

A hadron-structure-oriented approach to TMD phenomenology

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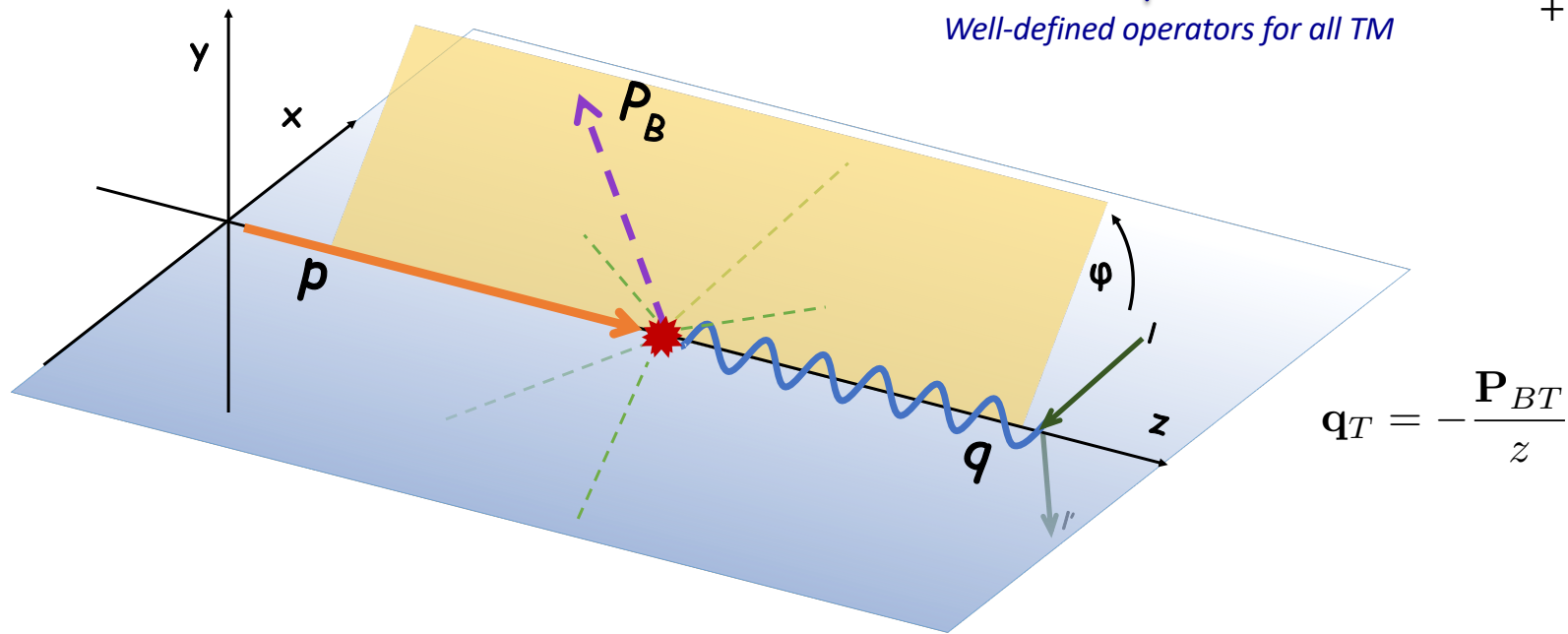
Based on:

- J.O. Gonzalez, TCR, N. Sato, Phys.Rev.D 106 (2022) 3, 034002
- J.O. Gonzalez, T. Rainaldi, TCR, arXiv:2303.04921 [hep-ph]

DIS 2023, March 30, 2023

TMD factorization & SIDIS (for $\frac{q_T}{Q} \ll 1$)

$$\frac{d\sigma}{dQ d^2\mathbf{q}_T dx dz} = H(Q/\mu_Q) \int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} \underbrace{f_{j/p}(x, \mathbf{k}_{1T}; \mu_Q, Q^2) D_{h/j}(z, z\mathbf{k}_{2T}; \mu_Q, Q^2)}_{\text{Well-defined operators for all TM}} \delta^{(2)}(\mathbf{q}_T + \mathbf{k}_{1T} - \mathbf{k}_{2T}) + \text{power suppressed}$$

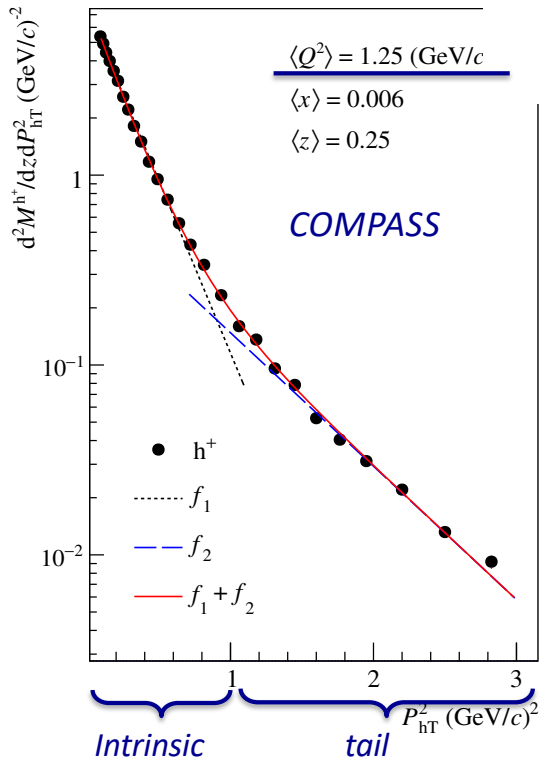


$$\frac{d\sigma}{dQ d^2\mathbf{q}_T dx dz} = H(Q/\mu_Q) \int \frac{d^2\mathbf{b}_T}{(2\pi)^2} e^{-i\mathbf{b}_T \cdot \mathbf{q}_T} \tilde{f}_{j/p}(x, \mathbf{b}_T; \mu_Q, Q^2) \tilde{D}_{h/j}(z, \mathbf{b}_T; \mu_Q, Q^2) + \text{power suppressed}$$

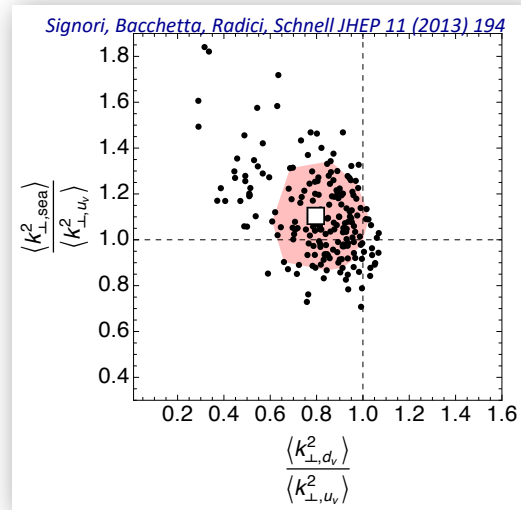
Nonperturbative structures in pheno

PHYSICAL REVIEW D 97, 032006 (2018)

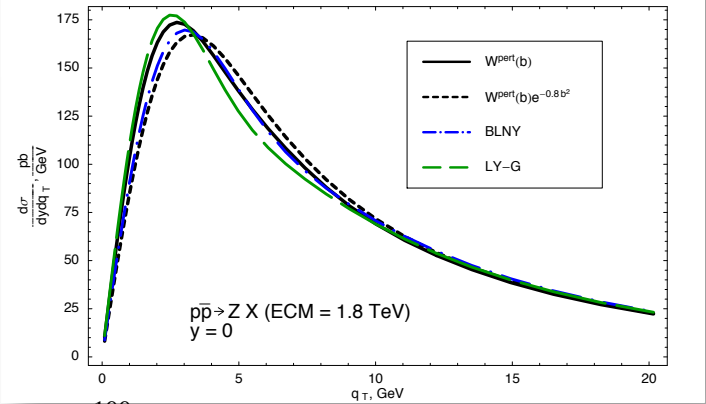
"While significant effort has been put into the study of $W(b)$ at large b [36, 42, 43, 44], none ... adequately describe the observed Z boson distribution without introducing free parameters."



