Nucleon Structure from Pseudo PDFs

SPIN 2023

Duke, September 25-29, 2023

Jefferson Lab
Thomas Jefferson National Accelerator Facility

Kostas Orginos, William & Mary / JLab

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2013 revolution
Go beyond moments

- Goal: Compute full x-dependence (generalized) parton distribution functions (GPDFs)
- Operator product: Mellin moments are local matrix elements that can be computed in Lattice QCD
    - Power divergent mixing limits us to few moments
- X. Ji suggested an approach for obtaining PDFs from Lattice QCD
- First calculations quickly became available
- Older approaches based on the hadronic tensor

X. Ji, Phys.Rev.Lett. 110, (2013)
Y.-Q. Ma J.-W. Qiu (2014) 1404.6860

H.-W. Lin, J.-W. Chen, S. D. Cohen, and X. Ji, Phys.Rev. D91, 054510 (2015)
C. Alexandrou, et al, Phys. Rev. D92, 014502 (2015)

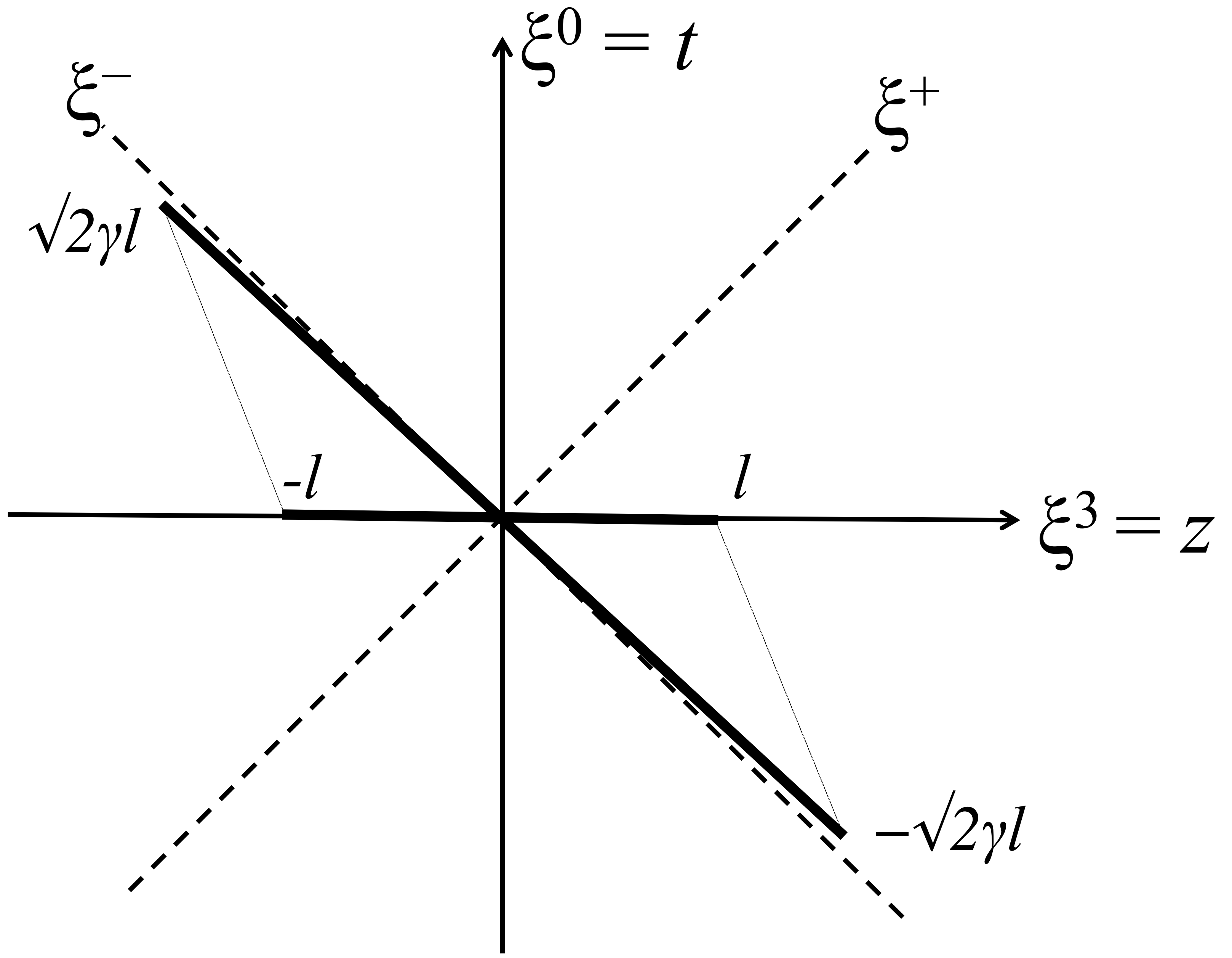
K-F Liu et al Phys. Rev. Lett. 72 (1994) , Phys. Rev. D62 (2000) 074501
Detmold and Lin 2005
M. T. Hansen et al arXiv:1704.08993.
UKQCD-QCDSF-CSSM Phys. Lett. B714 (2012), arXiv:1703.01153

X. Ji, Phys.Rev.Lett. 110, (2013)

X. Ji (2014) Sci. China Phys. Mech. Atron. 57 arXiv:1404.6680

Quasi-PDF
X. Ji’s Basic idea

- Lattice QCD computes equal time matrix elements
- Displace quarks in space-like interval
- Boost states to infinite momentum
- On the frame of the proton displacement becomes light-like
- Infinite momentum not possible on the lattice
  - Perurbative matching from finite momentum
  - LaMET



Renormalization of UV divergences is required

Good Lattice Cross sections

Current-Current Correlators

Y.-Q. Ma J.-W. Qiu (2014) arXiv:1404.6860
Y.-Q. Ma J.-W. Qiu (2017) arXiv:1709.3018

4-quark bi-local matrix elements:

$$\sigma_n(\nu, z^2) = \langle P \mid T\{\mathcal{O}_n(z)\} \mid P\rangle$$

equal time matrix element

Ex.

$$\begin{aligned} \mathcal{O}_S(z) &= (z^2)^2 Z_S^2 [\bar{\psi}_q \psi_q](z)[\bar{\psi}_q \psi](0) \\ \mathcal{O}_{V'}(z) &= z^2 Z_{V'}^2 [\bar{\psi}_q (z\cdot\gamma)\psi_{q'}](z)[\bar{\psi}_{q'} z\cdot\gamma\psi](0), \end{aligned}$$

Short distance factorization:

$$\sigma_n(\nu, z^2) = \sum_a \int_{-1}^{1} \frac{dx}{x} f_a(x,\mu^2)\, K_n^a(x\nu, z^2\mu^2) + \mathcal{O}(z^2\Lambda_{\rm QCD}^2),$$

