

Lattice QCD Calculations of Hadron Structure

Joe Karpie (JLab) part of the HadStruc Collaboration

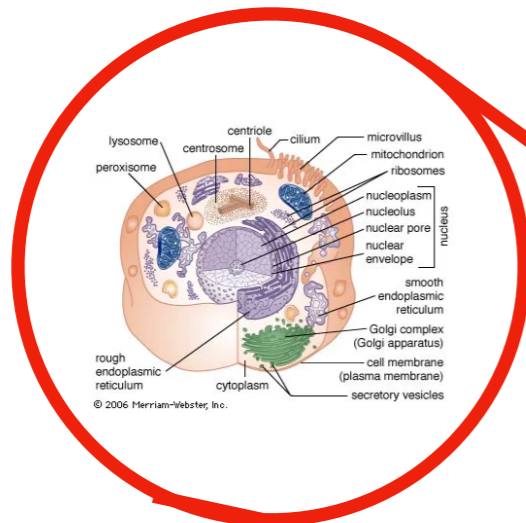
The logo for Jefferson Lab, featuring a red swoosh that starts from the bottom left, curves upwards and to the right, and then loops back down to the bottom left, ending in a red sphere. The text "Jefferson Lab" is written in a bold, black, sans-serif font across the middle of the swoosh.**Jefferson Lab**

Digging Deeper and Deeper



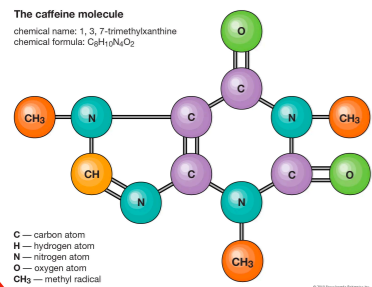
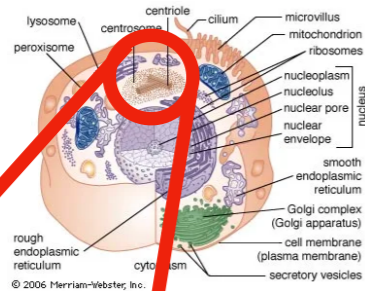
- Typical humans in Highline Park (NYC)

Digging Deeper and Deeper



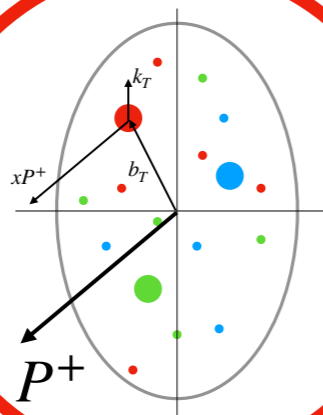
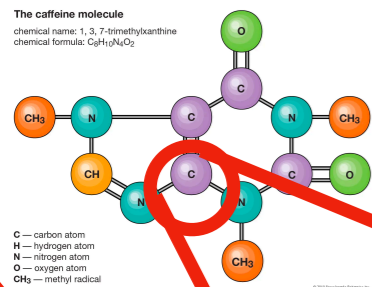
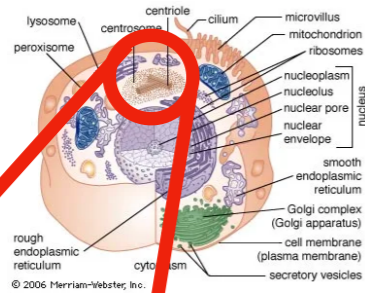
- Typical humans in Highline Park (NYC)

Digging Deeper and Deeper



- Typical humans in Highline Park (NYC)

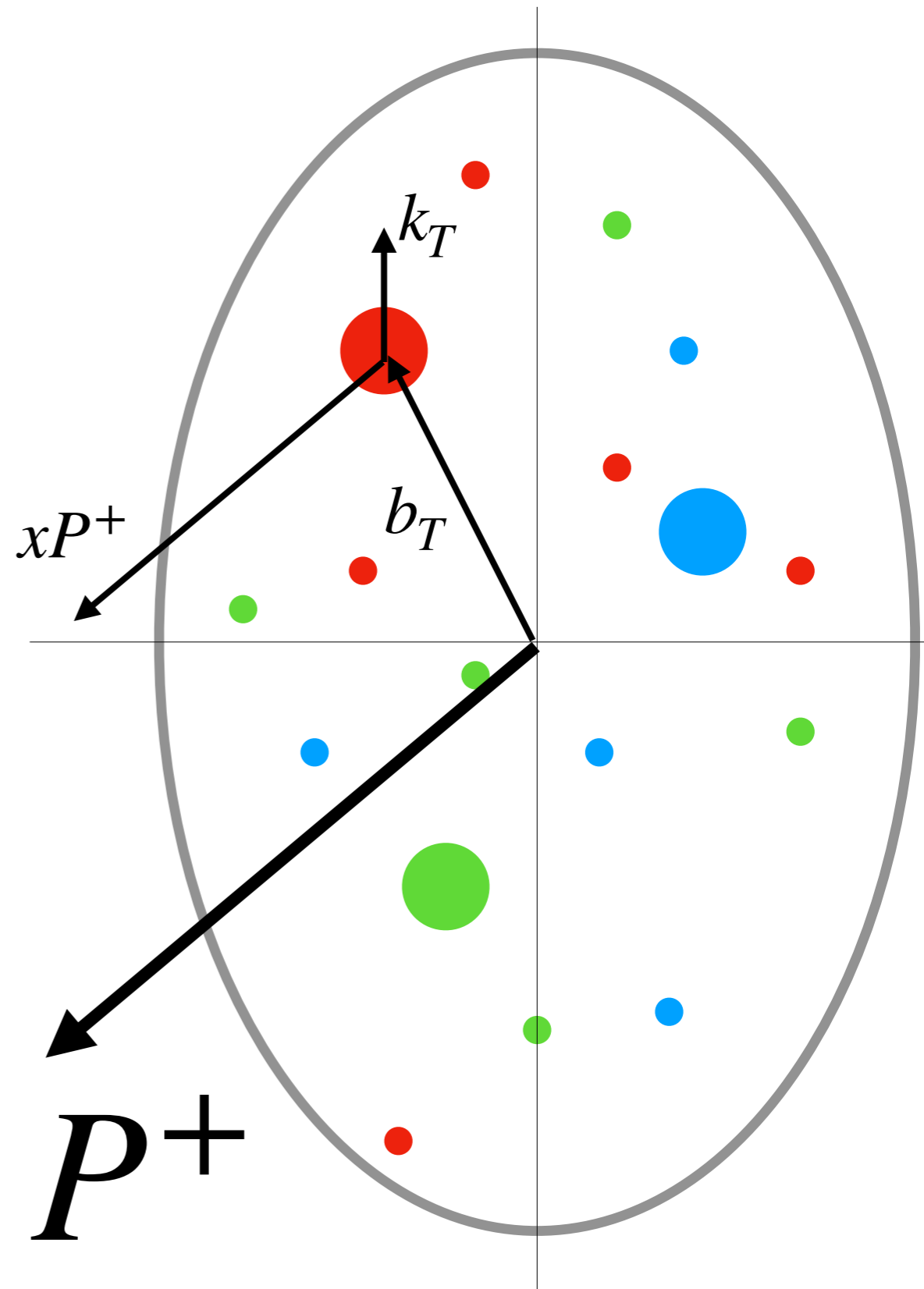
Digging Deeper and Deeper



- Typical humans in Highline Park (NYC)

Parton Structure

- Internal structure of hadrons
- How are quarks and gluons distributed?
- How do they contribute to cross sections, hadron mass, spin,?
- **Longitudinal** structure from momentum fraction x
- **Transverse** structure from
 - Impact parameter b_T
 - Transverse momentum k_T



Parton Structure

For various flavors and spin combinations

Wigner Distribution/
Generalized Transverse Momentum
Distribution (GTMD)

$$F(x, b_T, k_T)$$

$$\int d^2b_t$$

$$\int d^2k_t$$

Transverse Momentum
Distribution (TMD)

$$f(x, k_T)$$

Generalized Parton
Distribution (GPD)

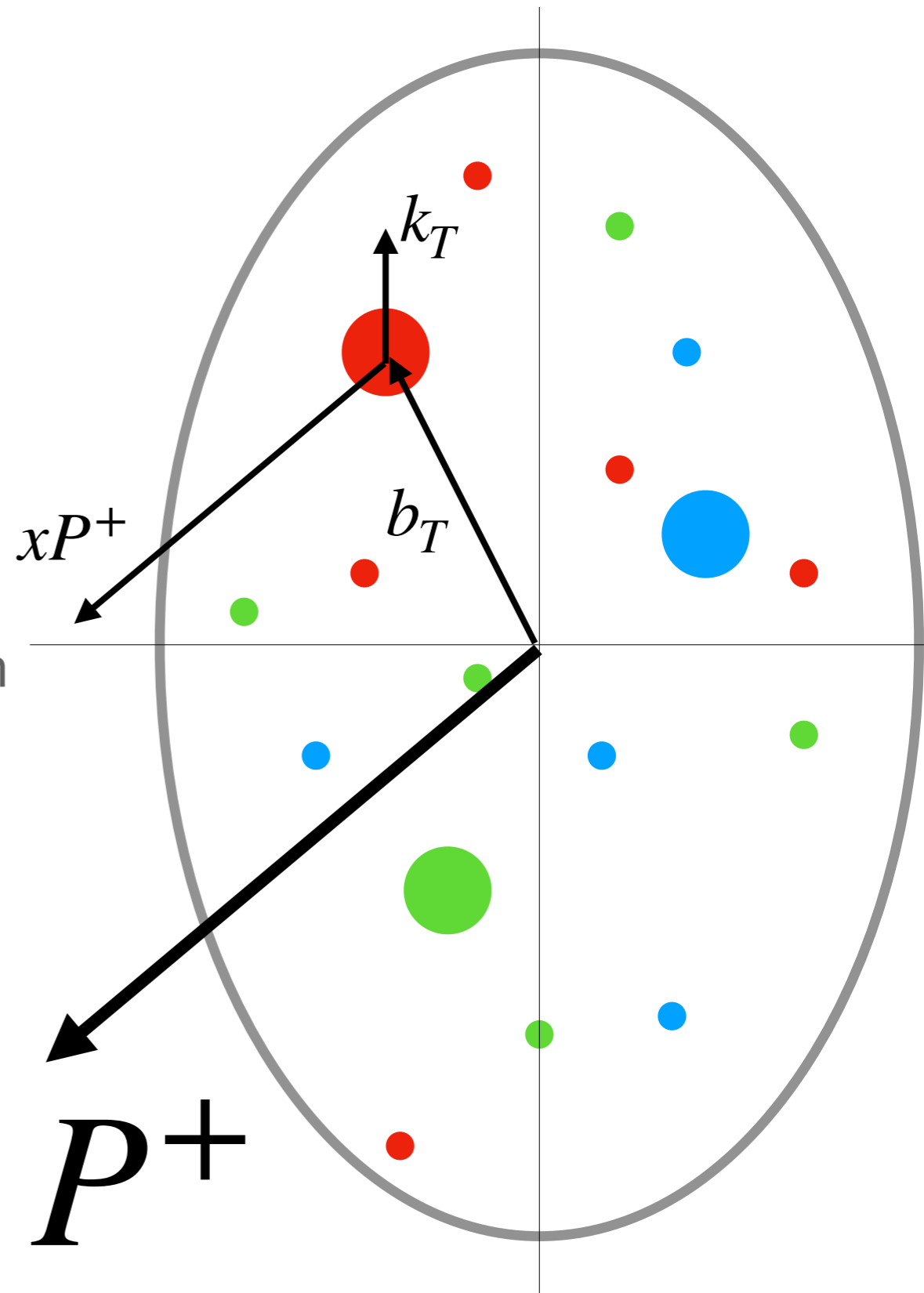
$$f(x, b_T)$$

$$\int d^2k_t$$

$$\int d^2b_t$$

Parton Distribution Function (PDF)

$$f(x)$$



Parton and Ioffe Time distributions

- Unpolarized Ioffe time distributions

Ioffe time: $\nu = p \cdot z$

V. Braun, P. Gornicki, L. Mankiewicz
Phys Rev D 51 (1995) 6036-6051

- $$I_q(\nu, \mu^2) = \frac{1}{2p^+} \langle p | \bar{\psi}_q(z^-) \gamma^+ W(z^-; 0) \psi_q(0) | p \rangle_\mu$$

$$z^2 = 0$$

- $$I_g(\nu, \mu^2) = \frac{1}{(2p^+)^2} \langle p | F_{+i}(z^-) W(z^-; 0) F_+^i(0) | p \rangle_\mu$$