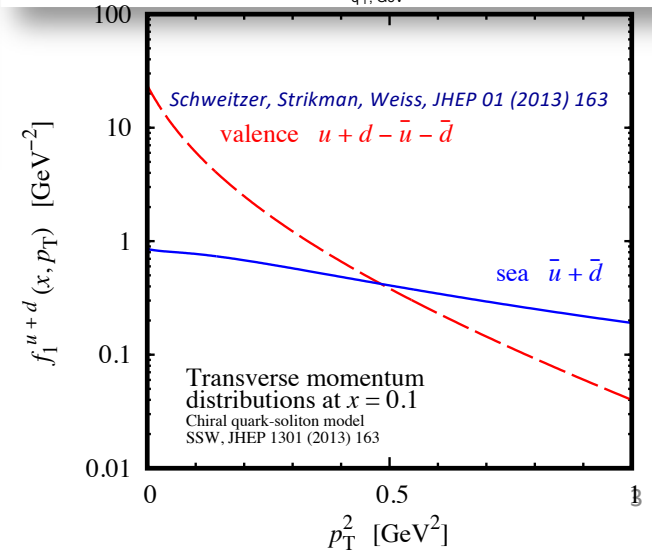
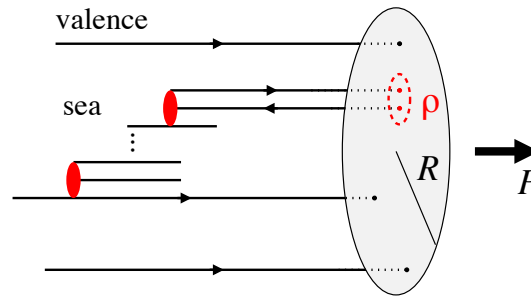
Transverse-momentum-dependent multiplicities of charged hadrons in muon-deuteron deep inelastic scattering



P. Nadolsky, (2004) Theory of W and Z Production

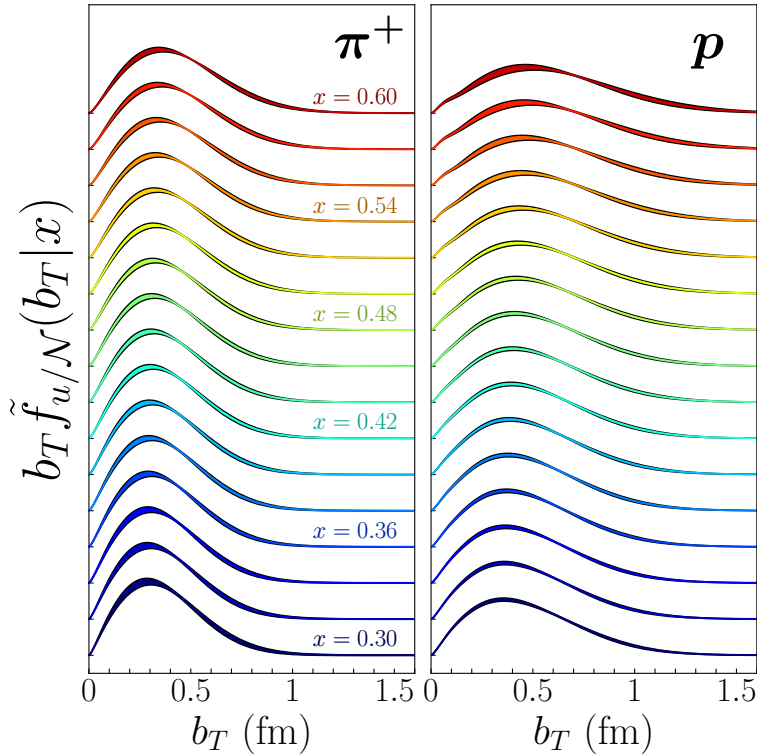


"... the two exponential functions in our parametrizations F_1 can be attributed to two completely different underlying physics mechanisms that overlap in the region $P_{hT} \approx 1 \text{ GeV}^2$."



Nonperturbative structures in pheno

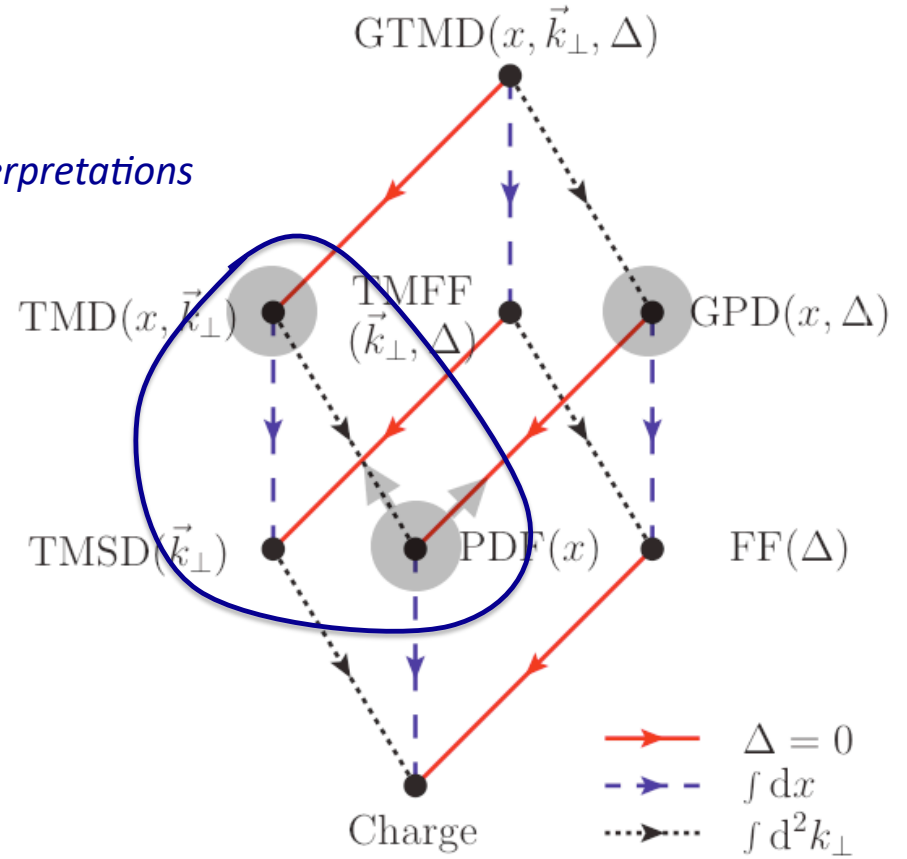
Barry, et al (2302.01192)



??

“Importantly, we have checked that the differences between the proton and pion $\langle b_T | x \rangle$ are completely due to the nonperturbative TMD structure, independent of the collinear PDFs.”

Interpretations



These integrals are often divergent

Conventional organization

$$\frac{\partial \ln \tilde{f}_{j/p}(x, b_T; \mu, \zeta)}{\partial \ln \sqrt{\zeta}} = \tilde{K}(b_T; \mu)$$

$$\frac{d \ln \tilde{f}_{j/p}(x, b_T; \mu, \zeta)}{d \ln \mu} = \gamma(\alpha_s(\mu); \zeta/\mu^2)$$

$$\frac{d \tilde{K}(b_T; \mu)}{d \ln \mu} = -\gamma_K(\alpha_s(\mu))$$

$$\tilde{f}_{j/p}(x, b_T; \mu, \zeta) = \int_x^1 \frac{d\xi}{\xi} \tilde{C}_{j/k}(x/\xi, b_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu) + O(b_T \Lambda_{\text{QCD}})$$

Or

$$f_{j/p}(x, k_T; \mu, \zeta) = \frac{1}{k_T^2} \left[\int_x^1 \frac{d\xi}{\xi} C_{j/k}(x/\xi, k_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu) + O\left(\frac{\Lambda_{\text{QCD}}}{k_T}\right) \right]$$

$$\tilde{K}(b_T; \mu) = \tilde{K}_{\text{pert}}(b_T \mu, \alpha_s(\mu)) + O(b_T \Lambda_{\text{QCD}})$$

Steps:

- 1) Solve evolution equations to relate overall SIDIS hard scale ($\mu_Q = Q$) to input scale ($\mu_{Q_0} = Q_0$)
- 2) How to use small $b_T \ll 1/\Lambda_{\text{QCD}}$ collinear factorization?
 - Partition small ($b_T < b_{\text{max}}$) & large ($b_T > b_{\text{max}}$) regions with a b_*
 - Define hard scale $\mu_{b_*} \sim 1/b_*$
- 3) Evolve again to connect Q_0 to μ_{b_*}
- 4) **Place remaining NP parts in an exponent:**

$$-g_{j/p}(x, b_T) \equiv \ln \left(\frac{\tilde{f}_{j/p}(x, \mathbf{b}_T; \mu_{Q_0}, Q_0^2)}{\tilde{f}_{j/p}(x, \mathbf{b}_*; \mu_{Q_0}, Q_0^2)} \right)$$

$$g_K(b_T) \equiv \tilde{K}(b_*; \mu) - \tilde{K}(b_T; \mu)$$

- 5) **Ansatz for g-functions**
- 6) **Perform small- b_T expansions & drop $O(\Lambda_{\text{QCD}} b_{\text{max}})$ errors**

Conventional organization

$$\int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} f_{j/p}(x, \mathbf{k}_{1T}; \mu_Q, Q^2) D_{h/j}(z, z\mathbf{k}_{2T}; \mu_Q, Q^2) \delta^{(2)}(\mathbf{q}_T + \mathbf{k}_{1T} - \mathbf{k}_{2T})$$



$$\int \frac{d^2\mathbf{b}_T}{(2\pi)^2} e^{-i\mathbf{q}_T \cdot \mathbf{b}_T} \tilde{f}_{j/p}^{\text{OPE}}(x_{bj}, \mathbf{b}_*; \mu_{b_*}, \mu_{b_*}^2) \tilde{D}_{h/j}^{\text{OPE}}(z_h, \mathbf{b}_*; \mu_{b_*}, \mu_{b_*}^2) \\ \times \exp \left\{ 2 \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[\gamma(\alpha_s(\mu'); 1) - \ln \frac{Q}{\mu'} \gamma_K(\alpha_s(\mu')) \right] + \ln \frac{Q^2}{\mu_{b_*}^2} \tilde{K}(b_*; \mu_{b_*}) \right\} \\ \times \exp \left\{ -g_{j/p}(x_{bj}, b_T) - g_{h/j}(z_h, b_T) - g_K(b_T) \ln \left(\frac{Q^2}{Q_0^2} \right) \right\} + O(b_{\max} \Lambda_{QCD})$$

$$\tilde{f}_{j/p}(x, b_T; \mu, \zeta) = \underbrace{\int_x^1 \frac{d\xi}{\xi} \tilde{C}_{j/k}(x/\xi, b_T; \zeta, \mu, \alpha_s(\mu)) f_{k/p}(\xi; \mu)}_{\tilde{f}_{j/p}^{\text{OPE}}(x, b_T; \mu, \zeta)} + O(b_T \Lambda_{QCD})$$

Steps:

- 1) Solve evolution equations to relate overall SIDIS hard scale ($\mu_Q = Q$) to input scale ($\mu_{Q_0} = Q_0$)
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 - Partition small ($b_T < b_{\max}$) & large ($b_T > b_{\max}$) regions with a b_*
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Check consistency

- $\int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} f_{j/p}(x, \mathbf{k}_{1T}; \mu_Q, Q^2) D_{h/j}(z, z\mathbf{k}_{2T}; \mu_Q, Q^2) \delta^{(2)}(\mathbf{q}_T + \mathbf{k}_{1T} - \mathbf{k}_{2T})$
is uniquely determined by its operator definition

- At $q_T \approx Q$

$$= \frac{1}{q_T^2} \left[\underbrace{C(q_T/\mu_Q, \alpha_s(\mu_Q)) \otimes f_{j/p}(x; \mu_Q) \otimes d_{h/j}(z; \mu_Q)}_{q_T \sim Q, Q \rightarrow \infty \text{ asymptote}} + O\left(\frac{\Lambda_{\text{QCD}}}{q_T}\right) \right]$$

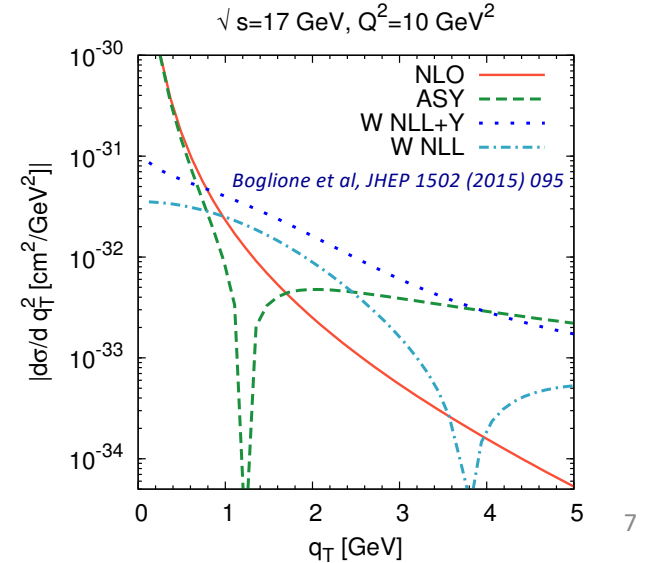
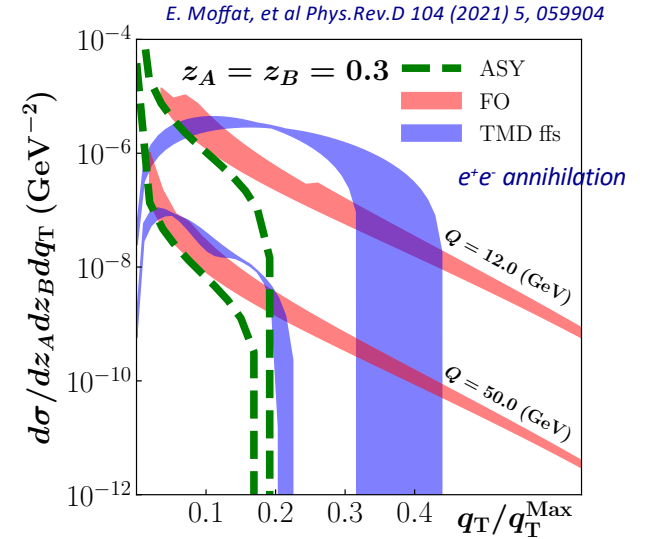
- For TMD pdfs & ffs

$$f_{j/p}(x, k_T \approx Q; \mu_Q, Q^2) = \frac{1}{k_T^2} \left[\int_x^1 \frac{d\xi}{\xi} C_{j/k}(x/\xi, k_T/Q, \alpha_s(Q)) f_{k/p}(\xi; Q) + O\left(\frac{\Lambda_{\text{QCD}}}{k_T}\right) \right]$$

&

$$\int d^2\mathbf{k}_T f_{j/p}(x, k_T; \mu_Q, Q^2) \approx f_{j/p}(x; \mu_Q)$$

- $\frac{d}{db_{\text{max}}} \left(\frac{d\sigma}{d^2\mathbf{q}_T \dots} \right) = 0$ for $b_{\text{max}} \ll 1/\Lambda_{\text{QCD}}$