PDFs can be obtained

Imitate scattering experiments: factorization

Renormalization of UV divergences of local operators is required

Pseudo-PDFs

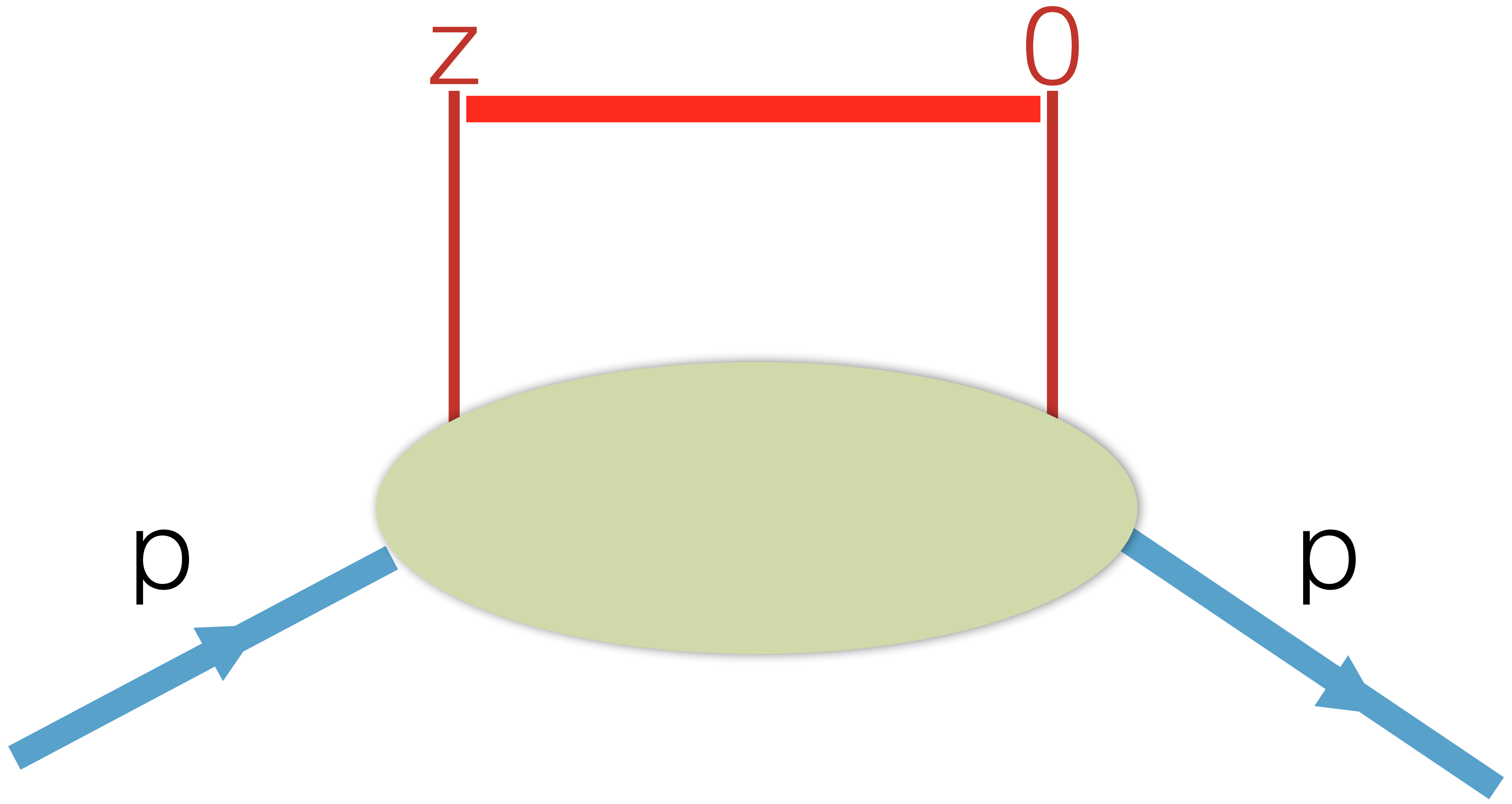
An alternative point of view

A. Radyushkin Phys.Lett. B767 (2017)

Unpolarized PDFs proton:

$$\mathcal{M}^{\alpha}(z,p) \equiv \langle p|\bar{\psi}(0)\,\gamma^{\alpha}\,\hat{E}(0,z;A)\psi(z)|p\rangle$$

$$\hat{E}(0,z;A) = \mathcal{P}\exp\left[-ig\int_0^z \mathrm{d}z'_{\mu}\,A^{\mu}_{\alpha}(z')T_{\alpha}\right]$$



space-like separation of quarks

Lorentz decomposition:

$$\mathcal{M}^{\alpha}(z,p) = 2p^{\alpha}\mathcal{M}_p(-(zp),-z^2) + z^{\alpha}\mathcal{M}_z(-(zp),-z^2)$$

A. Radyushkin Phys.Lett. B767 (2017)

Pseudo-PDFs

Connection to light cone PDFs

$$\mathcal{M}^\alpha(z,p) = 2p^\alpha \mathcal{M}_p(-(zp),-z^2) + z^\alpha \mathcal{M}_z(-(zp),-z^2)$$

Collinear PDFs: Choose

$$z = (0, z_-, 0)$$
$$p = (p_+, 0, 0)$$
$$\gamma^+$$

$$\mathcal{M}^+(z,p) = 2p^+ \mathcal{M}_p(-p_+ z_-, 0)$$

Definition of PDF:

$$\mathcal{M}_p(-p_+ z_-, 0) = \int_{-1}^{1} dx\, f(x)\, e^{-ixp_+ z_-}$$

Lorentz invariance allows for the computation of invariant form factors in any frame

Use equal time kinematics for LQCD

A. Radyushkin Phys.Lett. B767 (2017)

Lattice QCD calculation:

$$\mathcal{M}^\alpha(z,p) \equiv \langle p|\bar\psi(0)\,\gamma^\alpha\,\hat E(0,z;A)\psi(z)|p\rangle$$

Choose $$p = (p_0, 0, 0, p_3)$$ $$z = (0,0,0,z_3)$$ γ^0

On shell equal time matrix element
computable in Euclidean space

Briceno *et al* arXiv:1703.06072

Obtaining only the relevant

$$\mathcal{M}_p(\nu, z_3^2) = \frac{1}{2p_0}\mathcal{M}^0(z_3,p_3)$$

$$\mathcal{P}(x,-z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\,\mathcal{M}_p(\nu,-z^2)e^{-ix\nu}$$

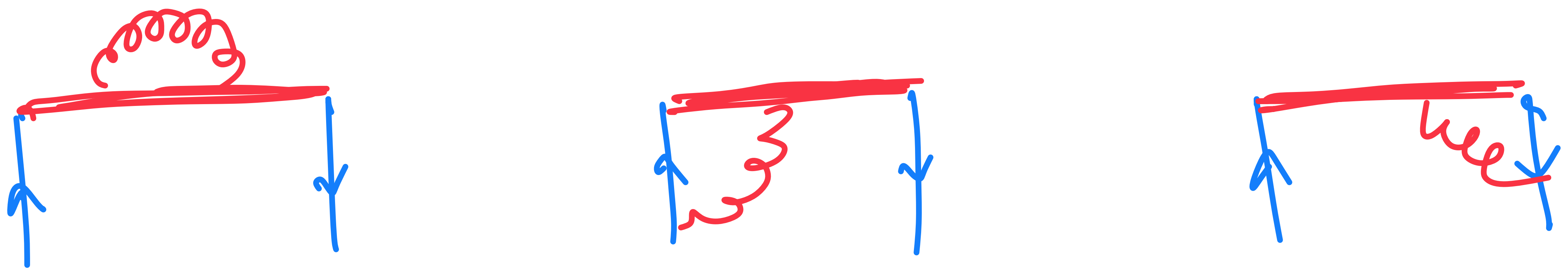
the pseudo-PDF $x \in [-1,1]$

Radyusking Phys.Lett. B767 (2017) 314-320

Choosing γ^0 was also suggested also by M. Constantinou at GHP2017 based
on an operator mixing argument for the renormalized matrix element

Alexandrou *et al* arXiv:1706.00265

- $\mathcal{M}_p(\nu, -z^2)$ is computable in LQCD with a lattice cutoff a
- Continuum limit $a \to 0$: UV divergences



$$\mathcal{M}_p(\nu, -z^2) \propto e^{-m|z|/a} \left(\frac{a}{z^2}\right)^{2\gamma_{end}}$$

UV divergences are multiplicative

- J.G.M.Gatheral,Phys.Lett.133B,90(1983)
- J.Frenkel, J.C.Taylor,Nucl.Phys.B246,231(1984),
- G.P.Korchemsky, A.V.Radyushkin,Nucl.Phys.B283,342(1987).

Constantinou, Panagopoulos *Phys.Rev.D* 107 (2023) 1, 014503
Alexandrou et al. Nucl.Phys. B923 (2017) 394

Consider the ratio $$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}_p(\nu, z_3^2)}{\mathcal{M}_p(0, z_3^2)}$$

UV divergences will cancel in this ratio resulting a renormalization group invariant (RGI) function

The lattice regulator can now be removed

$\mathfrak{M}^{cont}(\nu, z_3^2)$ Universal independent of the lattice

Its Fourier transformation with respect to ν is a particular definition of a PDF

It contains non-perturbative information about the structure of the proton

$\mathcal{M}_p(0, 0) = 1$ Isovector matrix element

Properties of $\mathcal{M}(\nu, -z^2)$

- Fourier Transform: $\mathcal{P}(x,-z^2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\nu\, \mathcal{M}(\nu,-z^2)\, e^{-i\nu x}$

  the Pseudo PDF $x \in [-1,1]$

- At $-z^2 \to 0$: Collinear divergences
  - The small $-z^2$ limit defines the twist-2 PDF
  - At small $-z^2$ it can be matched to the $\overline{MS}$ PDF

$$\mathcal{M}(\nu,z^2) = \underbrace{\int_0^1 d\alpha\, C(\alpha, z^2\mu^2)\, Q(\nu,\mu^2)}_{\text{twist-2}} + \underbrace{\mathcal{O}(z^2)}_{\text{higher twist}}$$

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

- DGLAP evolution $z^2 \frac{d}{dz^2}\mathcal{M}(\nu,-z^2) = \int_0^1 d\alpha\, B(\alpha,z^2)\, \mathcal{M}(\alpha\nu,-z^2) + \mathcal{O}(z^2)$

  DGLAP kernel

- $\mathcal{M}(\nu,-z^2)$ Computable for any z^2, ν

ν is called Ioffe time

B. Ioffe '64

Continuum limit matching to $\overline{MS}$ computed at 1-loop

Radyushkin Phys.Rev. D98 (2018) no.1, 014019
Zhang et al. Phys.Rev. D97 (2018) no.7, 074508

$$\mathfrak{M}(\nu, z^2) = \int_0^1 dx\, q_v(x,\mu)\mathcal{K}(x\nu, z^2\mu^2) \; + \sum_{k=1}^{\infty} \mathcal{B}_k(\nu)(z^2)^k\,.$$

$$\mathcal{K}(x\nu, z^2\mu^2) = \cos(x\nu) - \frac{\alpha_s}{2\pi} C_F \left[\ln(e^{2\gamma_E+1} z^2\mu^2/4)\tilde{B}(x\nu) + \tilde{D}(x\nu)\right]$$

$$\tilde{B}(x) = \frac{1-\cos(x)}{x^2} + 2\sin(x)\frac{x\mathrm{Si}(x)-1}{x} + \frac{3-4\gamma_E}{2}\cos(x) + 2\cos(x)\left[\mathrm{Ci}(x) - \ln(x)\right]$$

$$\tilde{D}(x) = x\mathrm{Im}\left[e^{ix}{}_3F_3(111;222;-ix)\right] - \frac{2-(2+x^2)\cos(x)}{x^2}$$