$$i = x, y$$

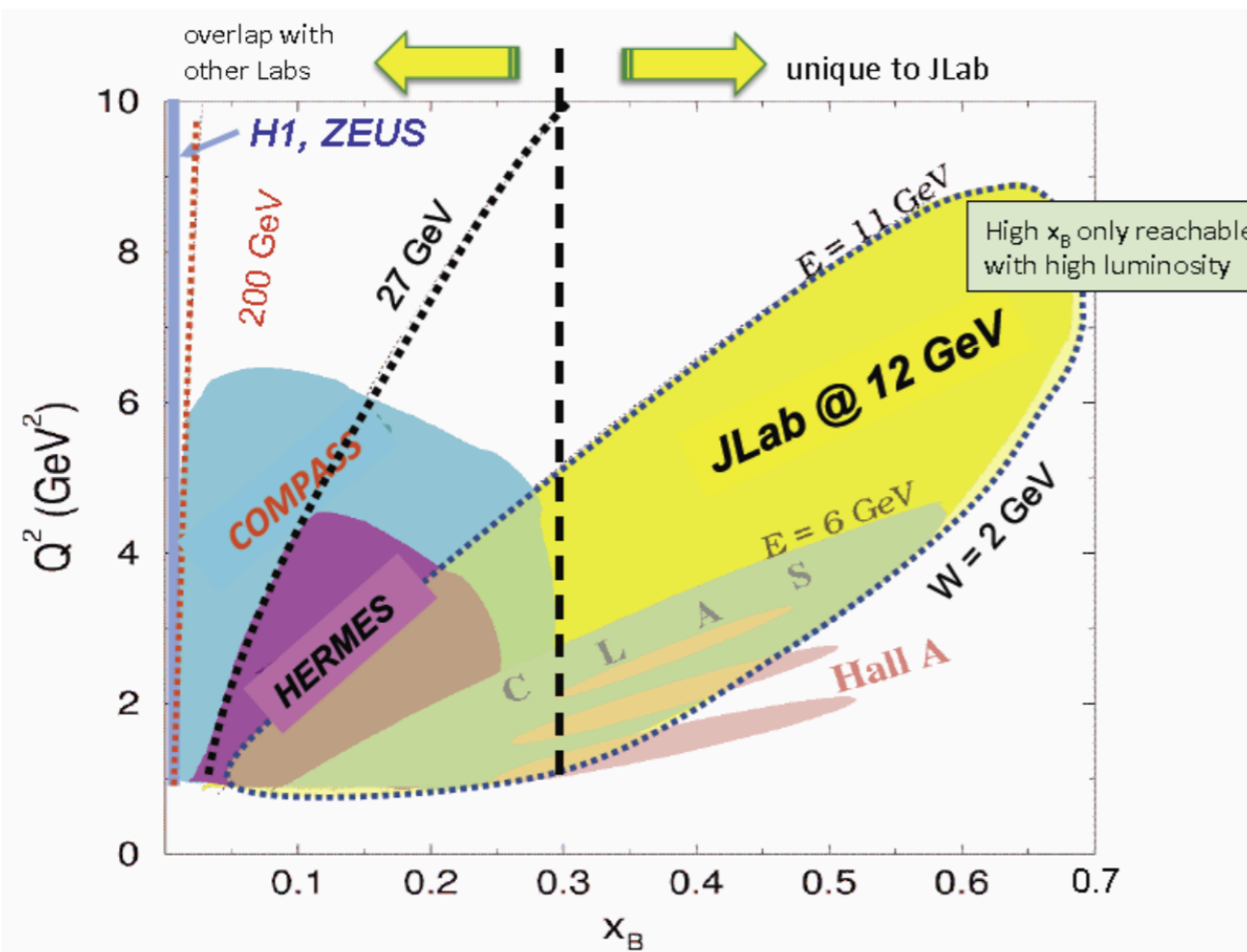
- Parton Distribution Functions

- $$I_q(\nu, \mu^2) = \int_{-1}^1 dx e^{ix\nu} f_q(x, \mu^2)$$

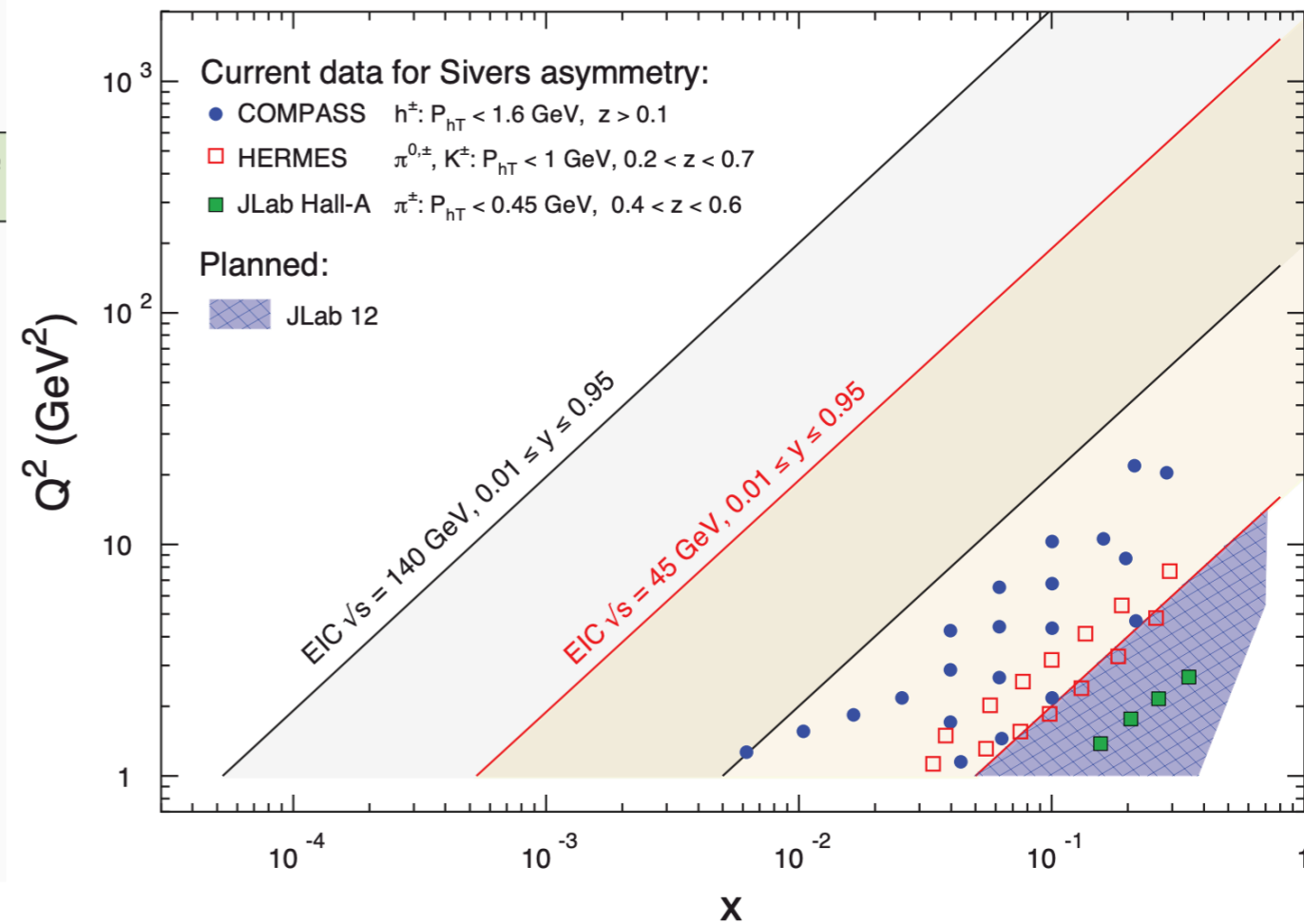
- $$I_g(\nu, \mu^2) = \int_0^1 dx \cos(x\nu) x f_g(x, \mu^2)$$

JLab 12 GeV and the EIC

- Expand Kinematic Coverage
 - JLab extends to large x (also where lattice is most sensitive)
 - Unpolarized gluon PDF dominates low x and is goal of EIC
- Expand to polarized and 3D distributions (lattice goal as well)



JLab 12 GeV White Paper:
Eur. Phys. J. A 48 (2012) 187



EIC White Paper:
Eur. Phys. J. A 52 (2016) 9, 268

Parton Distributions and the Lattice

- Parton Distributions are defined by operators with light-like separations

$$M(p, z) = \langle p | \bar{\psi}(z) \gamma^\alpha W(z; 0) \psi(0) | p \rangle$$

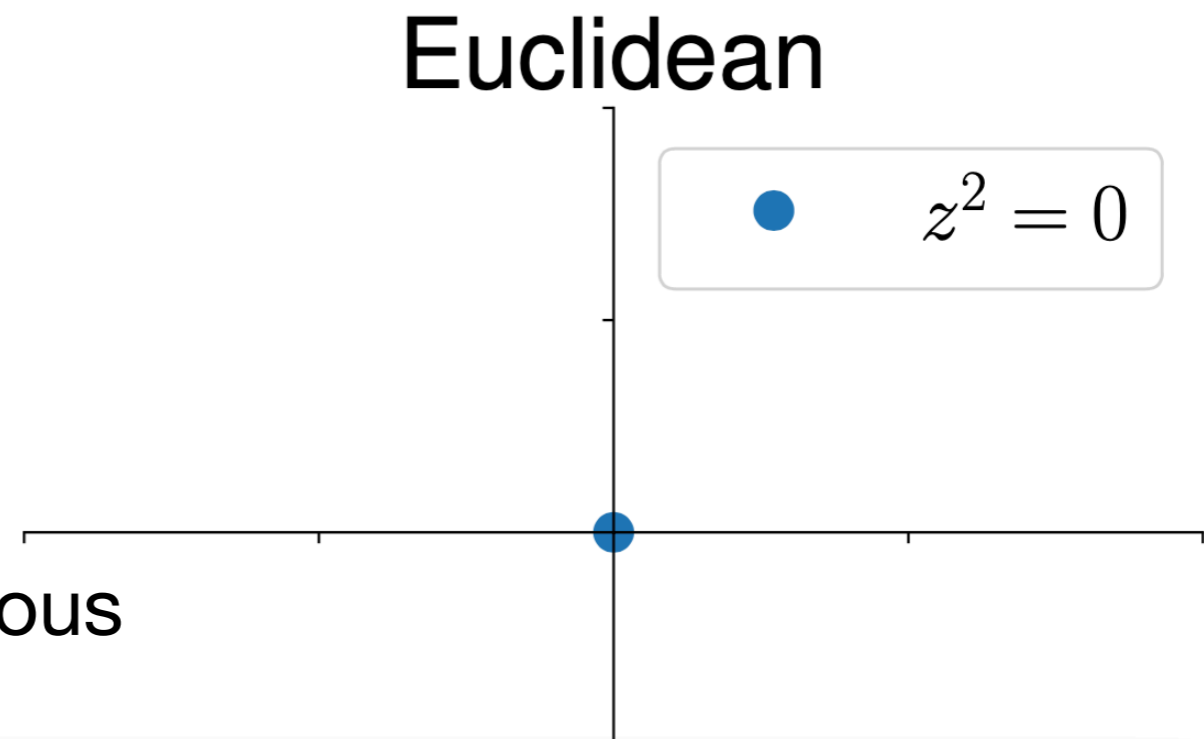
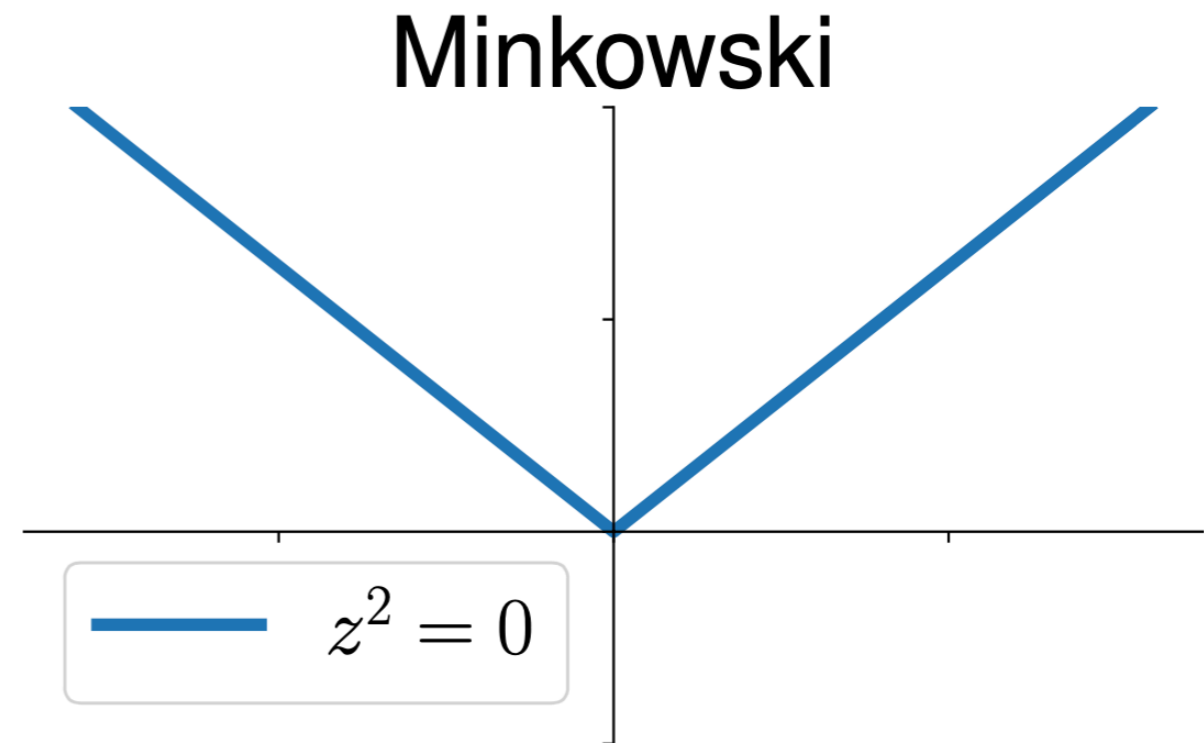
- Use space-like separations $z^2 \neq 0$

X. Ji Phys Rev Lett 110 (2013) 262002

- Wilson line operators or pairs of space-like separated currents

- Fourier transformations of matrix elements give PDF in certain limits

- Factorizable matrix elements analogous to cross sections



Many approaches

- **Wilson line operators**

$$O_{WL}(x; z) = \bar{\psi}(x + z)\Gamma W(x + z; x)\psi(x)$$

- LaMET *X. Ji Phys. Rev. Lett.* 110 (2013) 262002

- Pseudo-PDF *A. Radyushkin Phys. Rev. D* 96 (2017) 3, 034025

- **Two current correlators**

- **Hadronic Tensor**

K.-F. Liu et al Phys. Rev. Lett. 72 1790 (1994)

- **HOPE**

Phys. Rev. D 62 (2000) 074501

W. Detmold and C.-J. D. Lin, Phys. Rev. D 73 (2006) 014501

- **Short distance OPE**

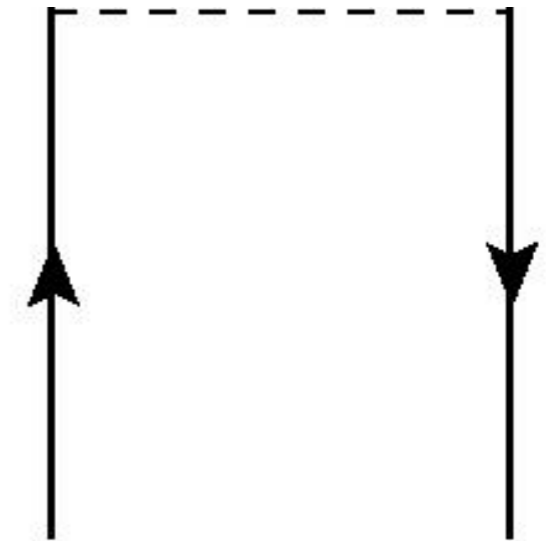
V. Braun and D. Muller Eur. Phys. J. C 55 (2008) 349