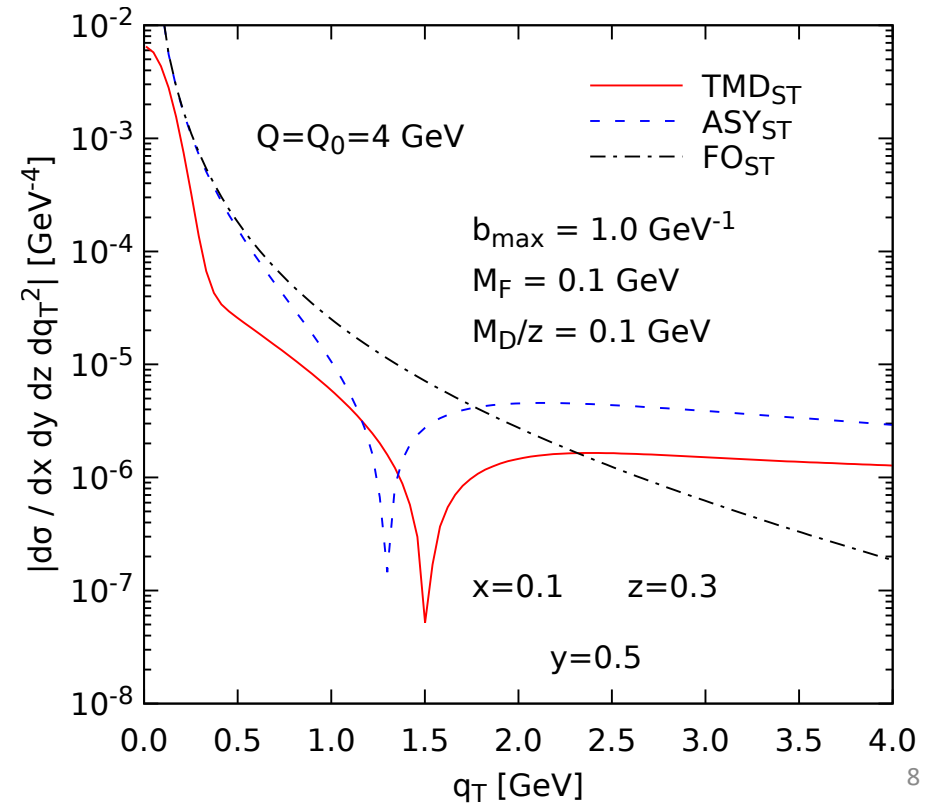
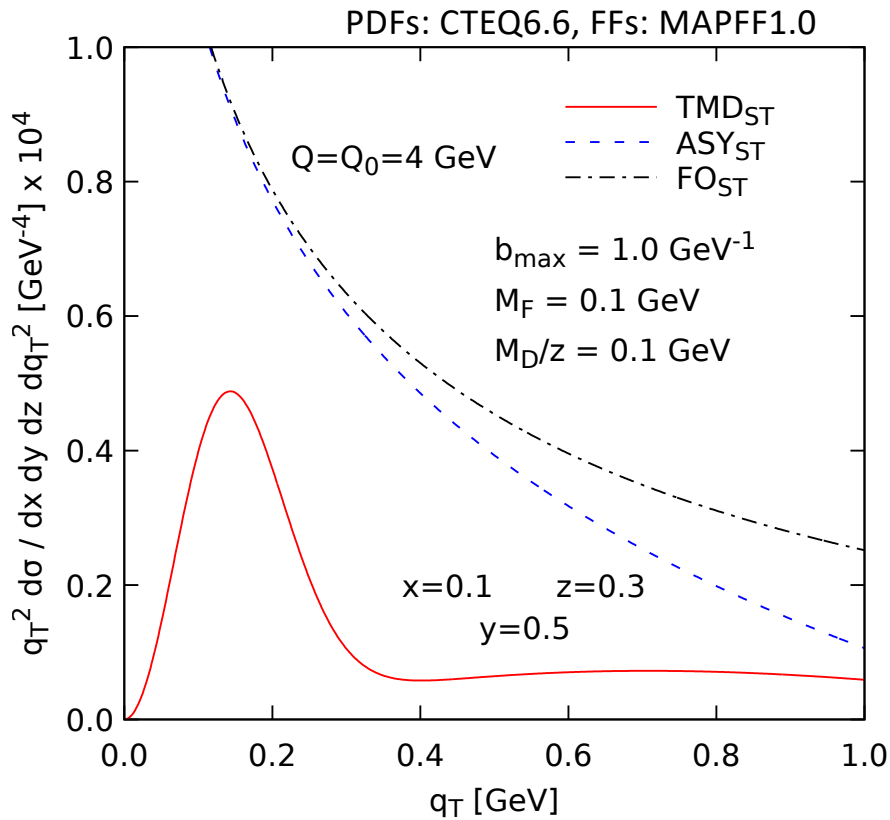


Conventional organization & problems

(An example typical of conventional approach)

$$g_{j/p}(x, b_T) = \frac{1}{4} M_F^2 b_T^2$$

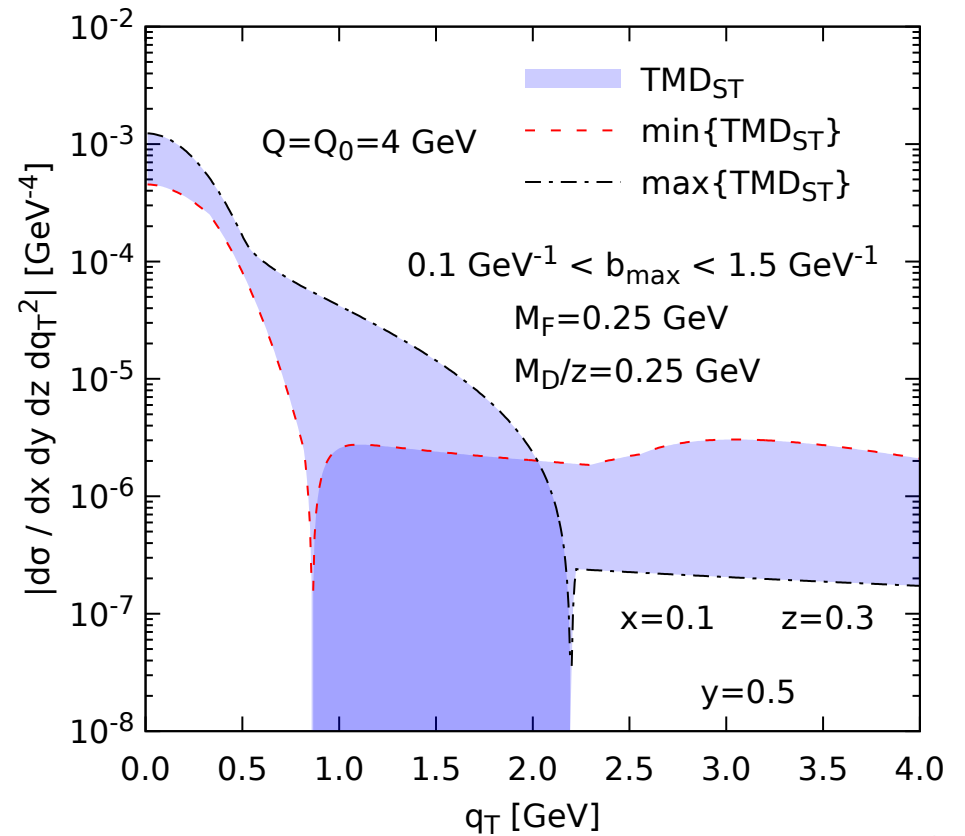
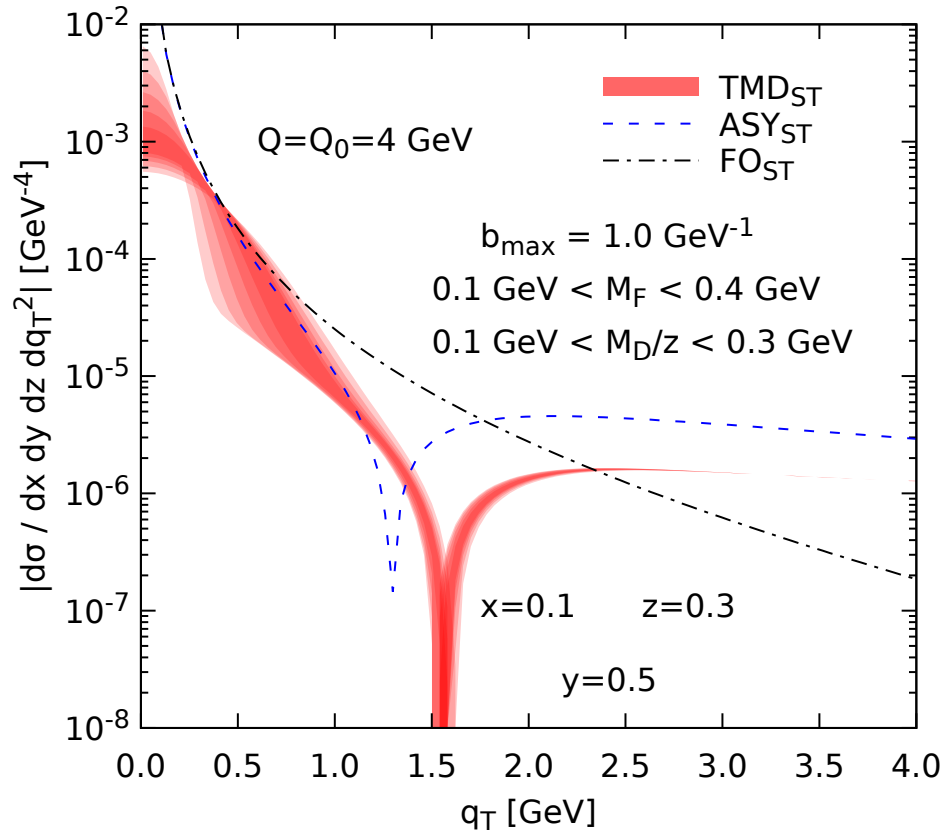
$$g_{h/j}(z, b_T) = \frac{1}{4 z^2} M_D^2 b_T^2$$