Polynomial corrections to the Ioffe time PDF may be suppressed

B. U. Musch, *et al* Phys. Rev. D 83, 094507 (2011)
M. Anselmino et al. 10.1007/JHEP04(2014)005
A. Radyushkin Phys.Lett. B767 (2017)

"Δός μοι πᾷ στῶ, καὶ τὰν γᾶν κινήσω" Αρχιμήδης

- Small lattice spacing for both continuum limit and small $-z^2$
- Large momentum to extend the range of ν

$$\text{large } \nu \longleftrightarrow \text{small } x$$

- Scaling $1/a^7$ (?)
- Large momentum $\longrightarrow$ bad signal to noise ratio

" Give me a big computer .... "

Leading twist extraction

$$\mathfrak{M}(p,z,a) = \mathfrak{M}_{\rm cont}(\nu, z^2) + \sum_{n=1}\left(\frac{a}{|z|}\right)^n P_n(\nu) + (a\Lambda_{\rm QCD})^n R_n(\nu)\,.$$

$$\mathrm{Re}\,\mathfrak{M}(\nu,z^2) = \int_0^1 dx\,\mathcal{K}_R(x\nu,\mu^2 z^2) q_-(x,\mu^2) + \mathcal{O}(z^2)$$

$$\mathrm{Im}\,\mathfrak{M}(\nu,z^2) = \int_0^1 dx\,\mathcal{K}_I(x\nu,\mu^2 z^2) q_+(x,\mu^2) + \mathcal{O}(z^2)\,,$$

- Obtain the PDF from a limited set of matrix elements obtained from lattice QCD
- z^2 is a physical length scale sampled on discrete values
- z^2 needs to be sufficiently small so that higher twist effects are under control
- ν is dimensionless also sampled in discrete values
- the range of ν is dictated by the range of z and the range of momenta available and is typically limited
- Parametrization of unknown functions

However on the Lattice after expanding in lattice spacing we have

$$\mathfrak{M}(p,z,a) = \mathfrak{M}_{\text{cont}}(\nu, z^2) + \sum_{n=1} \left(\frac{a}{|z|}\right)^n P_n(\nu) + (a\Lambda_{\text{QCD}})^n R_n(\nu)\,.$$

$$\mathfrak{M}(\nu, z^2) = \int_0^1 dx\, q_v(x,\mu)\mathcal{K}(x\nu, z^2\mu^2) + \sum_{k=1}^{\infty} \mathcal{B}_k(\nu)(z^2)^k\,.$$

Ioffe time $-z\cdot p = \nu$

- All coefficient functions respect continuum symmetries
- Lattice spacing corrections to higher twist effects are ignored
- On dimensional ground a/z terms must exist
- Additional O(a) effects (last term)

Bayesian Inference: Obtain q(x,μ) from the lattice matrix elements

see dicussion in J. Karpie *et al JHEP* 04 (2019) 057 and L. DelDebio *et al JHEP* 02 (2021) 138

Exploration of various methods for LO matching

Exploration of the NNPDF approach applied to lattice data

Jacobi Polynomials

Parametrization of Unknown functioins

$$q_+(x) = q(x) + \bar{q}(x)$$

$$q_-(x) = q(x) - \bar{q}(x)$$

PDF parametrization

$$q_\pm(x) = x^\alpha (1-x)^\beta \sum_{n=0}^{\infty} {}_\pm d_n^{(\alpha,\beta)} J_n^{(\alpha,\beta)}(x)$$

$J_n^{(\alpha,\beta)}(x)$ Jacobi Polynomials: Orthogonal and complete in the interval [0,1]

$$\int_0^1 dx\, x^\alpha (1-x)^\beta J_n^{(\alpha,\beta)}(x) J_m^{(\alpha,\beta)}(x) = N_n^{(\alpha,\beta)} \delta_{n,m}$$

Complete basis of functions in the interval [0,1] for any α and β

Bayesian Inference

Optimize model parameters

- Fix the expansion order in the Jacobi polynomial expansion
- Optimize α,β and the expansion of coefficients by maximizing the posterior probability
- Note that one could fix α,β at a reasonable value and vary the order of truncation in the Jacobi polynomial expansion
- Average over models using AICc

Posterior distribution

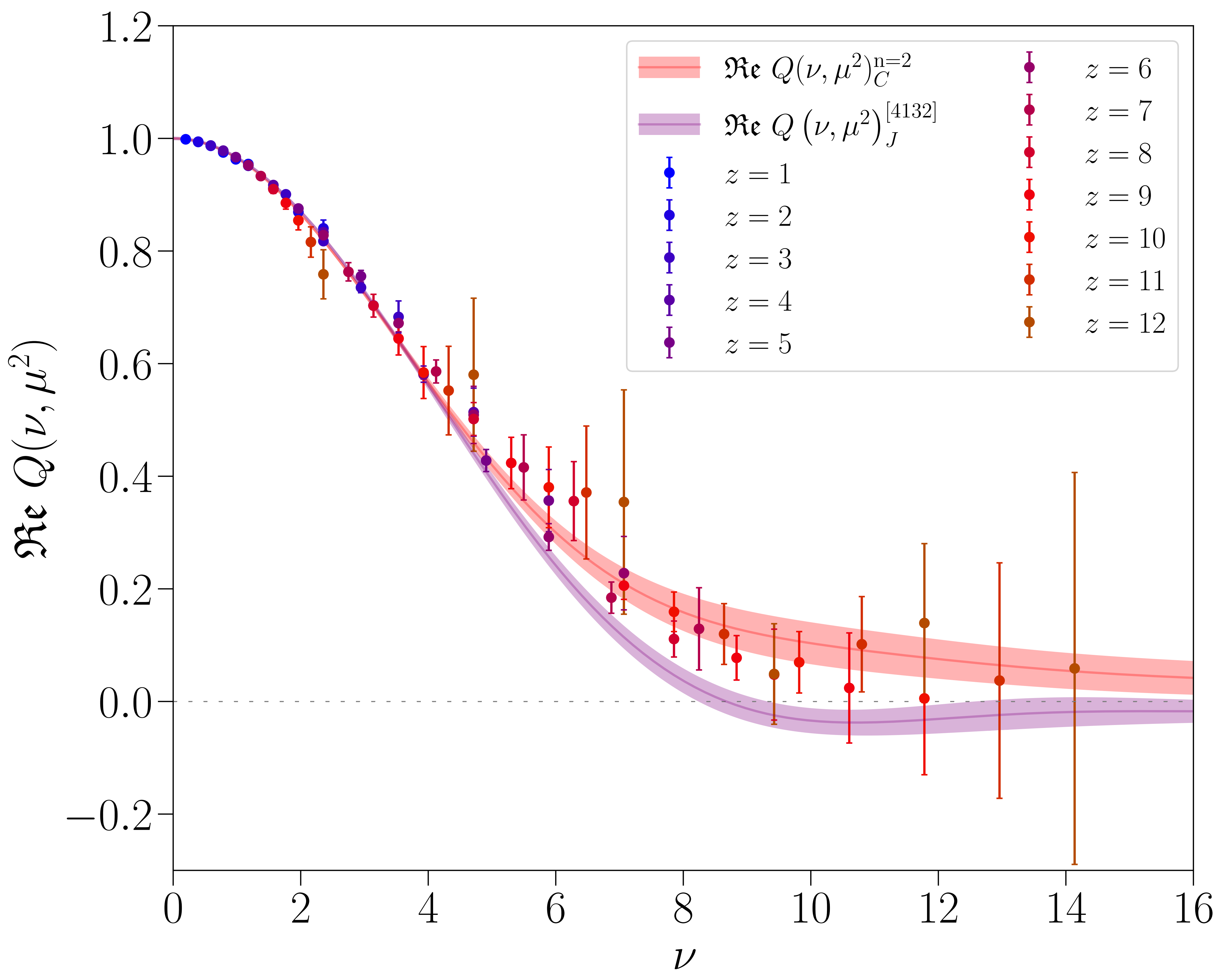
$$P\left[\theta|\mathfrak{M}^L, I\right] = \frac{P\left[\mathfrak{M}^L|\theta\right] P\left[\theta|I\right]}{P\left[\mathfrak{M}^L|I\right]}\,.$$