- **OPE-without-OPE**

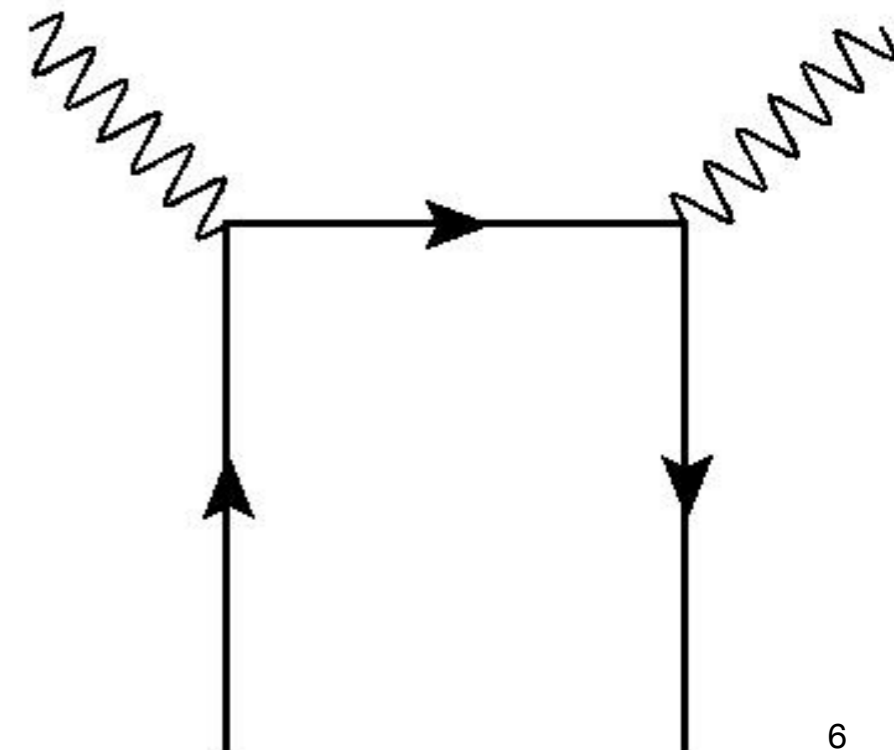
A. Chambers et al, Phys. Rev. Lett. 118 (2017) 242001

- **Good Lattice Cross Sections**

Y.-Q. Ma and J.-W. Qiu Phys. Rev. Lett. 120 (2018) 2, 022003



$$O_{CC}(x, y) = J_{\Gamma}(x)J_{\Gamma'}(y)$$



The Role of Separation and Momentum

- The separation and momenta will play the role x_B and Q^2 is DIS
- In **quasi-PDF** and **pseudo-PDF**, separation and momentum swap roles

Scale:

$$Q^2 / p_z^2 / z^2$$

Dynamical variable:

$$x_B / z / p_z, \text{ or } \nu = p \cdot z$$

- Scale for factorization to PDF
- Scale in power expansion
- Keep away from Λ_{QCD}^2
- Technically only requires single value
- Variable for inverse problem
- Can take large or small value
- Want as many as are available
- Wider range improves the inverse problem

**What do we do with this
data?**

Inverse Problems for pseudo-PDFs

- Limited range of z and p cannot approach $\nu \rightarrow \infty$ to integrate inverse

$$q(x) = \int_0^{\infty} d\nu C^{-1}(x\nu) \mathfrak{M}(\nu)$$

Inverse Problems for pseudo-PDFs

- Limited range of z and p cannot approach $\nu \rightarrow \infty$ to integrate inverse
- Forward integral to an ill posed matrix equation

$$q(x) = \int_0^{\infty} d\nu C^{-1}(x\nu) \mathfrak{M}(\nu)$$

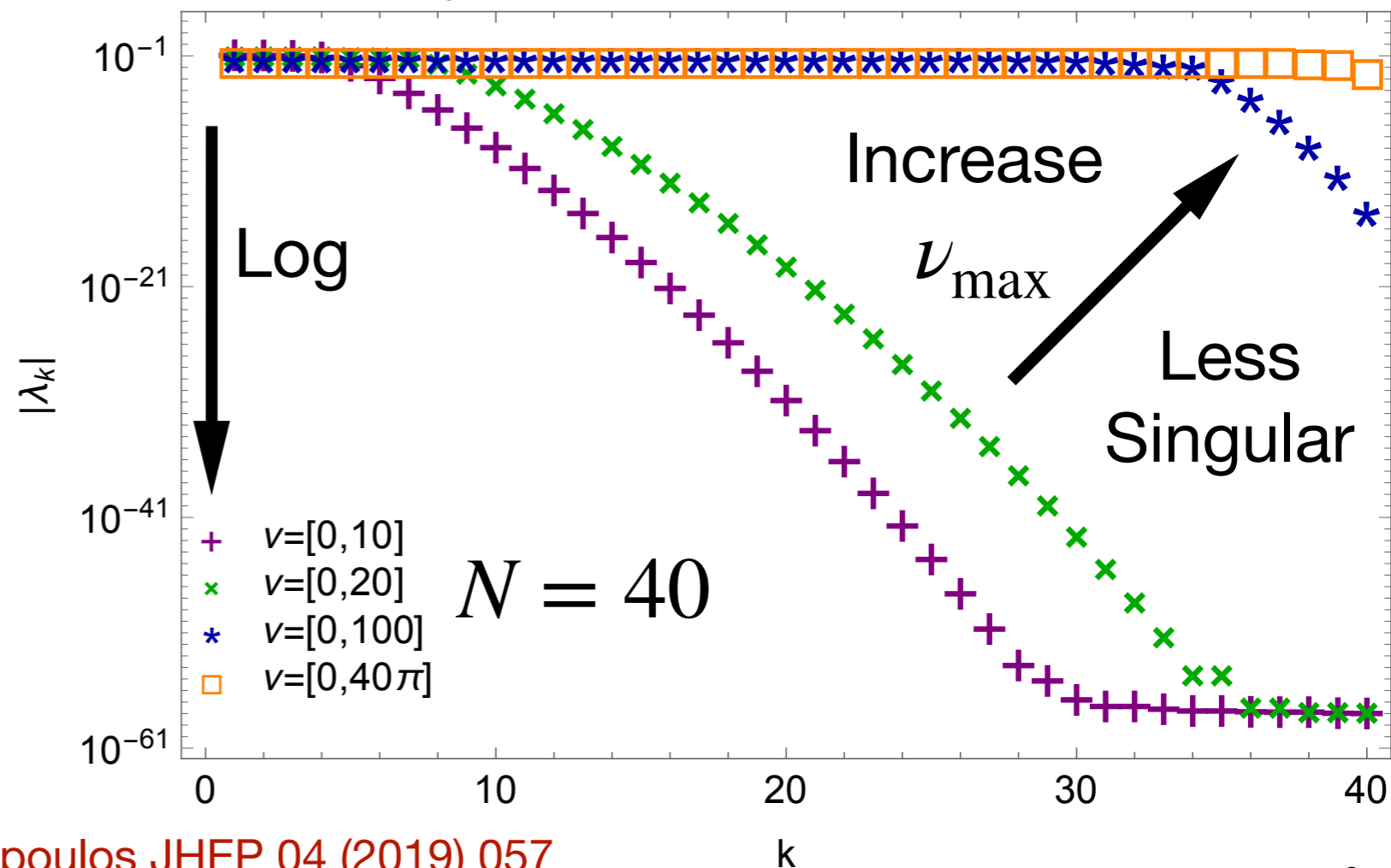
$$\mathfrak{M}(\nu) = \int_0^1 dx C(x\nu) q(x) \rightarrow [\mathbf{C}][\mathbf{q}]$$

Inverse Problems for pseudo-PDFs

- Limited range of z and p cannot approach $\nu \rightarrow \infty$ to integrate inverse
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$$q(x) = \int_0^\infty d\nu C^{-1}(x\nu) \mathfrak{M}(\nu)$$

$$\mathfrak{M}(\nu) = \int_0^1 dx C(x\nu) q(x) \rightarrow [\mathbf{C}][\mathbf{q}]$$



Inverse Problems for pseudo-PDFs

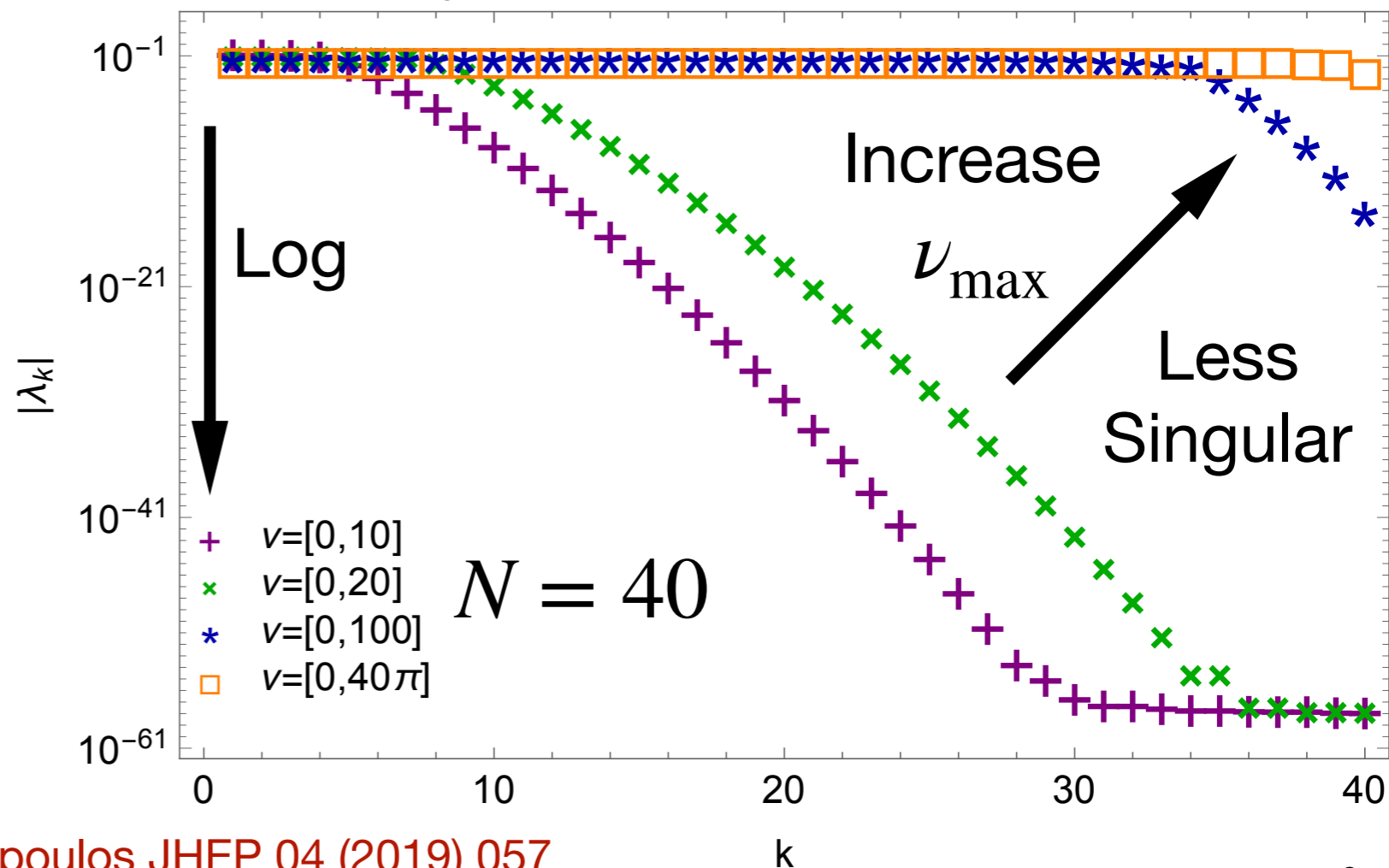
- Limited range of z and p cannot approach $\nu \rightarrow \infty$ to integrate inverse

$$q(x) = \int_0^\infty d\nu C^{-1}(x\nu) \mathfrak{M}(\nu)$$

- Forward integral to an ill-posed matrix equation

$$\mathfrak{M}(\nu) = \int_0^1 dx C(x\nu) q(x) \rightarrow [\mathbf{C}][\mathbf{q}]$$

- Must be regulated by additional information
 - Restricted functional form
 - Constraints on the PDF or parameters
 - Assumptions of smoothness, continuity,



Inverse Problems for Parton Physics

- **Structure Functions**

$$F_2(x, Q^2) = \sum_i \int_x^1 d\xi C(\xi, \frac{\mu^2}{Q^2}) q(\frac{x}{\xi}, \mu^2)$$

- **Local Matrix elements / HOPE / OPE-without-OPE**

- **LaMET (on the lattice)**

$$M(p_z, z) = \int_{-\infty}^{\infty} dy e^{iyp_z z} \tilde{q}(y, p_z^2)$$

$$a_n(\mu^2) = \int_{-1}^1 dx x^{n-1} q(x, \mu^2)$$

- **pseudo-Distributions / Good Lattice Cross Sections**

$$\mathfrak{M}(\nu, z^2) = \int_{-1}^1 dx C(x\nu, \mu^2 z^2) q(x, \mu^2)$$

- **Hadronic Tensor**

$$\tilde{W}_{\mu\nu}(\tau) = \int d\nu e^{-\nu\tau} W_{\mu\nu}(\nu)$$

Unknown Functions

- **Lattice systematic errors**

$$\mathfrak{M}^{\text{latt}}(p, z, a) = \mathfrak{M}^{\text{cont}}(\nu, z^2) + \sum_n \left(\frac{a}{|z|} \right)^n P_n(\nu) + \left(a\Lambda_{\text{QCD}} \right)^n R_n(\nu) + \dots$$

- **Power Corrections**

$$\mathfrak{M}^{\text{cont}}(p, z, a) = \mathfrak{M}^{\text{lt}}(\nu, z^2) + \sum_n \left(z^2 \Lambda_{\text{QCD}}^2 \right)^n B_n(\nu)$$