Conventional organization & problems

$$g_{j/p}(x, b_T) = \frac{1}{4} M_F^2 b_T^2$$

$$g_{h/j}(z, b_T) = \frac{1}{4 z^2} M_D^2 b_T^2$$



$$\frac{d\sigma}{dQ d^2\mathbf{q}_T dx dz} = H(Q/\mu_Q) \underbrace{\int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} f_{j/p}(x, \mathbf{k}_{1T}; \mu_Q, Q^2) D_{h/j}(z, z\mathbf{k}_{2T}; \mu_Q, Q^2) \delta^{(2)}(\mathbf{q}_T + \mathbf{k}_{1T} - \mathbf{k}_{2T})}_{\text{Well-defined operators for all TM}} + \text{power suppressed}$$

Diagnosis

- 1) Consistency tests will generally fail for g-function ansatz unless constraints are imposed

$$\frac{d\sigma}{dQ d^2\mathbf{q}_T dx dz} =$$

- 2) No region at input scale $Q = Q_0$ where $\Lambda_{QCD} \ll q_T \ll Q_0$

$$H(Q/\mu_Q) \int \frac{d^2\mathbf{b}_T}{(2\pi)^2} e^{-i\mathbf{q}_T \cdot \mathbf{b}_T} \tilde{f}_{j/p}(x, \mathbf{b}_T; \mu_{Q_0}, Q_0^2) \tilde{D}_{h/j}(z, \mathbf{b}_T; \mu_{Q_0}, Q_0^2) \times \exp \left\{ \tilde{K}(\mathbf{b}_T; \mu_{Q_0}) \ln \left(\frac{Q^2}{Q_0^2} \right) + \int_{\mu_{Q_0}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[2\gamma(\alpha_s(\mu'); 1) - \ln \frac{Q^2}{\mu'^2} \gamma_K(\alpha_s(\mu')) \right] \right\}$$

- 3) Backwards evolution...
No large, perturbative $\ln \frac{Q_0}{q_T}$.

- 4) $\int d^2\mathbf{k}_T f_{j/p}(x, \mathbf{k}_T; \mu_Q, Q^2) \approx f_{j/p}(x; \mu_Q)$
Very badly violated at moderate scales

See T. Rainaldi talk, 4:00 pm WG1,
for examples in toy model

$$\frac{d\sigma}{dQ d^2\mathbf{q}_T dx dz} = H(Q/\mu_Q) \underbrace{\int d^2\mathbf{k}_{1T} d^2\mathbf{k}_{2T} f_{j/p}(x, \mathbf{k}_{1T}; \mu_Q, Q^2) D_{h/j}(z, z\mathbf{k}_{2T}; \mu_Q, Q^2) \delta^{(2)}(\mathbf{q}_T + \mathbf{k}_{1T} - \mathbf{k}_{2T})}_{\text{Well-defined operators for all TM}} + \text{power suppressed}$$

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$$\frac{d\sigma}{dQ d^2\mathbf{q}_T dx dz} =$$

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$$H(Q/\mu_Q) \int \frac{d^2\mathbf{b}_T}{(2\pi)^2} e^{-i\mathbf{q}_T \cdot \mathbf{b}_T} \tilde{f}_{j/p}(x, \mathbf{b}_T; \mu_{Q_0}, Q_0^2) \tilde{D}_{h/j}(z, \mathbf{b}_T; \mu_{Q_0}, Q_0^2) \times \exp \left\{ \tilde{K}(\mathbf{b}_T; \mu_{Q_0}) \ln \left(\frac{Q^2}{Q_0^2} \right) + \int_{\mu_{Q_0}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[2\gamma(\alpha_s(\mu'); 1) - \ln \frac{Q^2}{\mu'^2} \gamma_K(\alpha_s(\mu')) \right] \right\}$$

- 3) Backwards evolution...
No large, perturbative $\ln \frac{Q_0}{q_T}$.

$$\tilde{D}_{h/j}(z, \mathbf{b}_T; \mu_{Q_0}, Q_0^2) = \tilde{D}_{\text{inpt}, h/j}(z, \mathbf{b}_T; \mu_{\bar{Q}_0}, \bar{Q}_0^2) E(\bar{Q}_0/Q_0, b_T)$$

$$\tilde{f}_{i/p}(x, \mathbf{b}_T; \mu_{Q_0}, Q_0^2) = \tilde{f}_{\text{inpt}, i/p}(x, \mathbf{b}_T; \mu_{\bar{Q}_0}, \bar{Q}_0^2) E(\bar{Q}_0/Q_0, b_T)$$

- 4) $\int d^2\mathbf{k}_T f_{j/p}(x, \mathbf{k}_T; \mu_Q, Q^2) \approx f_{j/p}(x; \mu_Q)$
Very badly violated at moderate scales

$$\lim_{b_T \rightarrow 0} \bar{Q}_0 \sim 1/b_T$$

See T. Rainaldi talk, 4:00 pm WG1,
for examples in toy model

*Instead, characterize the full behavior of
TMD functions at the input scale*

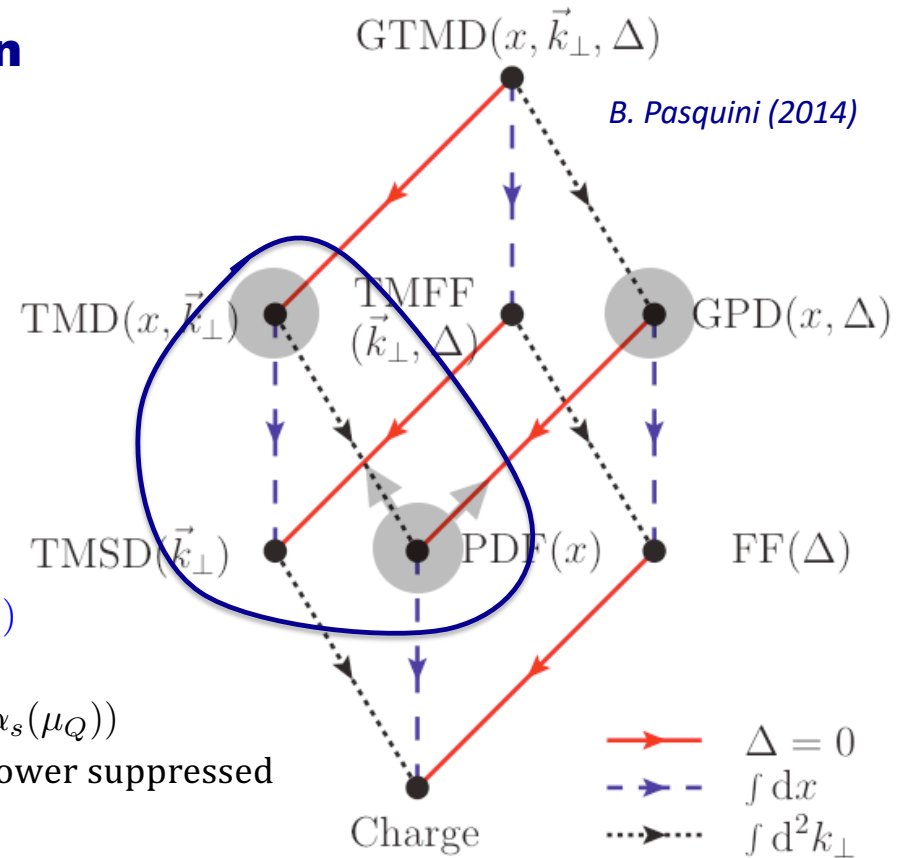
A hadron-structure-oriented reorganization