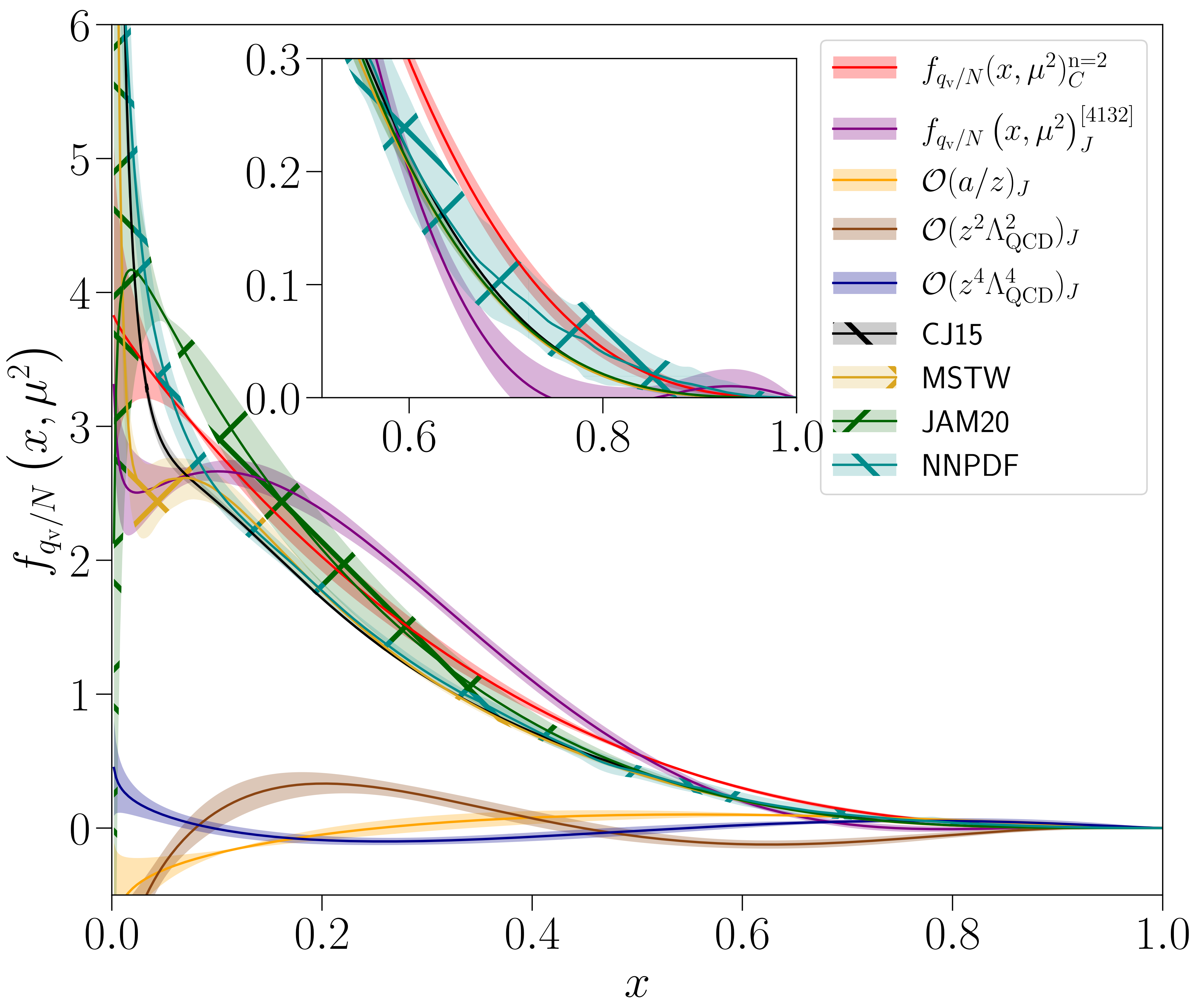
AICc model averaging

$$F_{AICc} = \sum_i F_i \frac{e^{-A_i/2}}{\sum_k e^{-A_k/2}} \quad \text{with} \quad A_i = -\,2\log P_i^{post} + 2p_i + \frac{2p_i(p_i+1)}{n_i - p_i - 1}$$

Unpolarized Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion





[arXiv:2107.05199](https://arxiv.org/abs/2107.05199) **[hep-lat] C. Egerer** *et. al.*

Helicity Isovector PDF

Matrix element: $$M^{\mu 5}(p,z) = \langle N(p,\lambda)|\,\overline{\psi}(z)\,\gamma^{\mu}\gamma^{5}W^{(f)}(z,0)\,\psi(0)\,|N(p,\lambda)\rangle$$

Lorentz decomposition:

$$M^{\mu 5}(p,z) = -2m_N S^{\mu}\mathcal{M}\left(\nu,z^2\right) - 2im_N p^{\mu}(z\cdot S)\,\mathcal{N}\left(\nu,z^2\right) + 2m_N^3 z^{\mu}(z\cdot S)\,\mathcal{R}\left(\nu,z^2\right)$$

$$S^{\mu} \equiv \frac{1}{2m_N}\overline{u}(p,\lambda)\,\gamma^{\mu}\gamma^{5}u(p,\lambda)$$

On the light-cone:

$$\begin{aligned} M^{+5}\left(p,z^{-}\right)_{\mathrm{Reg}_{\mu^2}} &= -2m_N S^{+}\left[\mathcal{M}\left(p^{+}z^{-},0\right) + ip^{+}z^{-}\mathcal{N}\left(p^{+}z^{-},0\right)\right]_{\mathrm{Reg}_{\mu^2}} \\ &= -2m_N S^{+}\left[\mathcal{M}(\nu,0) - i\nu\mathcal{N}(\nu,0)\right]_{\mathrm{Reg}_{\mu^2}} \equiv -2m_N S^{+}\mathcal{I}\left(\nu,\mu^2\right) \end{aligned}$$

$$g_{q/N}\left(x,\mu^2\right) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\nu}{2\pi}e^{-ix\nu}\mathcal{I}\left(\nu,\mu^2\right) \ .$$

arXiv:2211.04434 **[hep-lat] C. Egerer** *et. al.*

Helicity Isovector PDF

Space-like z: $$M^{35}\left(p, z_3\right) = -2m_N S^3\left[p_z \hat{z}\right]\left\{\mathcal{M}\left(\nu, z_3^2\right) - i p_z z_3 \mathcal{N}\left(\nu, z_3^2\right)\right\} - 2m_N^3 z_3^2 S^3\left[p_z \hat{z}\right] \mathcal{R}\left(\nu, z_3^2\right)$$

$$M^{35}\left(p, z_3\right) = -2m_N S^3\left[p_z \hat{z}\right]\left\{\mathcal{Y}\left(\nu, z_3^2\right) + m_N^2 z_3^2 \mathcal{R}\left(\nu, z_3^2\right)\right\}$$

$$\widetilde{\mathcal{Y}}\left(\nu, z_3^2\right) = \mathcal{Y}\left(\nu, z_3^2\right) + m_N^2 z_3^2 \mathcal{R}\left(\nu, z_3^2\right)$$

$$\mathfrak{Y}\left(\nu, z_3^2\right) = \left(\frac{\widetilde{\mathcal{Y}}\left(\nu, z_3^2\right)}{\widetilde{\mathcal{Y}}\left(0, z_3^2\right)|_{p_z=0}}\right) \Big/ \left(\frac{\widetilde{\mathcal{Y}}\left(\nu, 0\right)|_{z_3=0}}{\widetilde{\mathcal{Y}}\left(0, 0\right)|_{p_z=0, z_3=0}}\right)$$

Helicity distributions normalized by g_A :

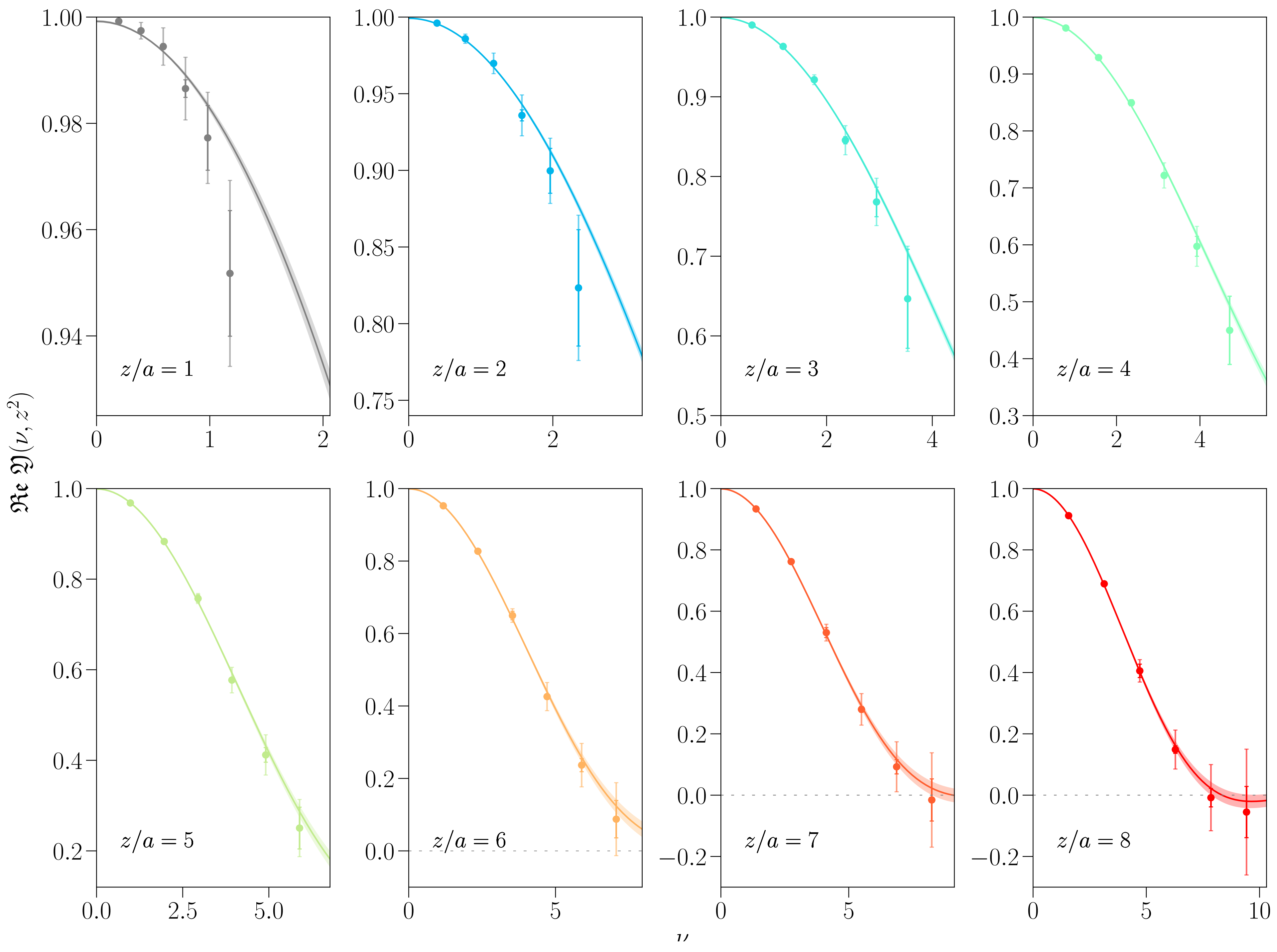
$$\mathrm{Re}\, \mathfrak{Y}\left(\nu, z^2\right) = g_A\left(\mu^2\right)^{-1} \int_0^1 \mathrm{d}x\ \mathcal{K}_-\left(x\nu, z^2\mu^2, \alpha_s\left(\mu^2\right)\right) g_{q_-/N}\left(x, \mu^2\right) + \mathcal{O}\left(z^2 \Lambda_{\mathrm{QCD}}^2\right)$$