- **Factorization**

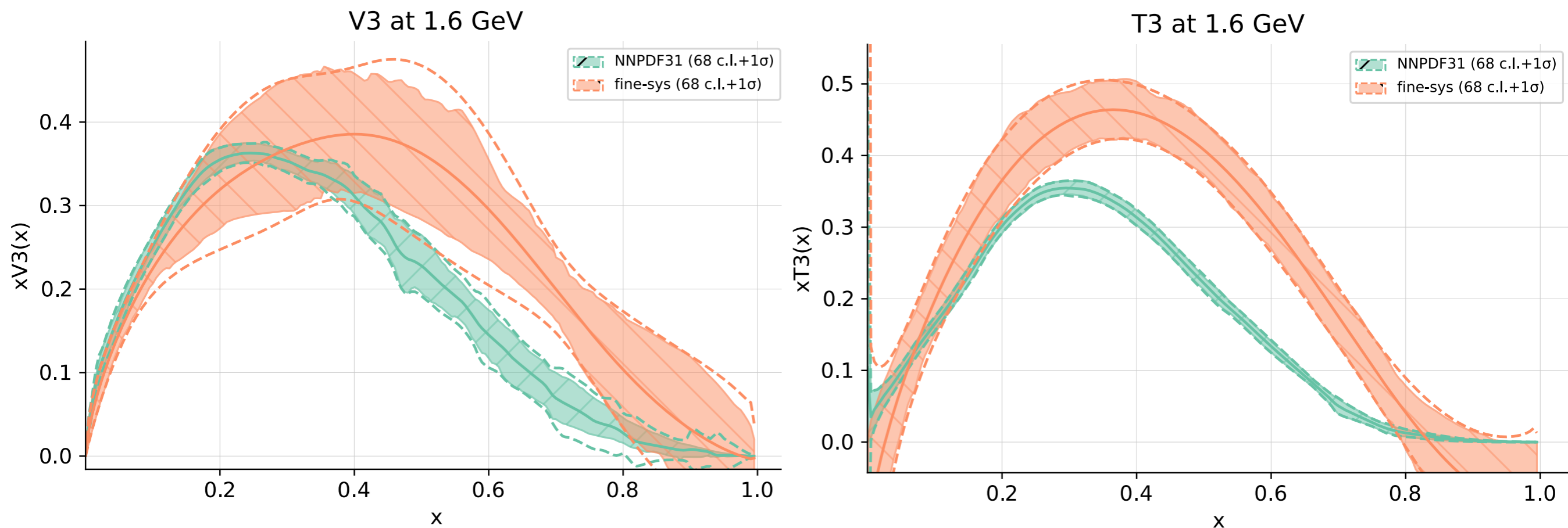
$$[\text{Re/Im}] \mathfrak{M}_{\text{lt}}(\nu, z^2) = \int_0^1 dx K_{\text{R/I}}(x\nu, \mu^2 z^2) q_{\mp}(x, \mu^2)$$

$$K_R(x\nu, \mu^2 z^2) = \cos(x\nu) + O(\alpha) \quad K_I(x\nu, \mu^2 z^2) = \sin(x\nu) + O(\alpha)$$

Neural Networks for Inverse Problems

- NNPDF Analysis with NN geom 2-5-3-1

L. Del Debbio, T. Giani, JK, K. Orginos, A. Radyushkin, S. Zafeiropoulos JHEP 02 (2021) 138



$$V_3 = q_- = q_v \quad T_3 = q_+$$

Jacobi Polynomials

- Change of variables from true Jacobi polynomials

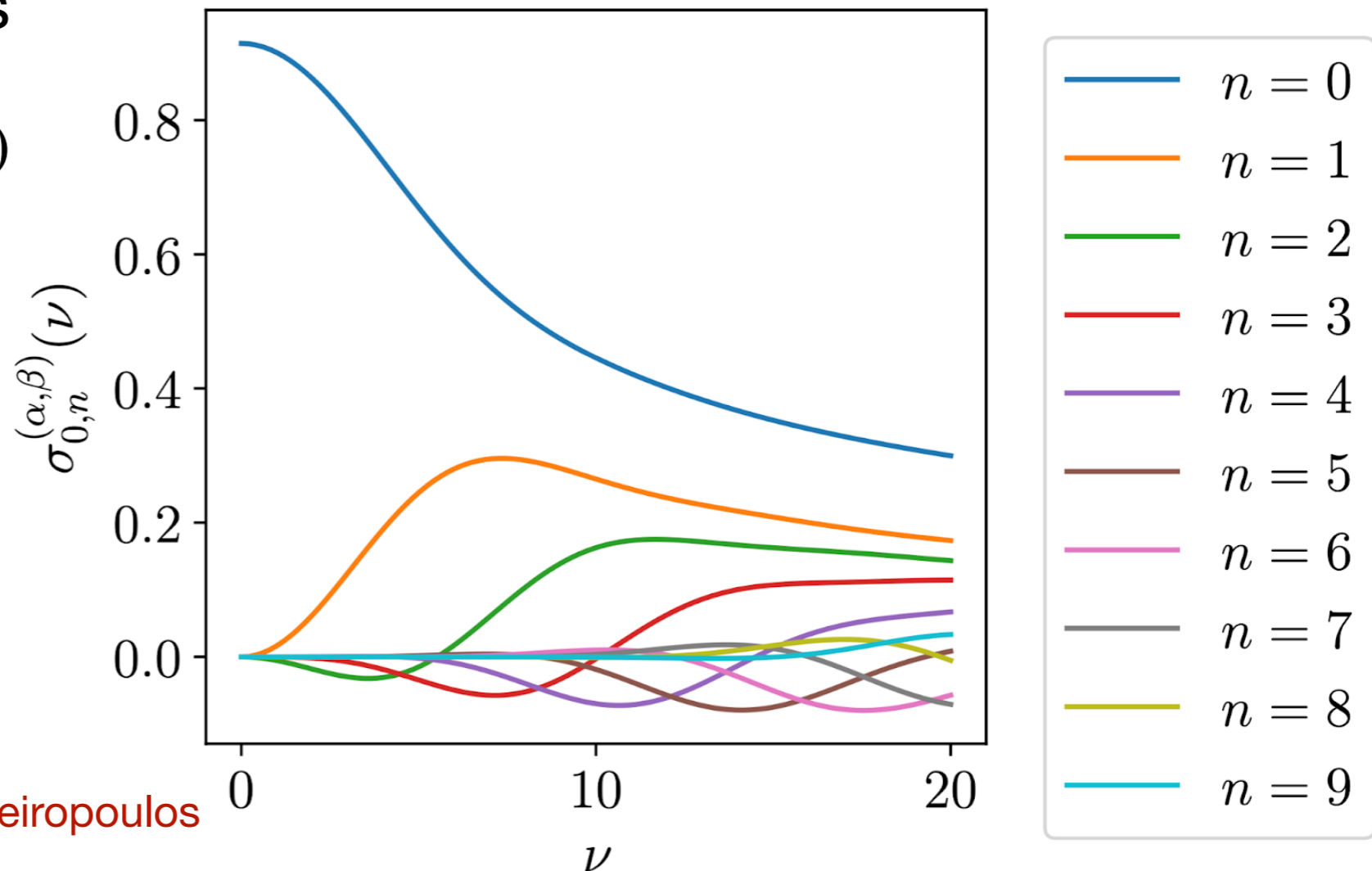
$$\int_{-1}^1 dz (1+z)^\alpha (1-z)^\beta P_n(z) P_m(z) = \delta_{nm} \tilde{N}_n \rightarrow \int_0^1 dx x^\alpha (1-x)^\beta J_n(x) J_m(x) = \delta_{nm} N_n$$

- PDF Parameterized as

$$q_- = x^\alpha (1-x)^\beta \sum_n -d_n P_n(x)$$

- ITD becomes

$$\text{Re } I_q(\nu) = \sum_n -d_n \sigma_n(\nu)$$



Quark pseudo-PDFs

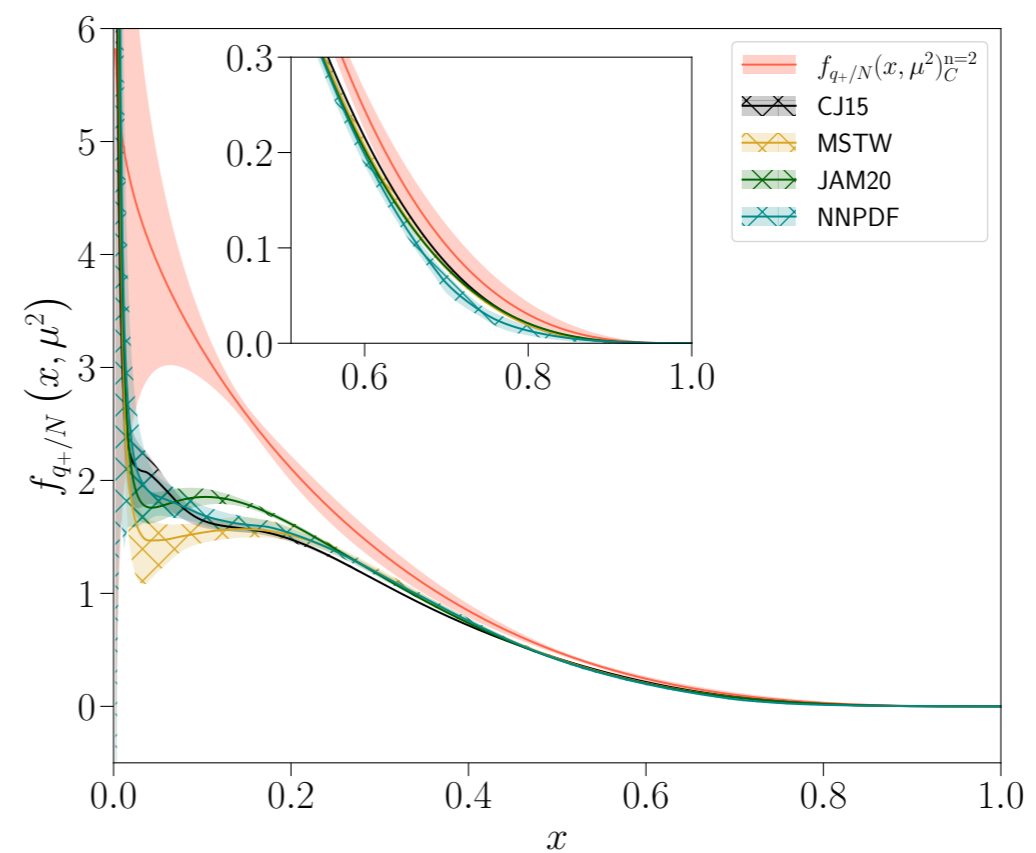
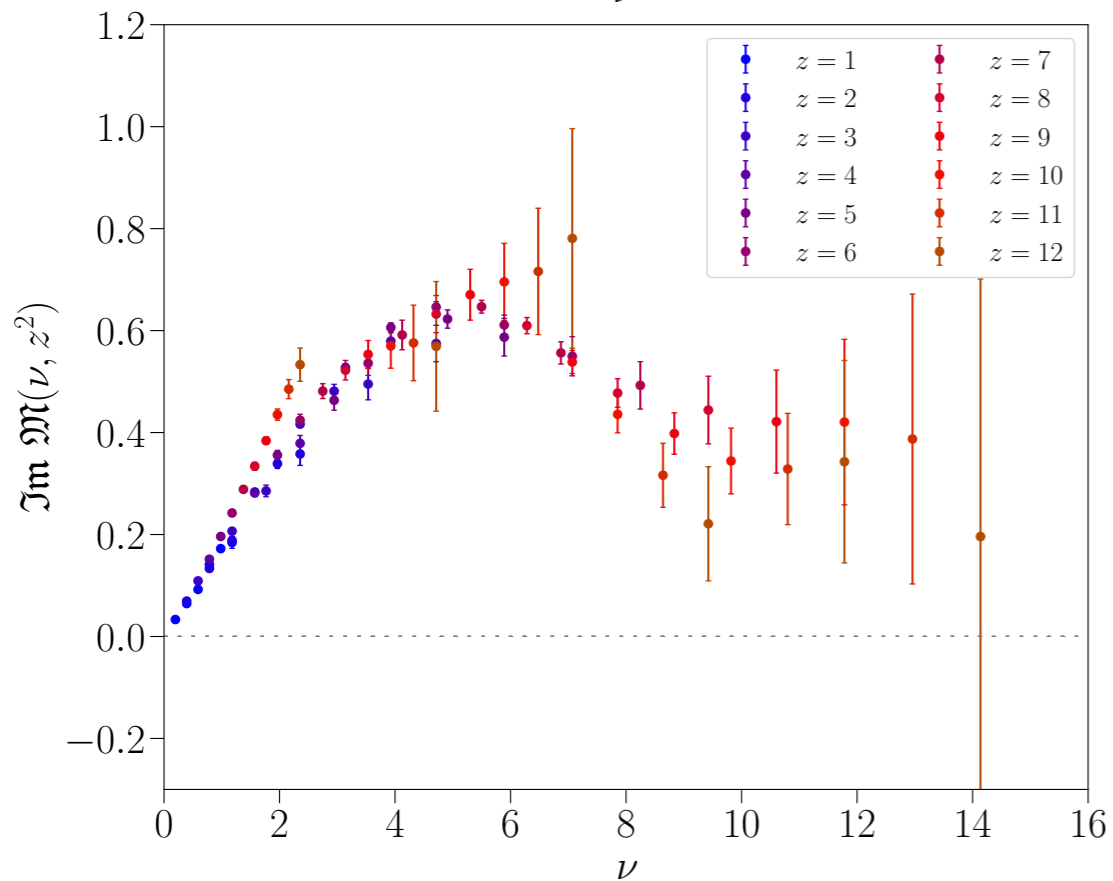
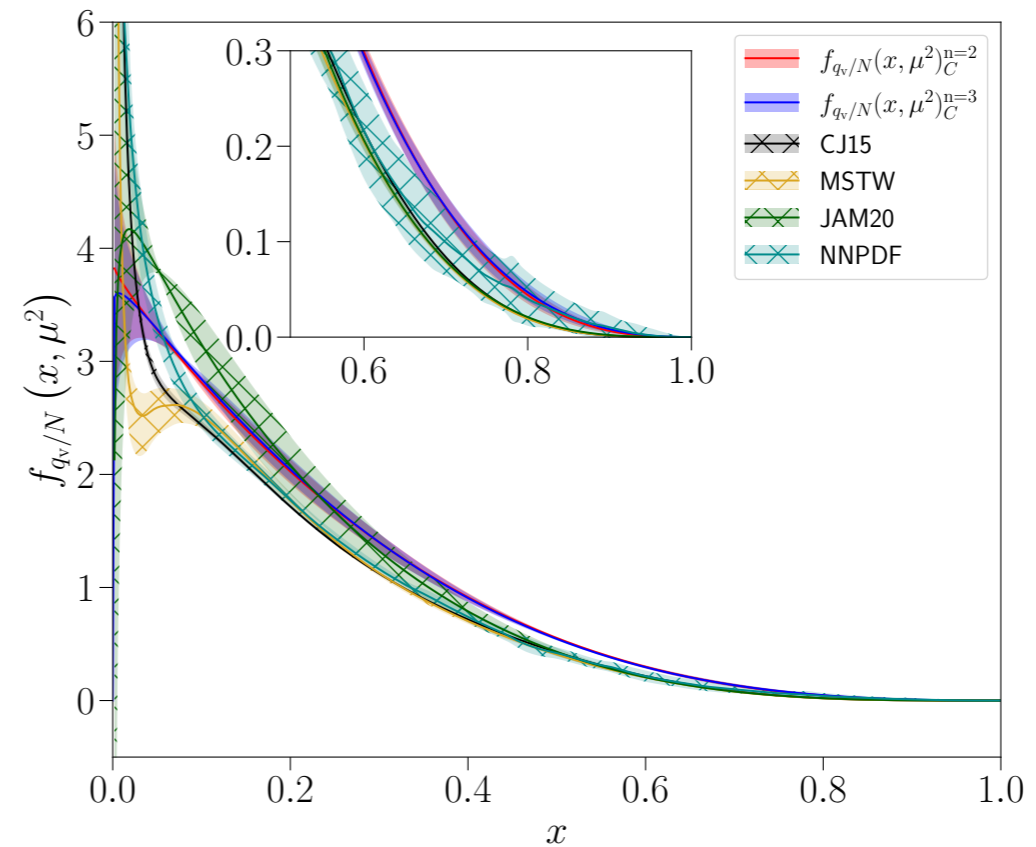
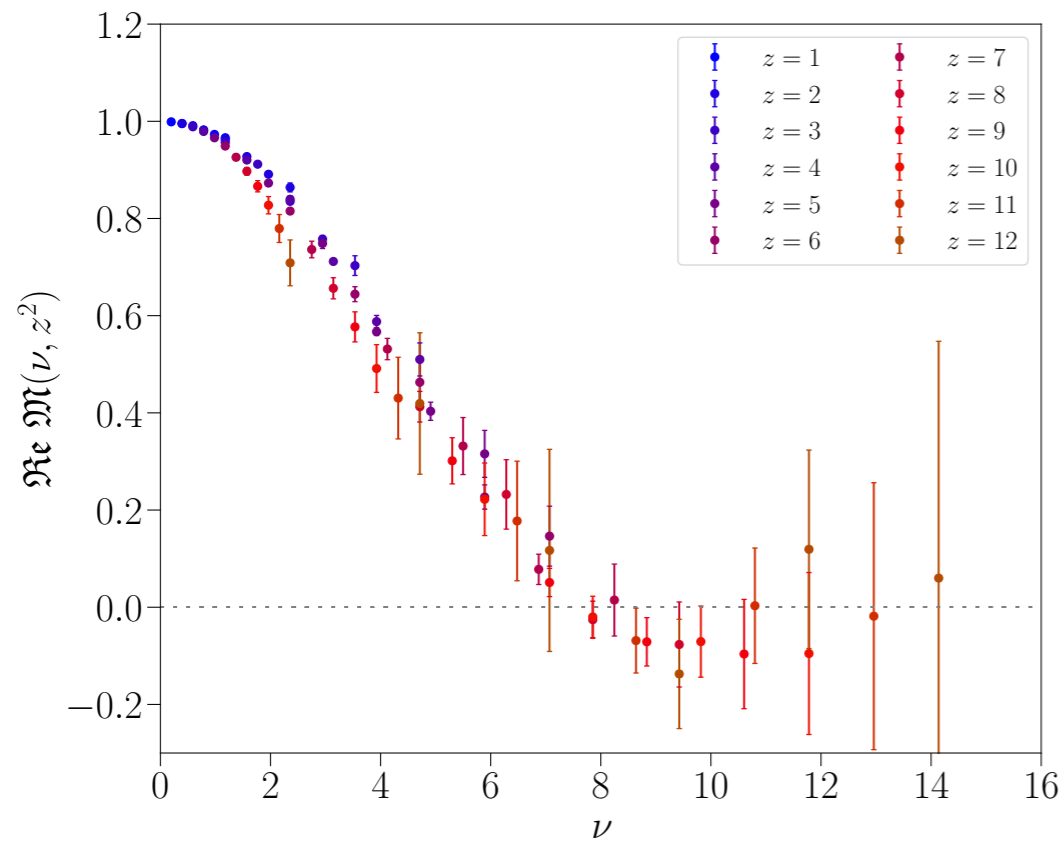
- Bare Matrix element: $M = \langle p | \bar{\psi}(z)\Gamma W(z; 0)\psi(0) | p \rangle$
- Choose multiplicatively renormalizable operators
 - Unpolarized PDF : $\Gamma = \gamma^t$
 - Helicity PDF : $\Gamma = \gamma^z \gamma^5$
 - Transversity PDF : $\Gamma = \gamma^5 \gamma^t \gamma^i$; $i = x, y$
- First study with distillation with unphysically heavy quarks and coarse ensemble

