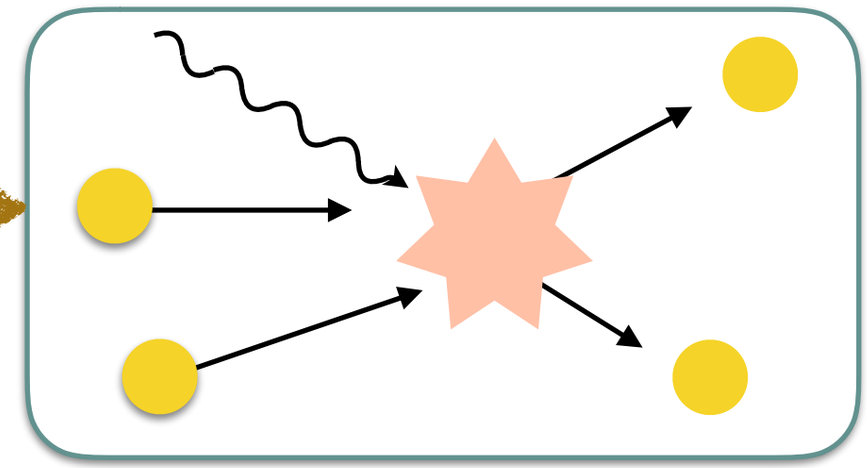
- The sum is “straightforward”
- The integral is highly not trivial (spectator particle goes on-shell)
  - integrand singularities are two surfaces in three-dimension
  - using mathematical trickery we can isolate the singularities, treat them with standard field theory techniques, and be left with a 3D smooth integral
  - Solution holds for generic momenta

Interesting… but how did you do it? where are some plots?

For equal initial and final momenta it has been shown that the above problem has an analytical solution

Outlook





$$\langle 2|\mathcal{J}|2\rangle|_L^2 = \frac{1}{L^6}\mathrm{Tr}\left[R(E_L,L)W_{L,\mathrm{df}}R(E_L,L)W_{L,\mathrm{df}}\right]$$

- ✓ Framework developed and under control
- Details of the implementation need to be fully understood.. (toy model analysis)
- Use this framework in an actual LQCD calculation

Form factors of resonances and bound states

Generalization to systems with spin
Similar study for baryons

Thank you!