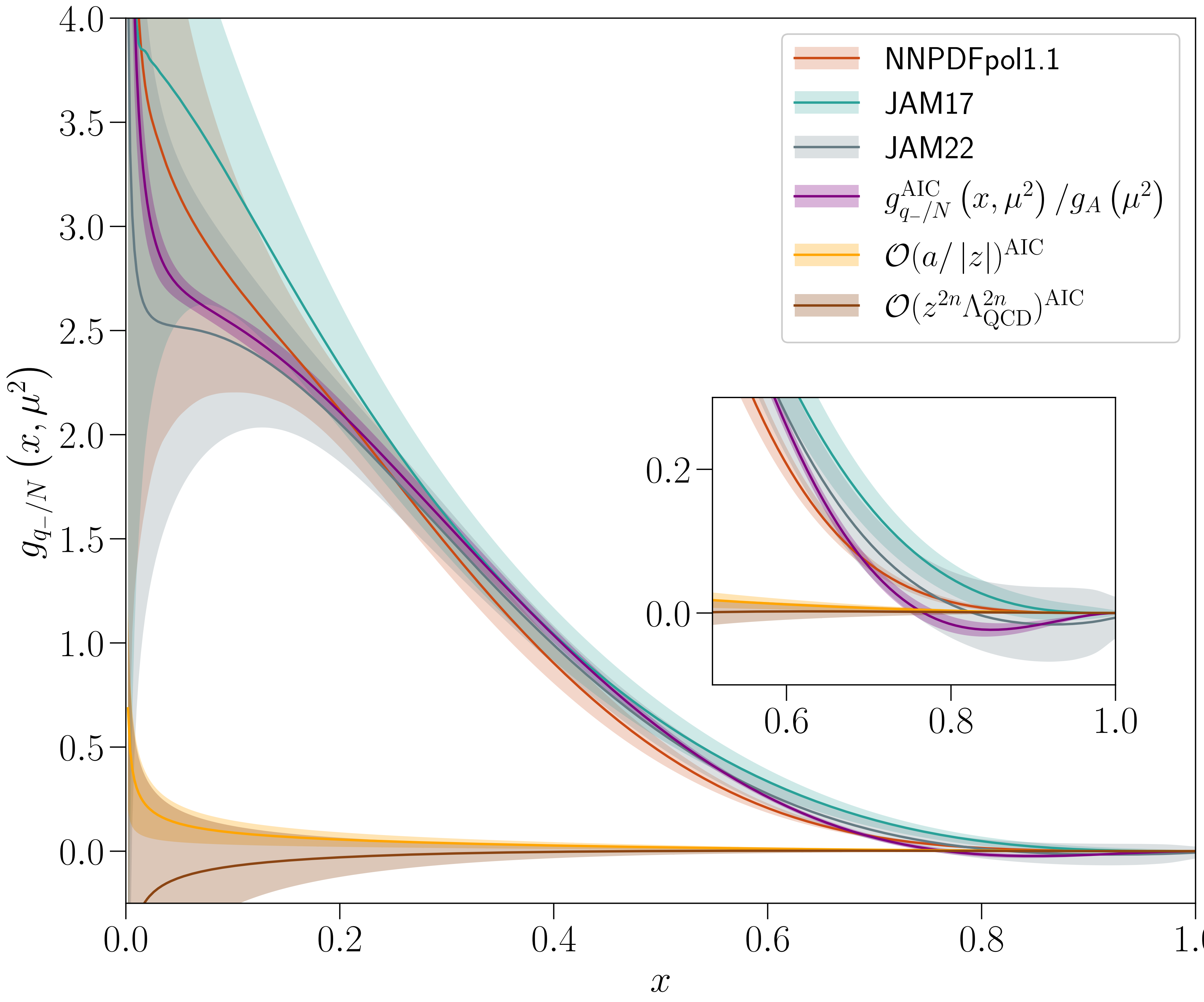
$$\mathrm{Im}\, \mathfrak{Y}\left(\nu, z^2\right) = g_A\left(\mu^2\right)^{-1} \int_0^1 \mathrm{d}x\ \mathcal{K}_+\left(x\nu, z^2\mu^2, \alpha_s\left(\mu^2\right)\right) g_{q_+/N}\left(x, \mu^2\right) + \mathcal{O}\left(z^2 \Lambda_{\mathrm{QCD}}^2\right).$$

arXiv:2211.04434 **[hep-lat] C. Egerer** *et. al.*

Helicity Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion





arXiv:2211.04434 **[hep-lat] C. Egerer** *et. al.*

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Transversity Isovector PDF

On the light-cone:

$$h(x,\mu) = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} e^{-ix\nu} \mathcal{I}(\nu,\mu) \qquad \text{with} \qquad ,$$
$$2P^+ S^{\rho_\perp} \mathcal{I}(P^+ z^-, \mu) = \left\langle P, S^{\rho_\perp} | \bar{\psi}(z^-) \gamma^+ \gamma^{\rho_\perp} \gamma_5 W_+(z^-, 0) \psi(0) | P, S^{\rho_\perp} \right\rangle ,$$

Lorentz decomposition:

$$\langle P, S^\perp | O_{\gamma_5 \gamma_\lambda \gamma_\rho}(z) | P, S^\perp \rangle =$$
$$2(P_\lambda S^\perp_\rho - P_\rho S^\perp_\lambda) \mathcal{M}(z \cdot P, z^2) + 2im_N^2 (z_\lambda S^\perp_\rho - z_\rho S^\perp_\lambda) \mathcal{N}(z \cdot P, z^2) + 2m_N^2 (z_\lambda P_\rho - z_\rho P_\lambda)(z \cdot S^\perp) \mathcal{R}(z \cdot P, z^2).$$

Space-like z:

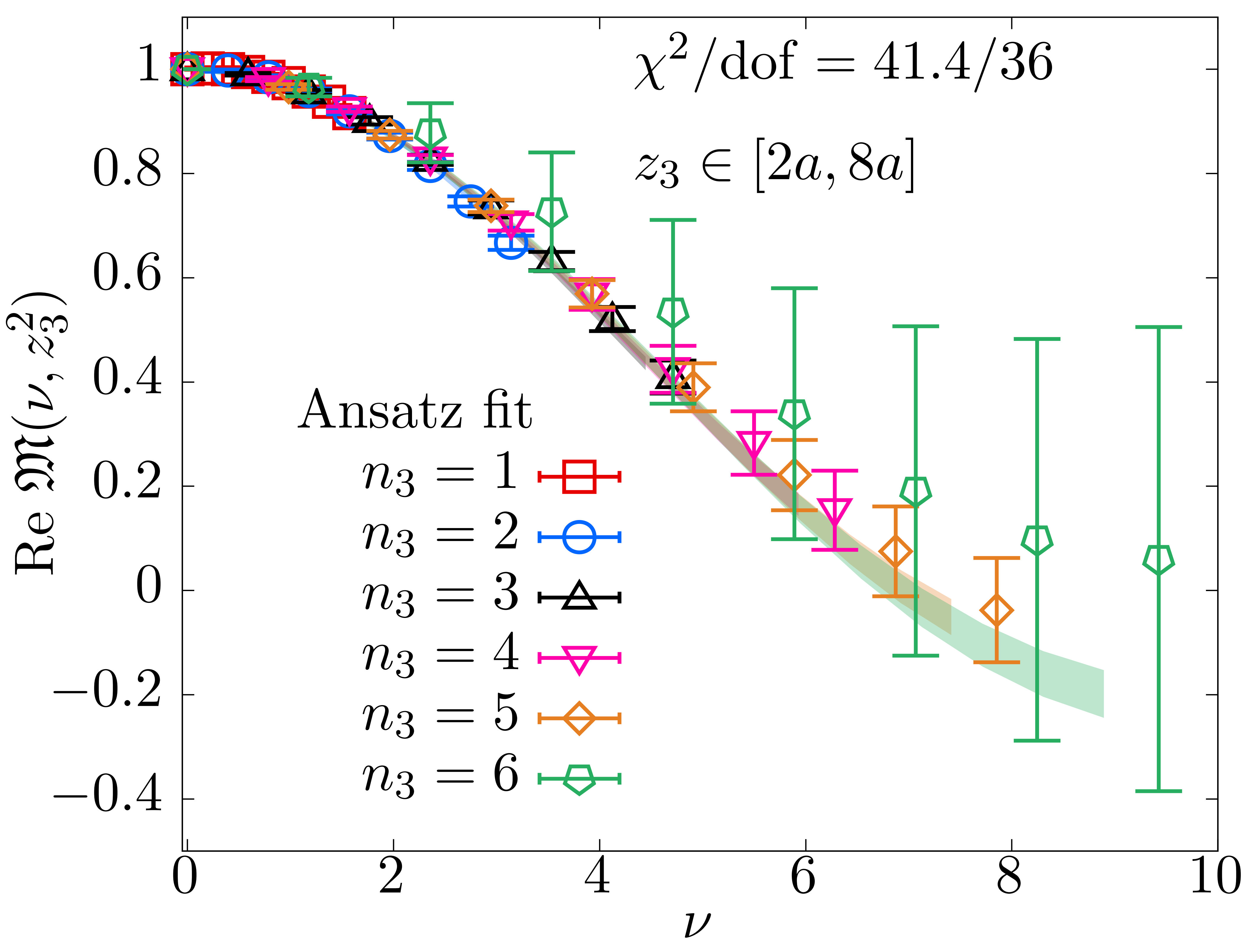
$$\mathcal{M}(z_3, P_3) = \frac{1}{4E(P_3)} \sum_{\rho=1}^{2} \langle P, S^\perp | O_{\gamma_5 \gamma_0 \gamma_\rho}(z) | P, S^\perp \rangle$$

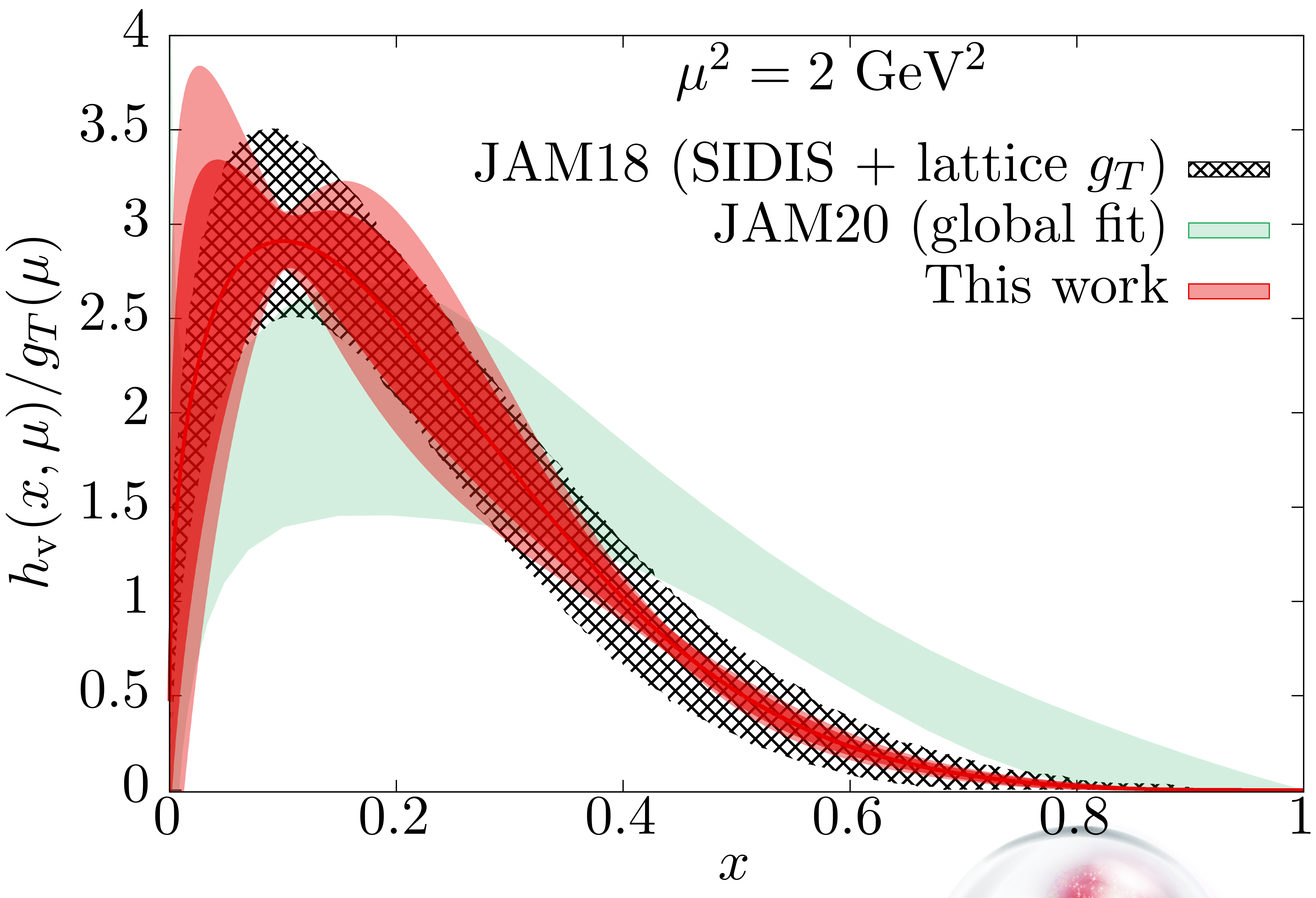
$$\mathfrak{M}(\nu, z_3^2) \equiv \frac{\mathcal{M}(z_3, P_3)}{\mathcal{M}(z_3, 0)} \frac{\mathcal{M}(0,0)}{\mathcal{M}(0, P_3)} \longrightarrow$$ Transversity normalized by g_T

arXiv:2111.01808 **[hep-lat] C. Egerer** *et. al.*

Transversity Isovector PDF

2+1 flavors single lattice spacing 350 MeV pion





arXiv:2111.01808 **[hep-lat] C. Egerer** *et. al.*

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Conclusions

Outlook

- The understanding hadronic structure is a major goal in nuclear physics
  - Large experimental effort: JLab 12 GeV and future EIC
- Lattice QCD calculations can in principle compute (Generalized) Parton distribution functions from first principles
- Controlling all systematics of the calculation is important and that complicates the solution of the inverse problem at hand
  - Both lattice spacing and higher twist effects need to be controlled
- New ideas are needed for pushing to higher momentum and improved sampling of the Ioffe time
  - The range of Ioffe time is essential for obtaining the x-dependence of distribution functions
- The synergy between lattice and experiment may be proven essential in providing precision estimates of (Generalized) Parton distribution functions

Back up — DGLAP

$$\mu^2 \frac{d}{d\mu^2} \mathcal{Q}(\nu, \mu^2) = -\frac{2}{3}\frac{\alpha_s}{2\pi} \int_0^1 du\, B(u)\, \mathcal{Q}(u\nu, \mu^2)$$