$$a = 0.094 \text{ fm}$$

$$m_\pi = 358 \text{ MeV}$$

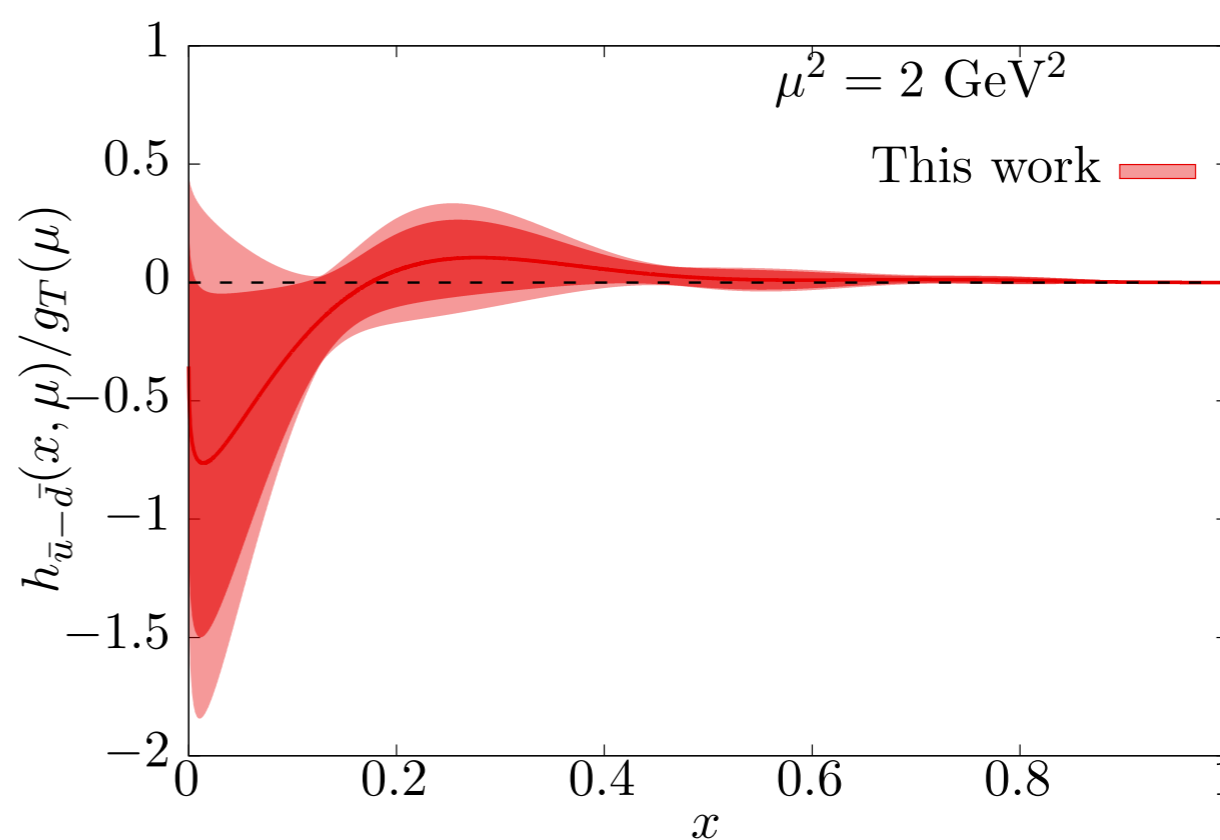
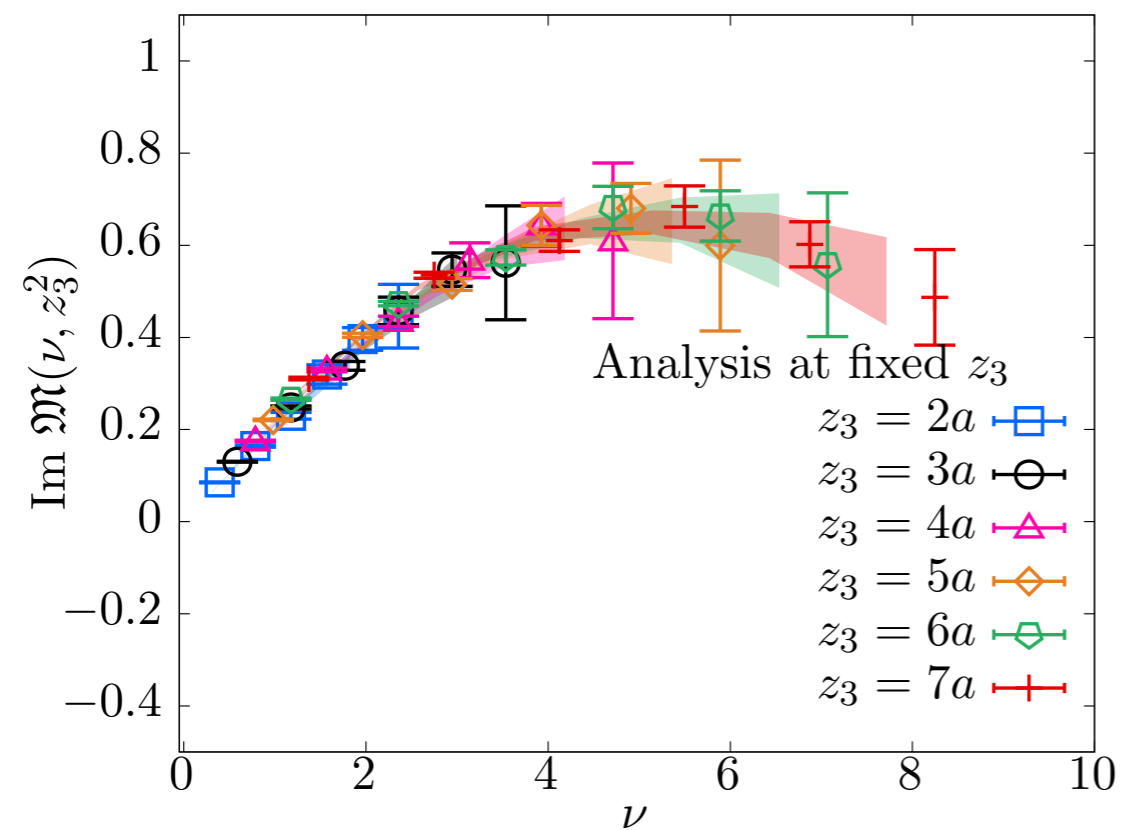
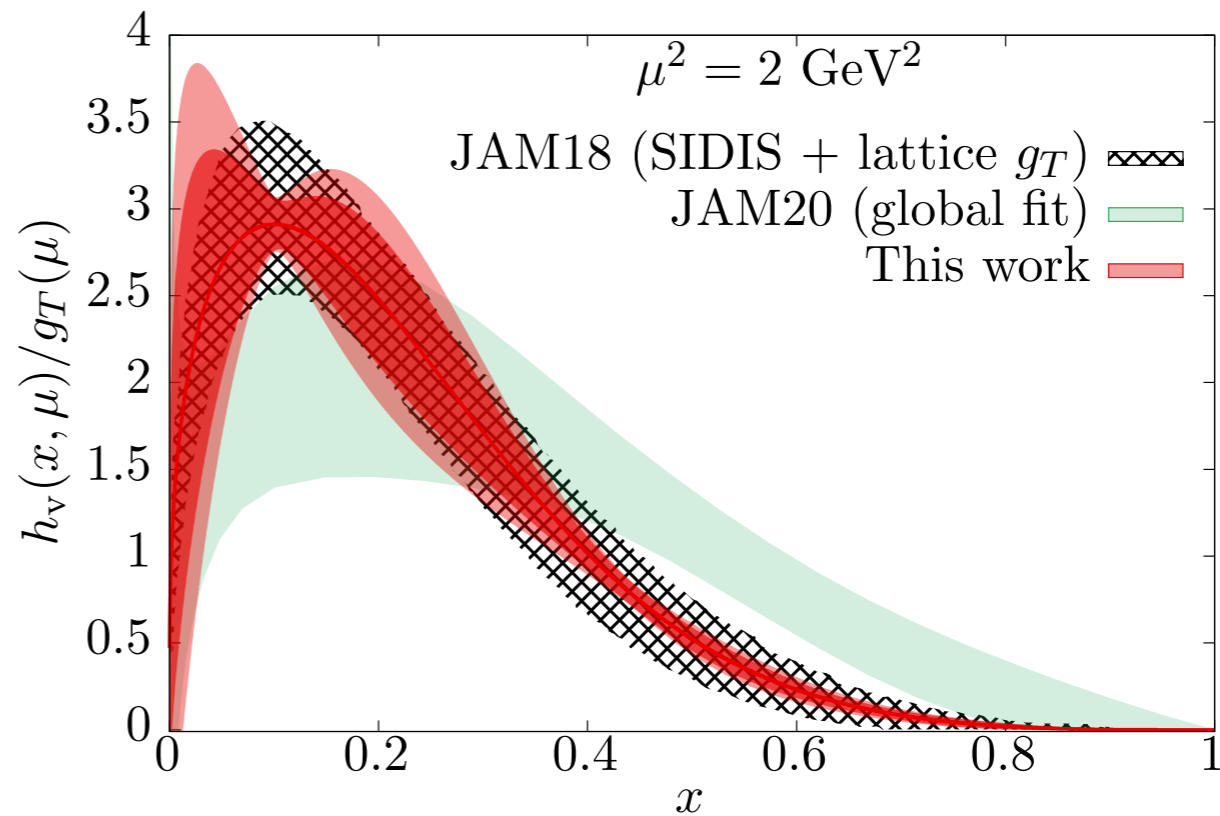
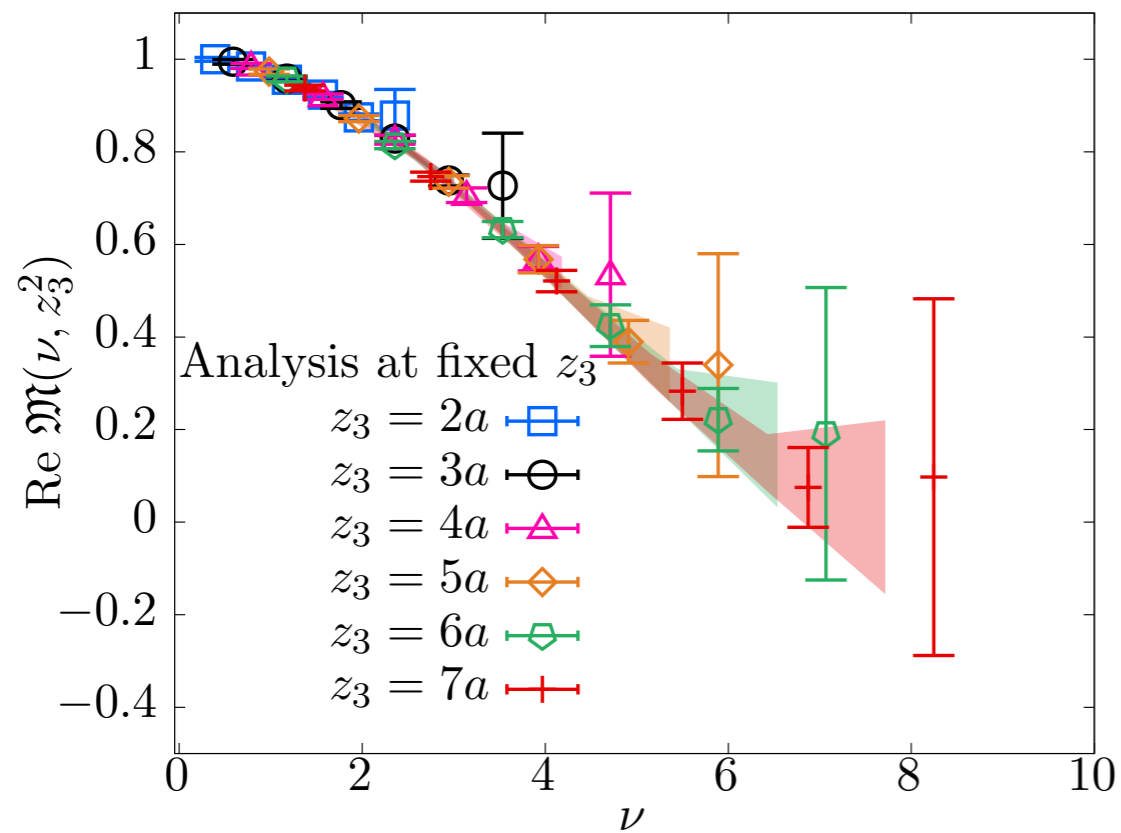
Unpolarized Quark PDF