- 1) Use the uniquely determined TMDs for all k_T
- 2) Smoothly transform directly between **nonperturbative** TM dependence at **small** TM ($k_T \approx \Lambda_{QCD}$) & **perturbative** (collinear) TM at **large** TM ($k_T \approx Q$)
- 3) (Approximate) probability interpretation

- Parton model: $\int d^2\mathbf{k}_T f_{j/p}(x, k_T; \mu_Q, Q^2) = f_{j/p}(x; \mu_Q)$

- QCD: $\pi \int^{\mu_Q^2} dk_T^2 f_{j/p}(x, k_T; \mu_Q, Q^2) = f_{j/p}(x; \mu_Q) + O(\alpha_s(\mu_Q))$
+ power suppressed

- 4) All should apply at input scale, Q_0
- 5) Pheno: Must be simple to swap one model/parametrization for another



A hadron-structure-oriented reorganization

- Recipe: Transform a model TMD parametrization into an evolved parametrization at other scales:
 - See Sec. VI of Phys.Rev.D 106 (2022) 3, 034002
- No b_{\max} , no b_*
- HSO approach is equivalent to standard TMD factorization, CSS, etc, just with constraints on the g-functions
- Straightforward to translate between standard treatment and HSO.
 - See Sec. IX of Phys.Rev.D 106 (2022) 3, 034002

An $O(\alpha_s)$ example

$$f_{\text{inpt},i/p}(x, \mathbf{k}_T; \mu_{Q_0}, Q_0^2) = \frac{1}{2\pi} \frac{1}{k_T^2 + m_{f_{i,p}}^2} \left[A_{i/p}^f(x; \mu_{Q_0}) + B_{i/p}^f(x; \mu_{Q_0}) \ln \frac{Q_0^2}{k_T^2 + m_{f_{i,p}}^2} \right] + \frac{1}{2\pi} \frac{1}{k_T^2 + m_{f_{g,p}}^2} A_{i/p}^{f,g}(x; \mu_{Q_0})$$

$$+ C_{i/p}^f f_{\text{core},i/p}(x, \mathbf{k}_T; Q_0^2)$$

$$D_{\text{inpt},h/j}(z, z\mathbf{k}_T; \mu_{Q_0}, Q_0^2) = \frac{1}{2\pi z^2} \frac{1}{k_T^2 + m_{D_{h,j}}^2} \left[A_{h/j}^D(z; \mu_{Q_0}) + B_{h/j}^D(z; \mu_{Q_0}) \ln \frac{Q_0^2}{k_T^2 + m_{D_{h,j}}^2} \right] + \frac{1}{2\pi z^2} \frac{1}{k_T^2 + m_{D_{h,g}}^2} A_{h/j}^{D,g}(z; \mu_{Q_0})$$

$$+ C_{h/j}^D D_{\text{core},h/j}(z, z\mathbf{k}_T; Q_0^2)$$

- C^f & C^D constrained by:

$$f_{i/p}^c(x; \mu_{Q_0}) \equiv 2\pi \int_0^{\mu_{Q_0}} dk_T k_T f_{i/p}(x, \mathbf{k}_T; \mu_{Q_0}, Q_0^2) = f_{i/p}(x; \mu_{Q_0}) + \mathcal{H}_{i/i'} \otimes f_{i'/p} + \text{p.s.}$$

$$d_{h/j}^c(z; \mu_{Q_0}) \equiv 2\pi z^2 \int_0^{\mu_{Q_0}} dk_T k_T D_{h/j}(z, z\mathbf{k}_T; \mu_{Q_0}, Q_0^2) = d_{h/j}(z; \mu_{Q_0}) + \mathcal{H}_{j'/j} \otimes d_{h/j'} + \text{p.s.}$$

An $O(\alpha_s)$ example

- Parametrizing the very small transverse momentum

A. Gaussian model

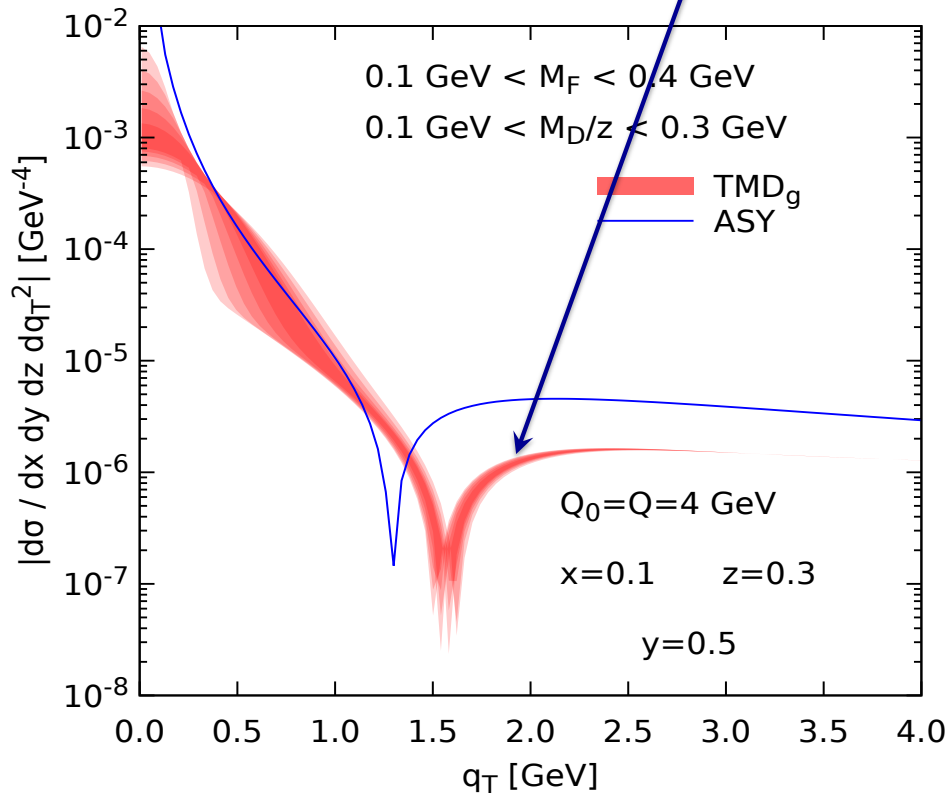
$$f_{\text{core},i/p}^{\text{Gauss}}(x, \mathbf{k}_T; Q_0^2) = \frac{e^{-k_T^2/M_F^2}}{\pi M_F^2}, \quad D_{\text{core},h/j}^{\text{Gauss}}(z, z\mathbf{k}_T; Q_0^2) = \frac{e^{-z^2 k_T^2/M_D^2}}{\pi M_D^2}$$