$$B(u) = \left[\frac{1+u^2}{1-u}\right]_+$$

DGLAP kernel in position space

V. Braun, et. al Phys. Rev. D 51, 6036 (1995)

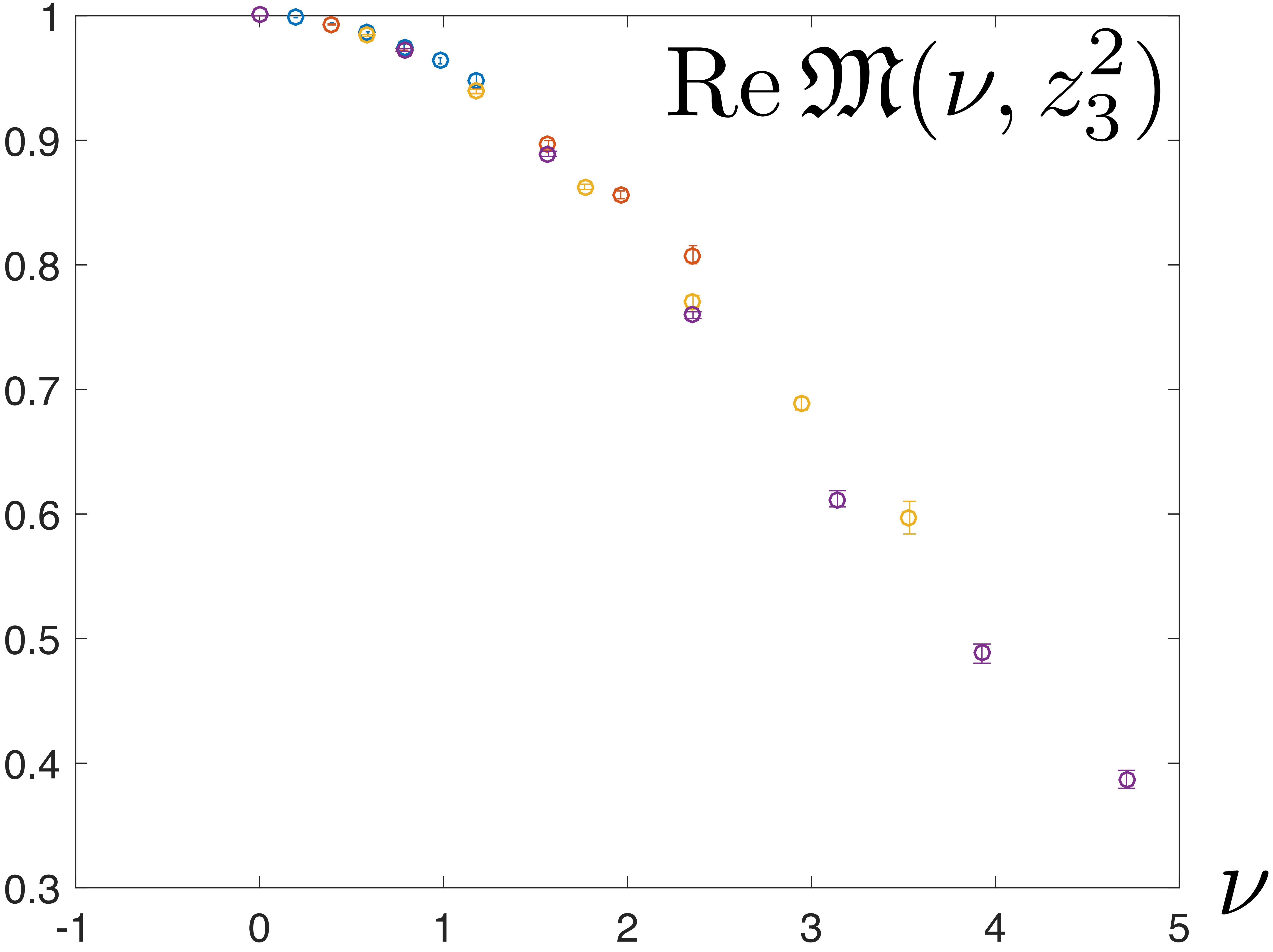
At 1-loop

$$\mathcal{Q}(\nu, \mu'^2) = \mathcal{Q}(\nu, \mu^2) \ -\frac{2}{3}\frac{\alpha_s}{2\pi} \ln(\mu'^2/\mu^2) \int_0^1 du\, B(u)\, \mathcal{Q}(u\nu, \mu^2)$$

Which implies (ignoring higher twist)

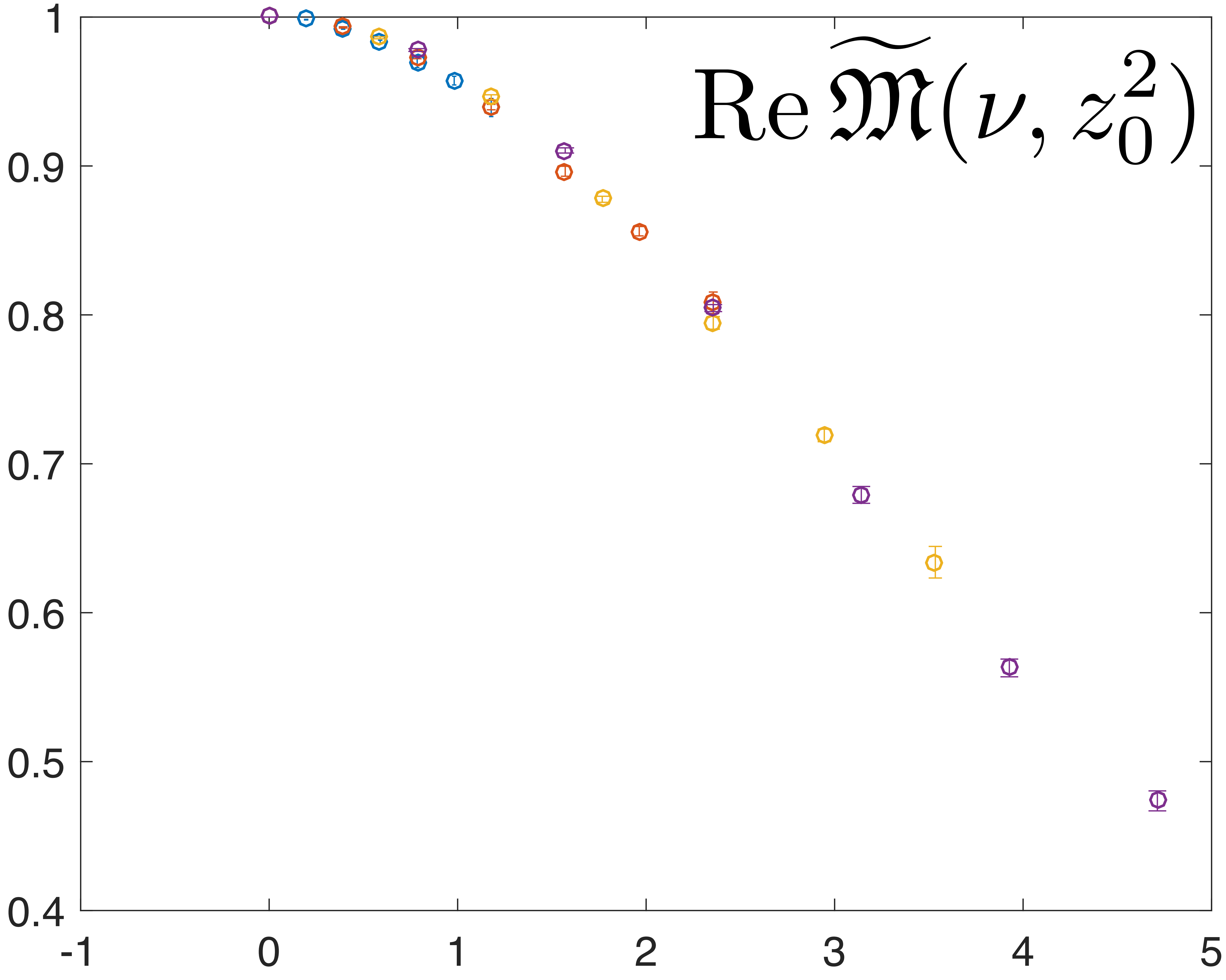
$$\mathfrak{M}(\nu, z'^2) = \mathfrak{M}(\nu, z^2) \ -\frac{2}{3}\frac{\alpha_s(z^2)}{\pi} \ln(z'^2/z^2) \int_0^1 du\, B(u)\, [\mathfrak{M}(u\nu, z^2)$$