C. Egerer et al (HadStruc)
JHEP 11 (2021) 148



Transversity Quark PDF

C. Egerer et al (HadStruc)
Phys Rev D 105 (2022) 3, 034507



Gluon Matrix Elements

- **General Matrix Element**

$$M^{\mu\alpha;\nu\beta}(z, p, s) = \langle p, s | \text{Tr} [F^{\mu\alpha}(z) W(z; 0) F^{\nu\beta}(0)] | p, s \rangle$$

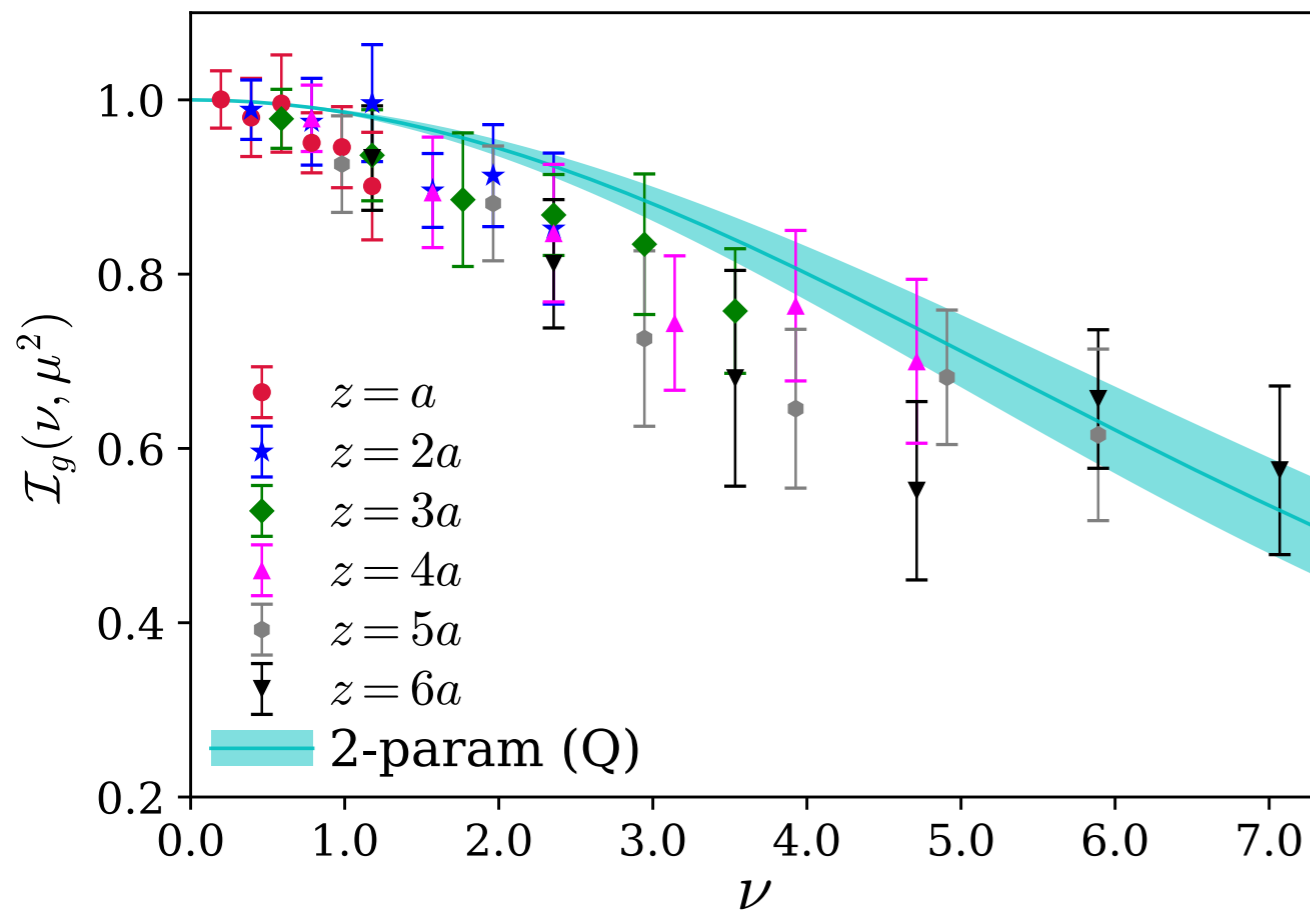
- Assume z is along cardinal direction (eventually lattice axis)

- **Renormalization**

Z-Y. Li, Y-Q. Ma, J-W. Qiu.
Phys. Rev. Lett. 122 (2019) 6, 062002

- Multiplicatively renormalizable
- Depends on how many of μ, ν, ρ, σ are in z direction.
- Matrix element has complicated Lorentz decomposition in terms of p^μ, z^μ, s^μ
 - Need to isolate amplitudes with leading twist contributions

Unpolarized Gluon PDF

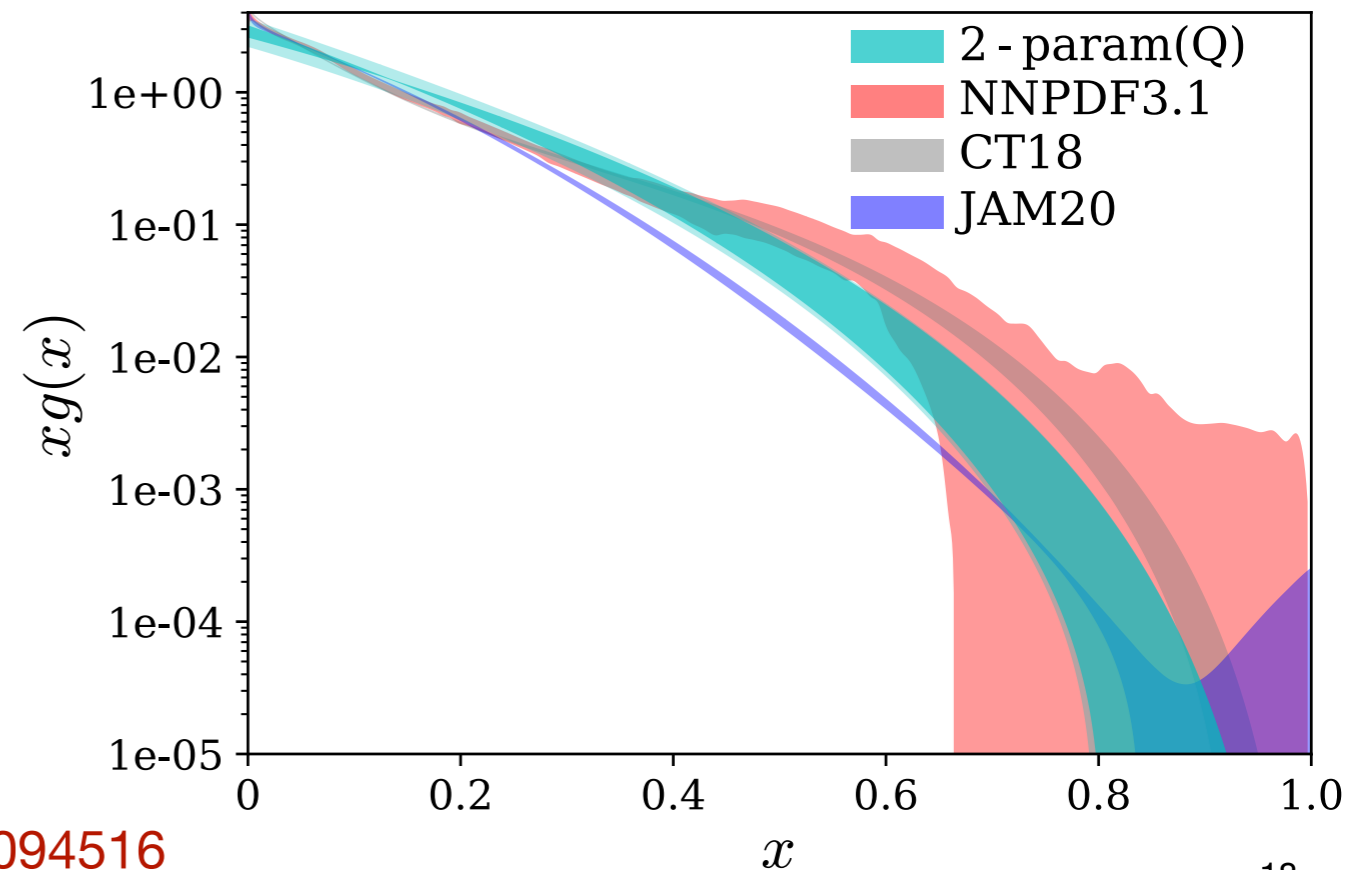


- Data modified by NLO formula ignoring quark mixing
- Data normalized by $\langle x \rangle_g$

$$a = 0.094 \text{ fm}$$

$$m_\pi = 358 \text{ MeV}$$

- ITD fit to cosine transform of $xg(x) = x^a(1-x)^b/B(a+1, b+1)$
- Qualitative agreement with global analysis



Helicity Gluon matrix element

I. Balitsky, W. Morris, A. Radyushkin JHEP 02 (2022) 193
 C. Egerer et al (HadStruc) arXiv:2207.08733

- Helicity Gluon Matrix Element:

$$\widetilde{M}_{\mu\alpha;\nu\beta}(z, p, s) = \frac{1}{2} \epsilon_{\nu\beta\rho\sigma} M_{\mu\alpha;\rho\sigma} = \langle p, s | \text{Tr} [F^{\mu\alpha}(z) W(z; 0) \widetilde{F}^{\nu\beta}(0)] | p, s \rangle$$

- Useful Combination $\widetilde{\mathcal{M}}(z, p) = [\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij}]$

- Gives **two** amplitudes, one has no leading twist contribution

- Use ratio with finite continuum limit

$$\widetilde{\mathfrak{M}}(\nu, z^2) = i \frac{[\widetilde{\mathcal{M}}(z, p)/p_z p_0] / Z_L(z/a)}{\mathcal{M}(0, z^2)/m^2}$$