B. Spectator model

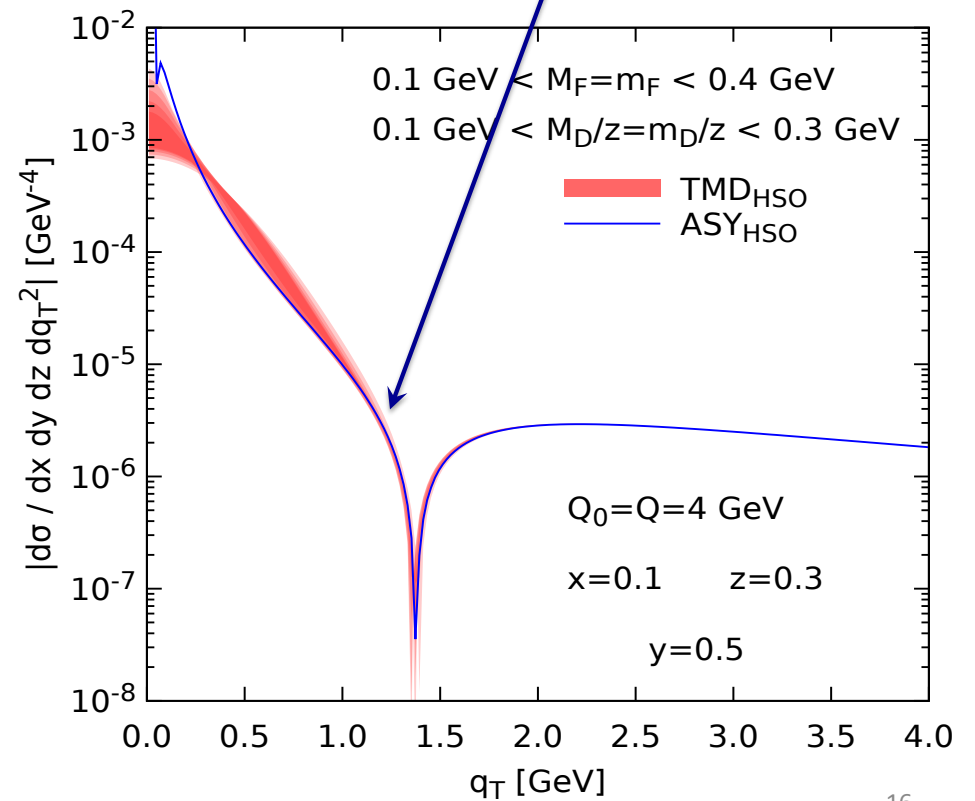
$$f_{\text{core},i/p}^{\text{Spect}}(x, \mathbf{k}_T; Q_0^2) = \frac{6M_{0F}^6}{\pi(2M_F^2 + M_{0F}^2)} \frac{M_F^2 + k_T^2}{(M_{0F}^2 + k_T^2)^4}, \quad D_{\text{core},h/j}^{\text{Spect}}(z, z\mathbf{k}_T; Q_0^2) = \frac{2M_{0D}^4}{\pi(M_D^2 + M_{0D}^2)} \frac{M_D^2 + k_T^2 z^2}{(M_{0D}^2 + k_T^2 z^2)^3}$$

Compare standard with constrained

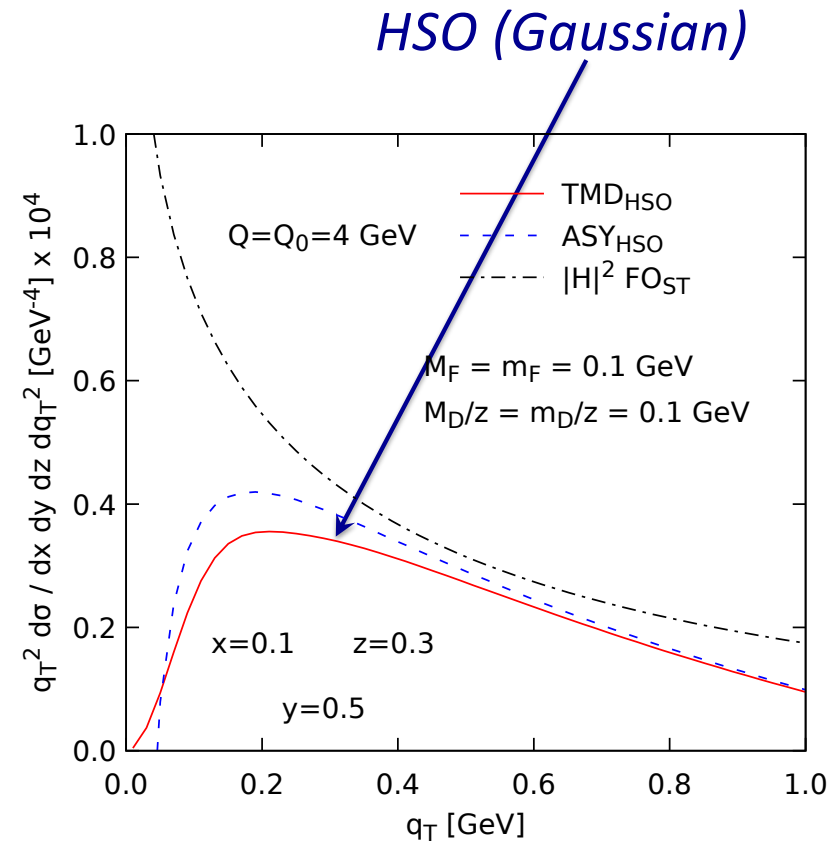
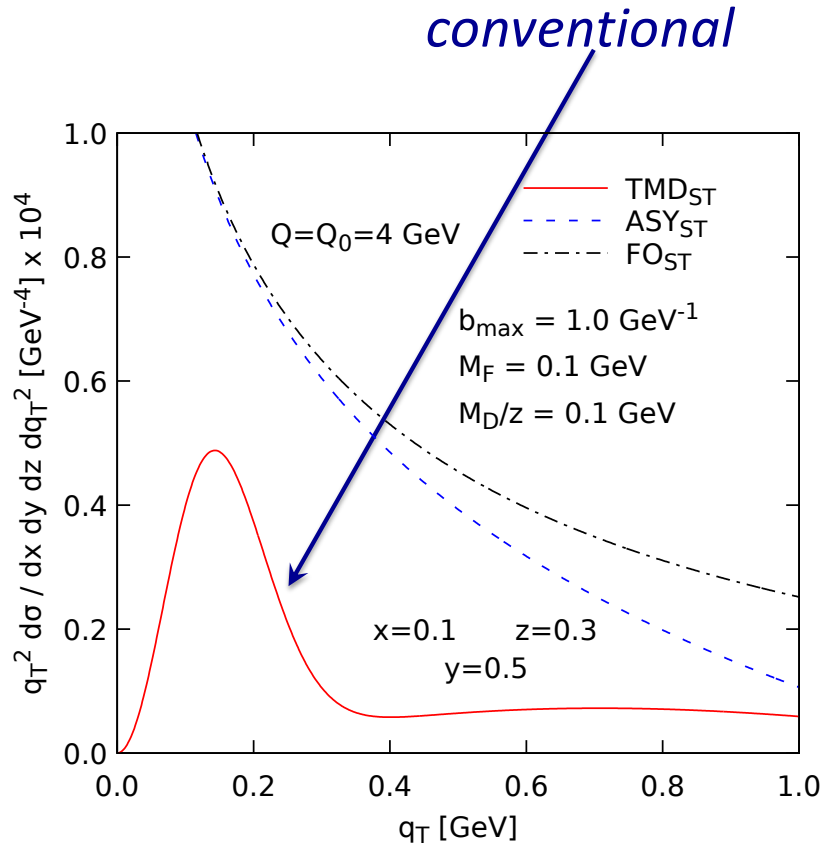
Typical/conventional



HSO (Gaussian)