Quenched QCD



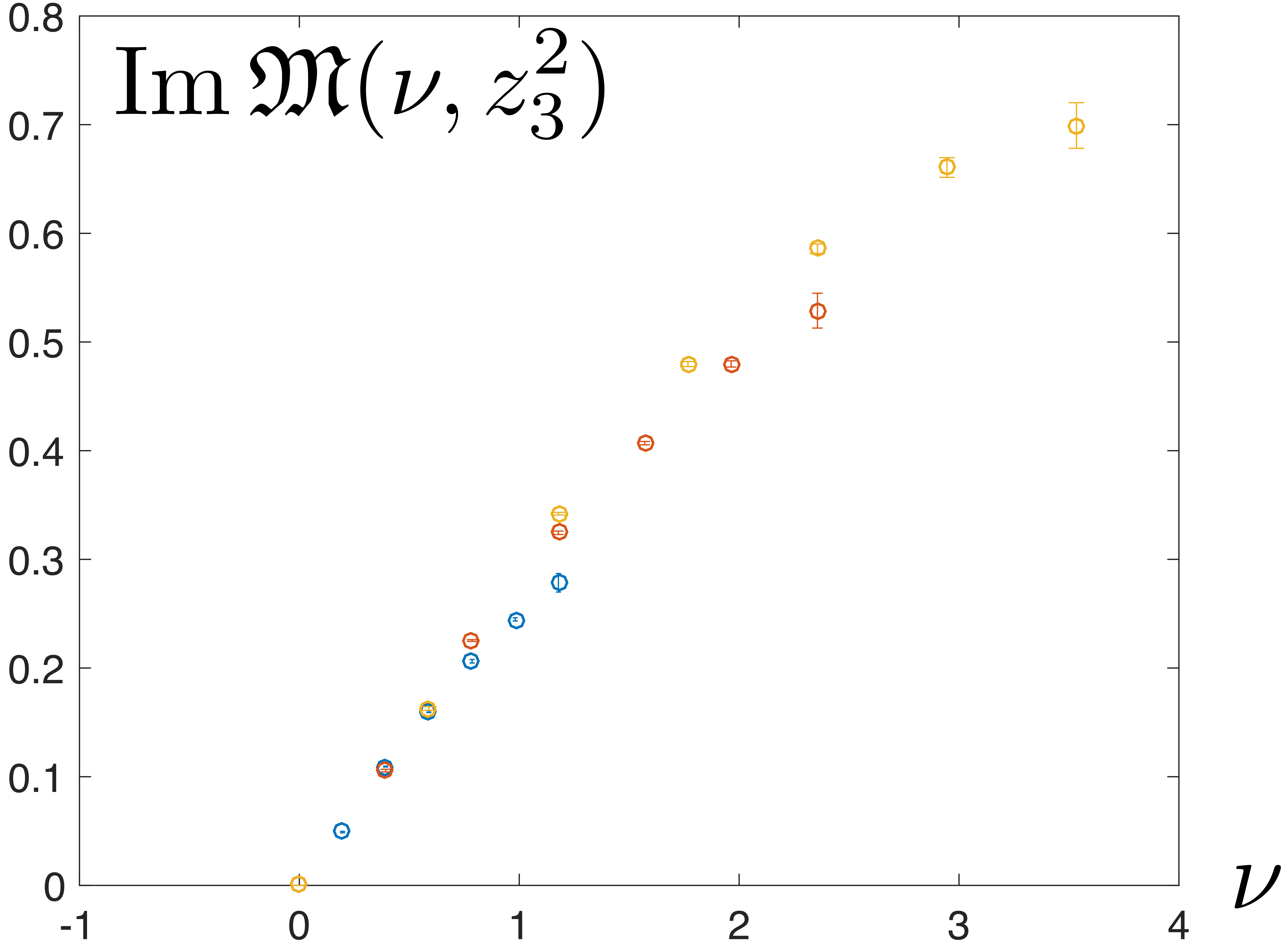
Data corresponding to z/a= 1, 2, 3, 4

Quenched QCD



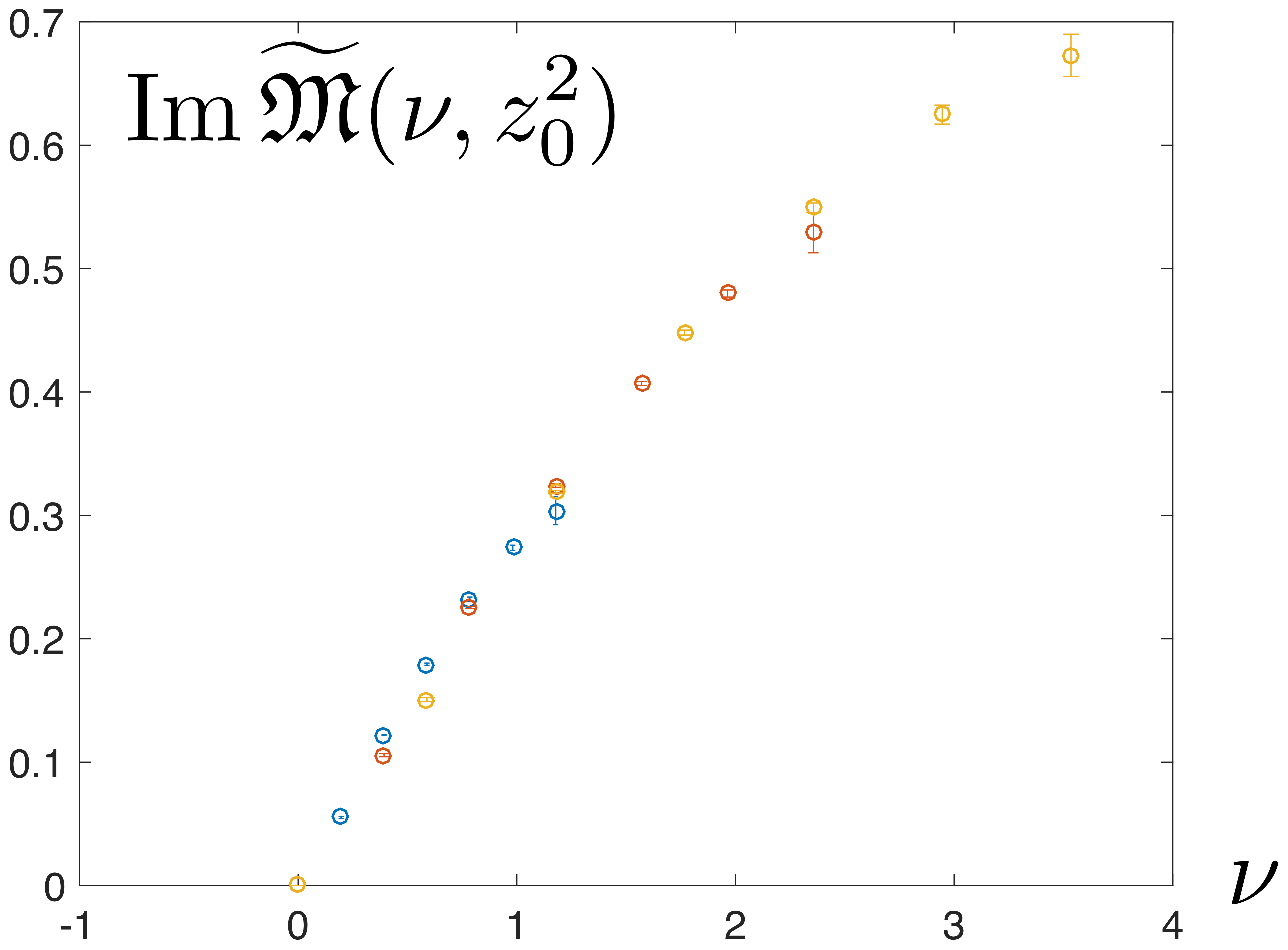
Evolved to 1GeV

Quenched QCD



Data corresponding to z/a= 1, 2, 3, 4

Quenched QCD

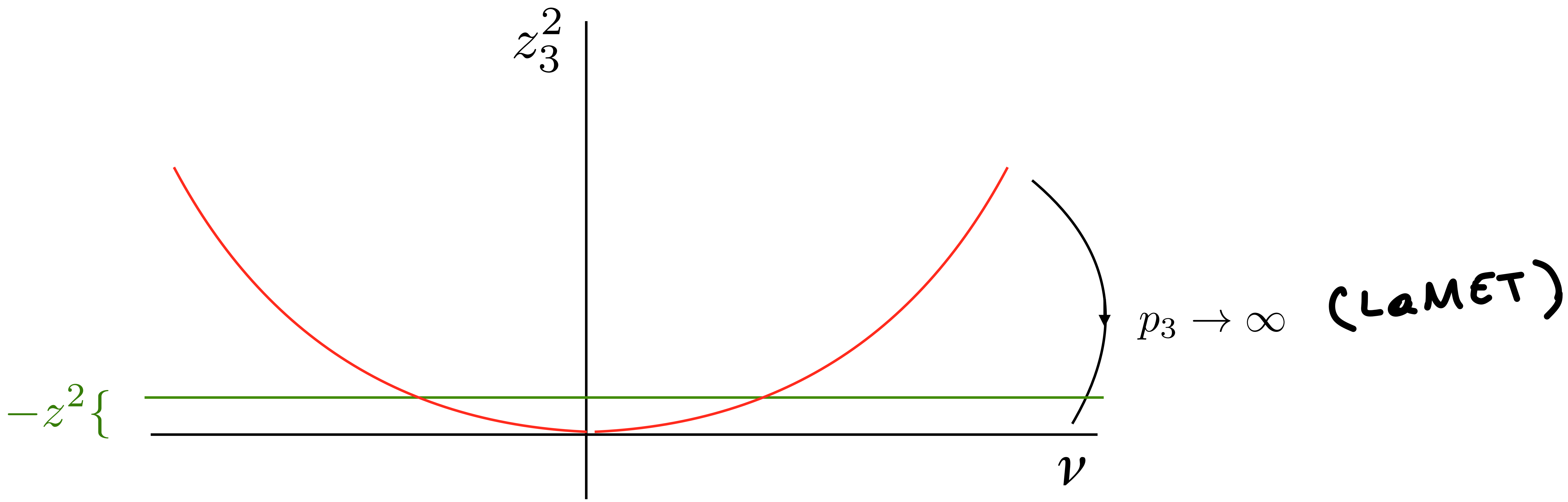


Evolved to 1GeV

$$Q(y, p_3) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \mathcal{M}_p(\nu, \nu^2/p_3^2) e^{-iy\nu}$$ Ji's quasi-PDF $y \in (-\infty, \infty)$

Large values of $z_3 = \nu/p_3$ are problematic

Alternative approach to the light-cone:



$$\mathcal{P}(x, -z^2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu \, \mathcal{M}_p(\nu, -z^2) e^{-ix\nu}$$

PDFs can be recovered $-z^2 \to 0$

Note that $x \in [-1, 1]$

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