Lorentz decomposition

I. Balitsky, W. Morris, A. Radyushkin
JHEP 02 (2022) 193

$$\begin{aligned}\widetilde{M}_{\mu\alpha;\lambda\beta}^{(2)}(z, p) = & (sz) \left(g_{\mu\lambda} p_\alpha p_\beta - g_{\mu\beta} p_\alpha p_\lambda - g_{\alpha\lambda} p_\mu p_\beta + g_{\alpha\beta} p_\mu p_\lambda \right) \widetilde{\mathcal{M}}_{pp} \\ & + (sz) \left(g_{\mu\lambda} z_\alpha z_\beta - g_{\mu\beta} z_\alpha z_\lambda - g_{\alpha\lambda} z_\mu z_\beta + g_{\alpha\beta} z_\mu z_\lambda \right) \widetilde{\mathcal{M}}_{zz} \\ & + (sz) \left(g_{\mu\lambda} z_\alpha p_\beta - g_{\mu\beta} z_\alpha p_\lambda - g_{\alpha\lambda} z_\mu p_\beta + g_{\alpha\beta} z_\mu p_\lambda \right) \widetilde{\mathcal{M}}_{zp} \\ & + (sz) \left(g_{\mu\lambda} p_\alpha z_\beta - g_{\mu\beta} p_\alpha z_\lambda - g_{\alpha\lambda} p_\mu z_\beta + g_{\alpha\beta} p_\mu z_\lambda \right) \widetilde{\mathcal{M}}_{pz} \\ & + (sz) \left(p_\mu z_\alpha - p_\alpha z_\mu \right) \left(p_\lambda z_\beta - p_\beta z_\lambda \right) \widetilde{\mathcal{M}}_{ppzz} \\ & + (sz) \left(g_{\mu\lambda} g_{\alpha\beta} - g_{\mu\beta} g_{\alpha\lambda} \right) \widetilde{\mathcal{M}}_{gg}\end{aligned}$$

$$\begin{aligned}\widetilde{M}_{\mu\alpha;\lambda\beta}^{(1)}(z, p) = & \left(g_{\mu\lambda} s_\alpha p_\beta - g_{\mu\beta} s_\alpha p_\lambda - g_{\alpha\lambda} s_\mu p_\beta + g_{\alpha\beta} s_\mu p_\lambda \right) \widetilde{\mathcal{M}}_{sp} \\ & + \left(g_{\mu\lambda} p_\alpha s_\beta - g_{\mu\beta} p_\alpha s_\lambda - g_{\alpha\lambda} p_\mu s_\beta + g_{\alpha\beta} p_\mu s_\lambda \right) \widetilde{\mathcal{M}}_{ps} \\ & + \left(g_{\mu\lambda} s_\alpha z_\beta - g_{\mu\beta} s_\alpha z_\lambda - g_{\alpha\lambda} s_\mu z_\beta + g_{\alpha\beta} s_\mu z_\lambda \right) \widetilde{\mathcal{M}}_{sz} \\ & + \left(g_{\mu\lambda} z_\alpha s_\beta - g_{\mu\beta} z_\alpha s_\lambda - g_{\alpha\lambda} z_\mu s_\beta + g_{\alpha\beta} z_\mu s_\lambda \right) \widetilde{\mathcal{M}}_{zs} \\ & + (p_\mu s_\alpha - p_\alpha s_\mu) (p_\lambda z_\beta - p_\beta z_\lambda) \widetilde{\mathcal{M}}_{pspz} \\ & + (p_\mu z_\alpha - p_\alpha z_\mu) (p_\lambda s_\beta - p_\beta s_\lambda) \widetilde{\mathcal{M}}_{pzps} \\ & + (s_\mu z_\alpha - s_\alpha z_\mu) (p_\lambda z_\beta - p_\beta z_\lambda) \widetilde{\mathcal{M}}_{szpz} \\ & + (p_\mu z_\alpha - p_\alpha z_\mu) (s_\lambda z_\beta - s_\beta z_\lambda) \widetilde{\mathcal{M}}_{pzs z}\end{aligned}$$

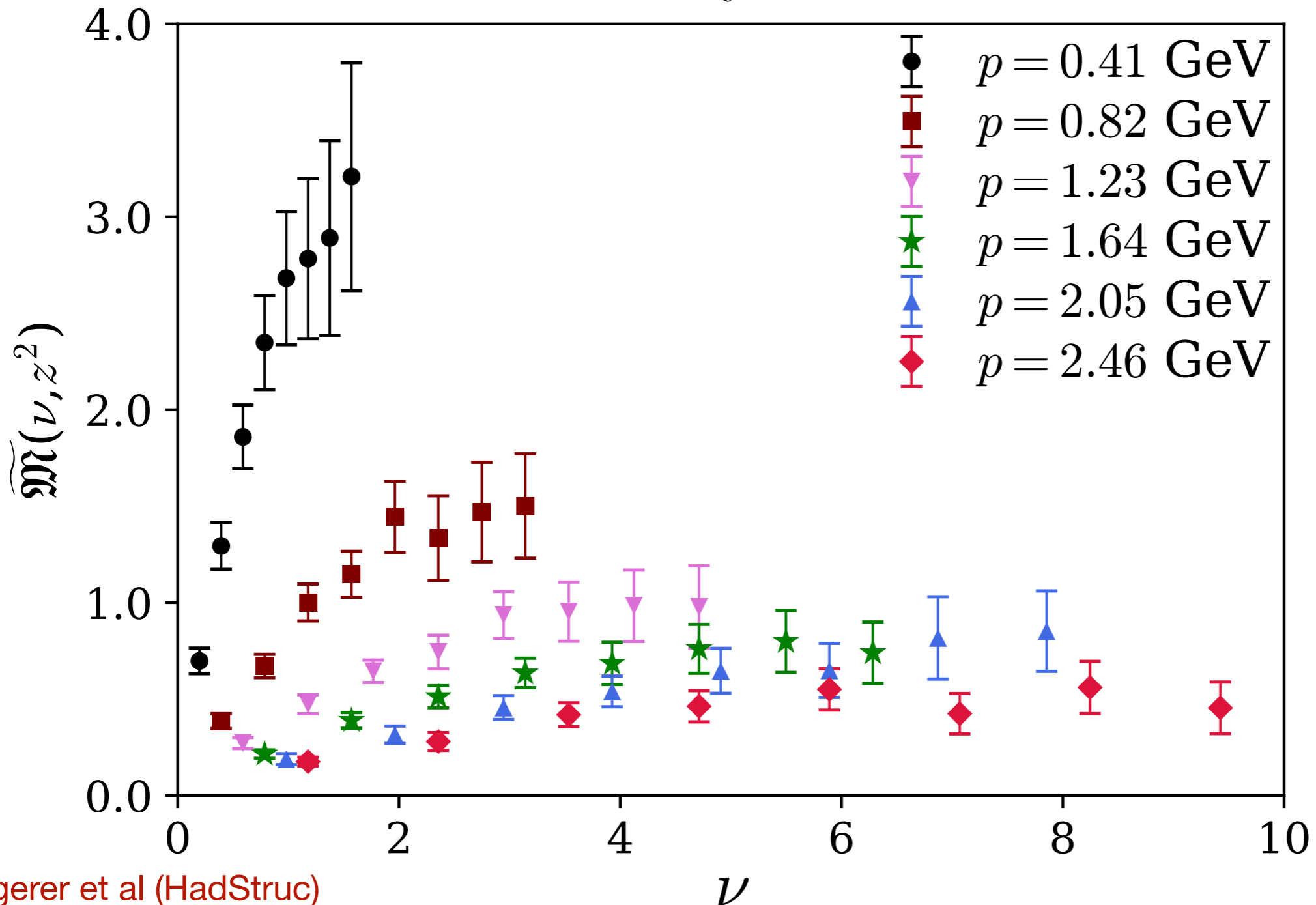
Want: $M_{\Delta g}(\nu, z^2) = \left[\widetilde{\mathcal{M}}_{sp}^{(+)} - \nu \widetilde{\mathcal{M}}_{pp} \right]$

Can get: $\widetilde{\mathcal{M}}(z, p) = \left[\widetilde{M}_{ti;it} + \widetilde{M}_{ij;ij} \right]$

$$\begin{aligned}& = M_{\Delta g} - \frac{m^2 z^2}{\nu} \widetilde{\mathcal{M}}_{pp} \\ & = M_{\Delta g} - \frac{m^2}{p_z^2} \nu \widetilde{\mathcal{M}}_{pp}\end{aligned}$$

Helicity Gluon Matrix Element

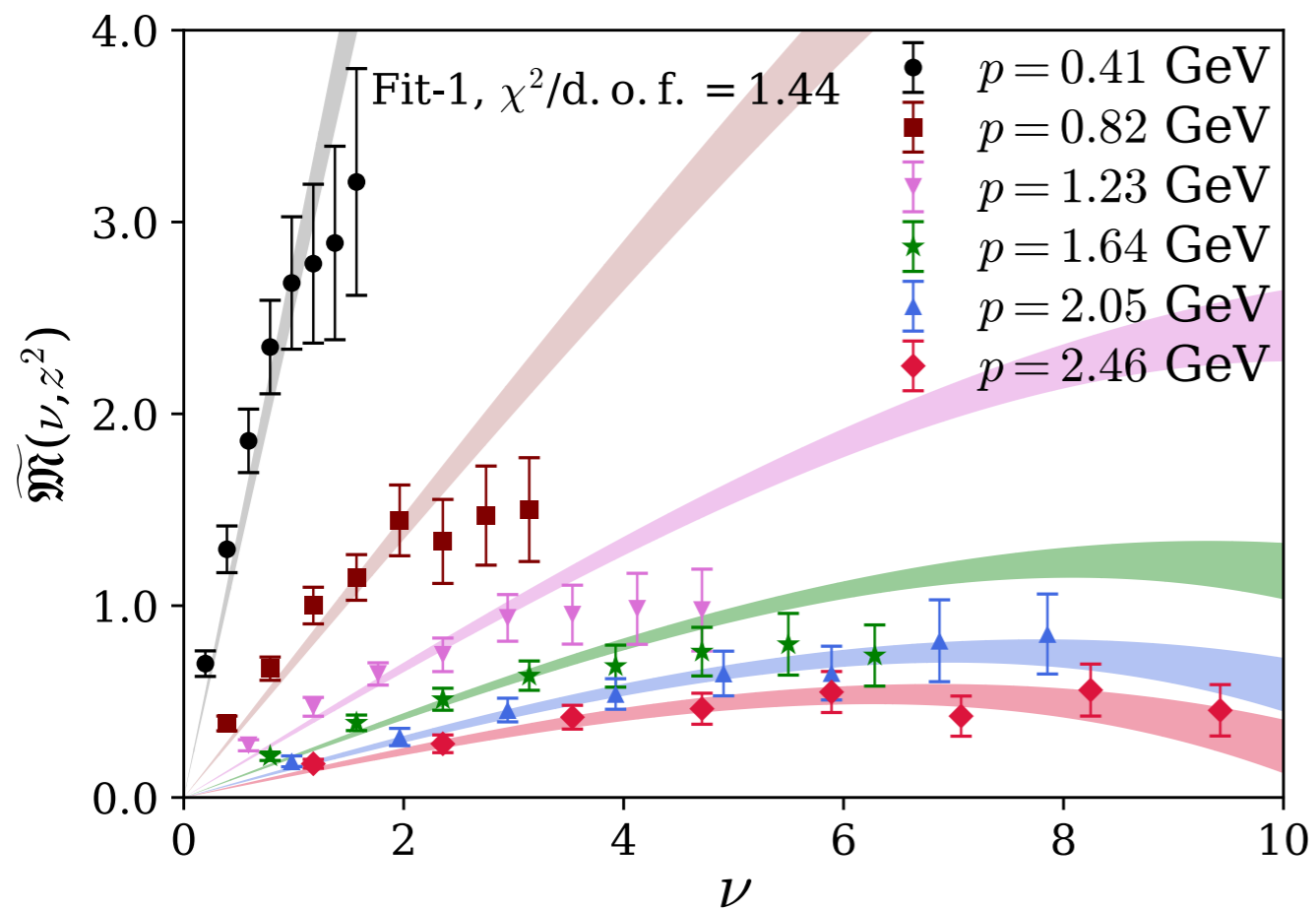
- Large contamination from $\frac{m^2}{p_z^2} \nu \widetilde{\mathcal{M}}_{pp}$ will need to be removed



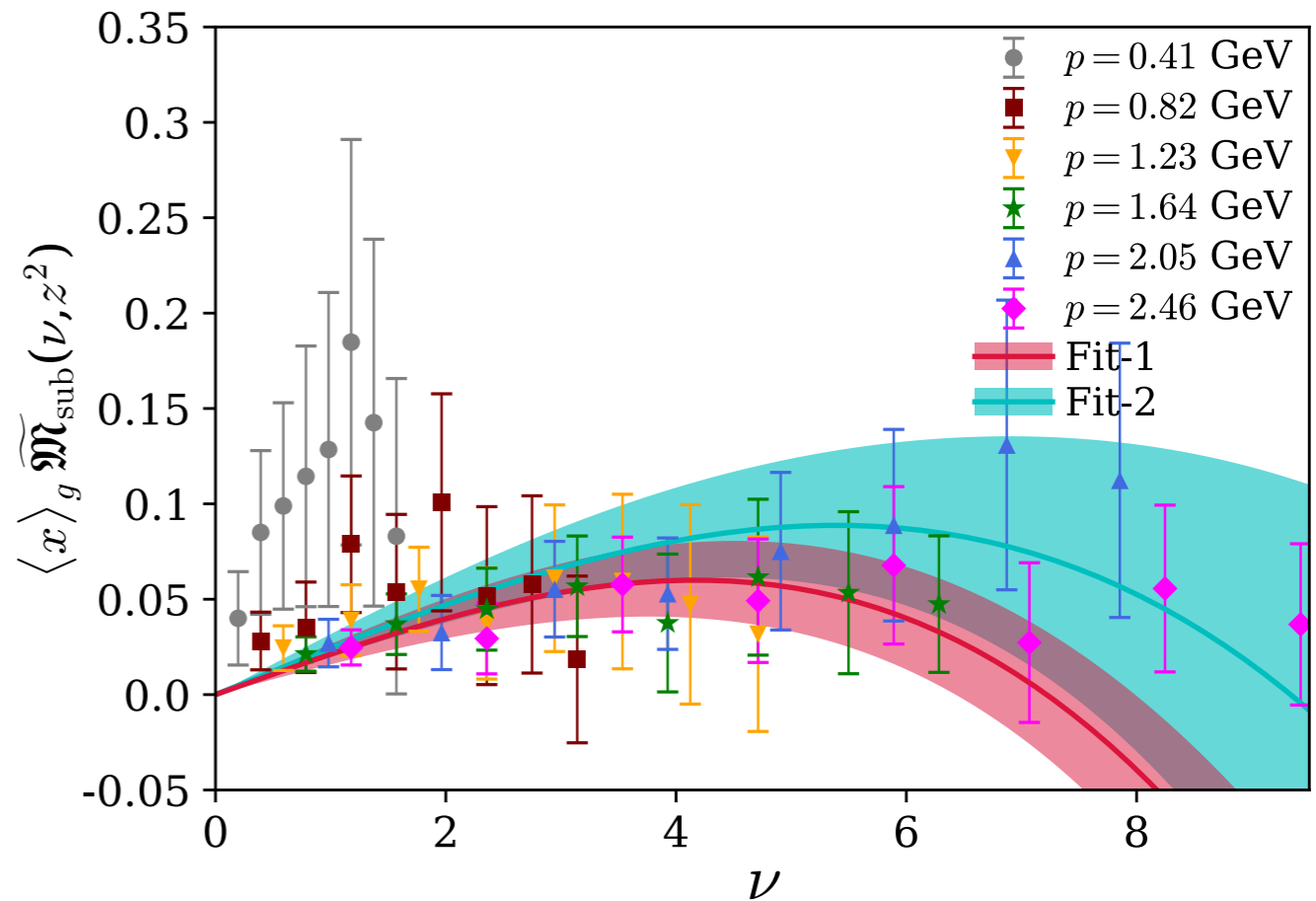
$a = 0.094$ fm
 $m_\pi = 358$ MeV

Helicity Gluon PDF

- Model both terms



- Subtract rest frame



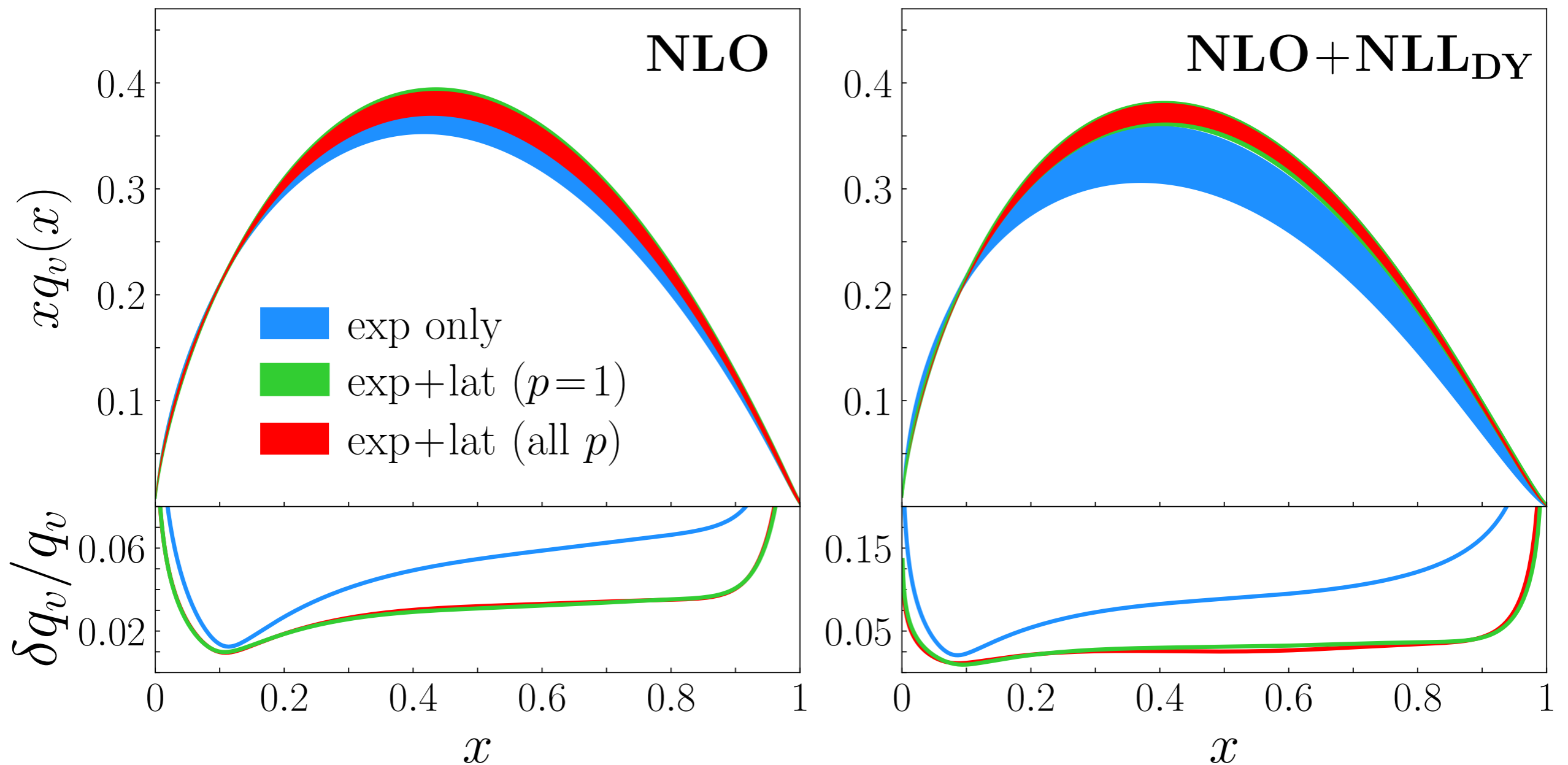
$$a = 0.094 \text{ fm}$$

$$m_\pi = 358 \text{ MeV}$$

**Lattice data aren't that
different from
experimental data**

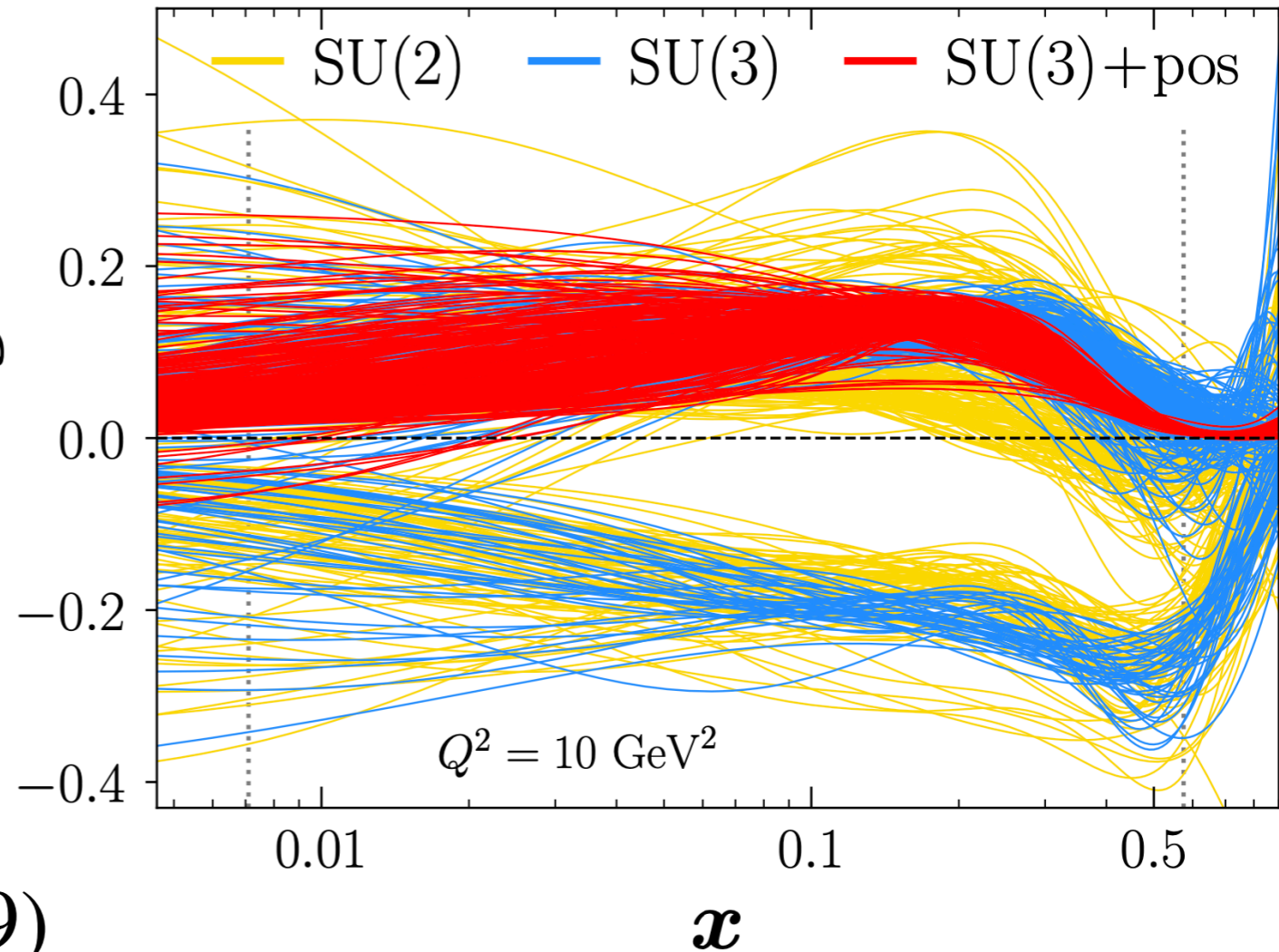
Combining Lattice and Experiment

- Simultaneously fit Lattice and Experimental pion PDF data
- Each gives unique information complementing each other



Spinning gluons (revisited)

Y. Zhou et al (JAM) Phys. Rev. D 105, 074022 (2022)



R. Jaffe and A. Manohar, Nucl. Phys. B 337, 509 (1990)

$$J = \frac{1}{2} \Delta \Sigma + L_q + L_G + \Delta G$$

$$\Delta G = \int dx \Delta g(x)$$

- Positivity removed from JAM helicity gluon PDF

$$|\Delta g| \leq g(x)$$

- Reveals new band of solutions

- With constraint: $\Delta G = 0.39(9)$

- Without constraint: $\Delta G = 0.3(5)$

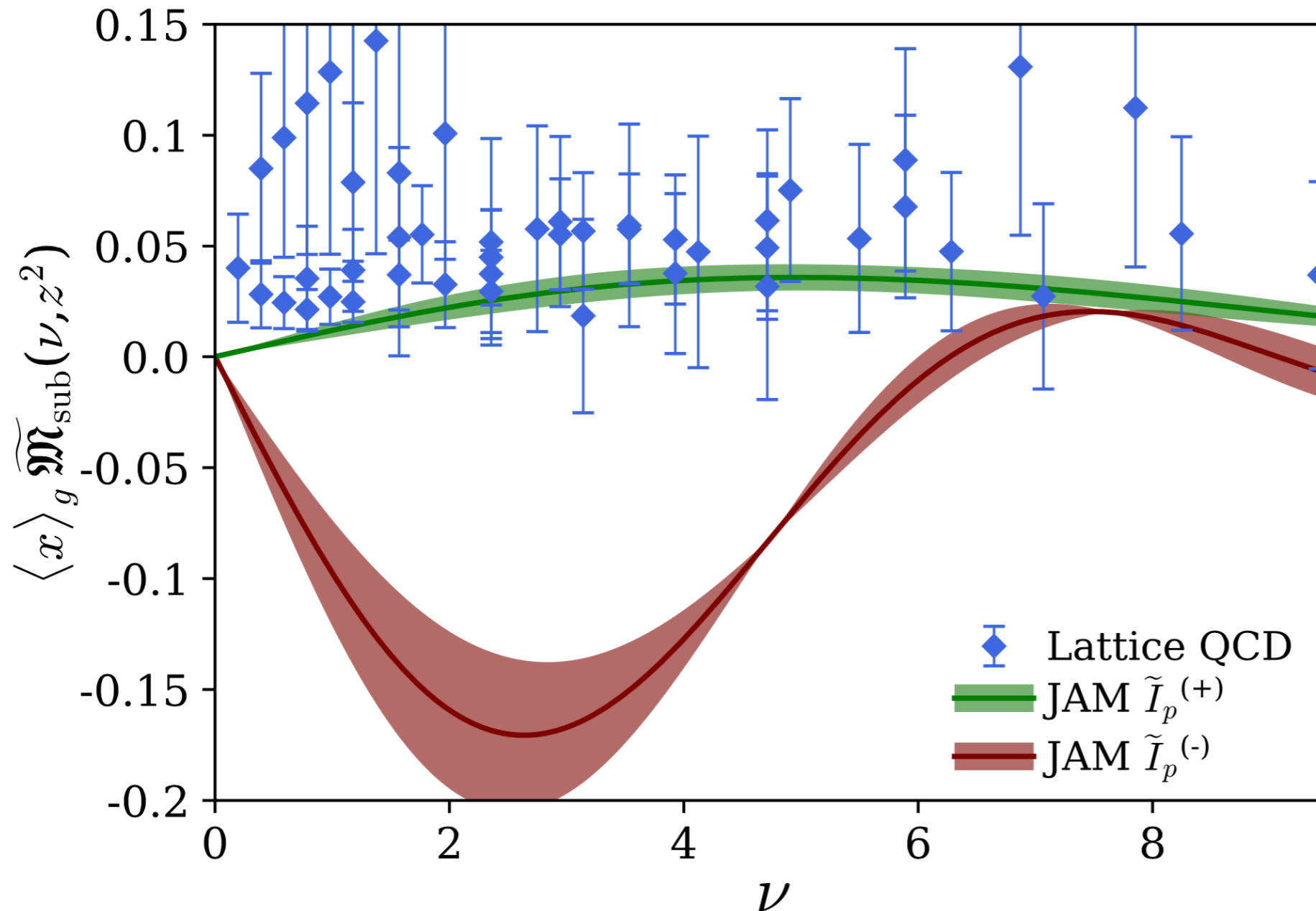
- Lattice: $\Delta G = 0.251(47)(16)$

Y-B. Yang et al (χ -QCD) Phys. Rev. Lett. 118, 102001 (2017)

K-F. Liu arXiv: 2112.08416

Spinning gluons (revisited)

Can lattice data affect phenomenological polarized gluon analysis?



$$\Delta G = \int d\nu I_g(\nu)$$

- The positive and negative solutions without positivity constraints plotted in ν space
- Only positive band consistent with lattice data

Y. Zhou et al Phys. Rev. D 105, 074022 (2022)
C. Egerer et al (HadStruc) arXiv:2207.08733

Conclusions

- Modern Lattice QCD can find parton x or ν dependent structure
 - High precision results require complex techniques
 - Precision needed to impact phenomenology is possible
 - Need to improve systematic errors at this precision
- Possibly impact phenomenological PDF analyses
 - **Today:** with pion PDFs and polarized gluon PDF
 - **Future:** JLab 12GeV and EIC data on PDF and on new distributions

Thank you and the organizers!