Compare standard with constrained

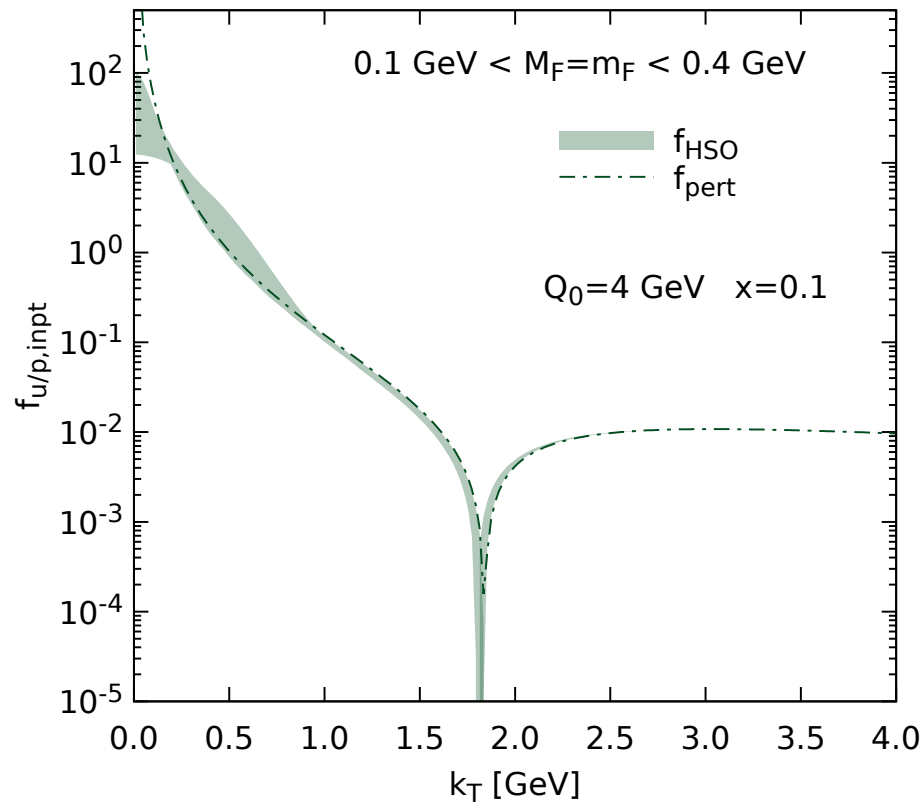


Summary

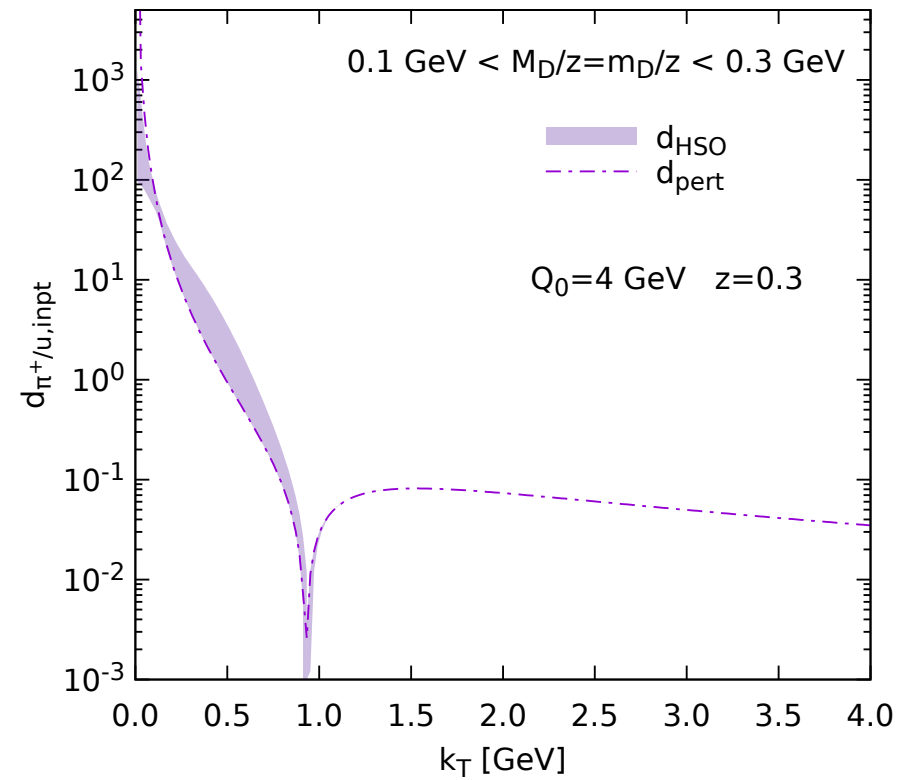
- Switching to a hadron-structure-oriented approach to pheno with TMD factorization improves consistency in the large transverse behavior of TMD correlation functions
- Necessary for understanding the shapes of nonperturbative distributions, separating perturbative and nonperturbative parts, etc
- Next:
 - Applications
 - Higher orders
 - Incorporating NP calculations (lattice, EFTs, models etc)
 - Spin dependent TMDs

Backup

up-quark TMD pdfs



π^+ from up-quark TMD ff



HSO (Gaussian)

Backup

$$A_{i/p}^f(x; \mu_{Q_0}) \equiv \sum_{ii'} \delta_{ii'} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ [(P_{i'i} \otimes f_{i'/p})(x; \mu_{Q_0})] - \frac{3C_F}{2} f_{i'/p}(x; \mu_{Q_0}) \right\},$$

$$B_{i/p}^f(x; \mu_{Q_0}) \equiv \sum_{i'i} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} f_{i'/p}(x; \mu_{Q_0}),$$

$$A_{i/p}^{f,g}(x; \mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} [(P_{ig} \otimes f_{g/p})(x; \mu_{Q_0})],$$

$$C_{i/p}^f \equiv \frac{1}{N_{i/p}^f} \left[f_{i/p}(x; \mu_{Q_0}) - A_{i/p}^f(x; \mu_{Q_0}) \ln \left(\frac{\mu_{Q_0}}{m_{f_{i,p}}} \right) - B_{i/p}^f(x; \mu_{Q_0}) \ln \left(\frac{\mu_{Q_0}}{m_{f_{i,p}}} \right) \ln \left(\frac{Q_0^2}{\mu_{Q_0} m_{f_{i,p}}} \right), \right. \\ \left. - A_{i/p}^{f,g}(x; \mu_{Q_0}) \ln \left(\frac{\mu_{Q_0}}{m_{f_{g,p}}} \right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{ii'} \delta_{i'i} [C_{\Delta}^{i/i'} \otimes f_{i'/p}](x; \mu_{Q_0}) + [C_{\Delta}^{i/g} \otimes f_{g/p}](x; \mu_{Q_0}) \right\} \right].$$

$$P_{ig}(x) = T_F [x^2 + (1-x)^2],$$

$$C_{\Delta}^{i/i}(x) = C_F(1-x) - C_F \frac{\pi^2}{12} \delta(1-x),$$

$$C_{\Delta}^{g/p}(x) = 2T_F x(1-x),$$

$$N_{i/p}^f \equiv 2\pi \int_0^{\infty} dk_{\text{T}} k_{\text{T}} f_{\text{core},i/p}(x, \mathbf{k}_{\text{T}}; Q_0^2)$$

Backup

$$\begin{aligned}
A_{h/j}^D(z; \mu_{Q_0}) &\equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ [(P_{jj'} \otimes d_{h/j'})(z; \mu_{Q_0})] - \frac{3C_F}{2} d_{h/j'}(z; \mu_{Q_0}) \right\}, \\
B_{h/j}^D(z; \mu_{Q_0}) &\equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} d_{h/j'}(z; \mu_{Q_0}), \\
A_{h/j}^{D,g}(z; \mu_{Q_0}) &\equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} [(P_{gj} \otimes d_{h/g})(z; \mu_{Q_0})], \\
C_{h/j}^D &\equiv \frac{1}{N_{h/j}^D} \left[d_{h/j}(z; \mu_{Q_0}) - A_{h/j}^D(z; \mu_{Q_0}) \ln \left(\frac{\mu_{Q_0}}{m_{D_{h,j}}} \right) - B_{h/j}^D(z; \mu_{Q_0}) \ln \left(\frac{\mu_{Q_0}}{m_{D_{h,j}}} \right) \ln \left(\frac{Q_0^2}{\mu_{Q_0} m_{D_{h,j}}} \right) \right. \\
&\quad \left. - A_{h/j}^{D,g}(z; \mu_{Q_0}) \ln \left(\frac{\mu_{Q_0}}{m_{D_{h,g}}} \right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{jj'} \delta_{j'j} [C_{\Delta}^{j'/j} \otimes d_{h/j'}](z; \mu_{Q_0}) + [C_{\Delta}^{g/j} \otimes d_{h/g}](z; \mu_{Q_0}) \right\} \right].
\end{aligned}$$

$$P_{qq}(z) = P_{\bar{q}\bar{q}}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2} \delta(1-z) \right],$$

$$P_{gq}(z) = C_F \frac{1+(1-z)^2}{z},$$

$$C_{\Delta}^{q/q}(z) = 2P_{qq}(z) \ln z + C_F(1-z) - C_F \frac{\pi^2}{12} \delta(1-z),$$

$$C_{\Delta}^{g/q}(z) = 2P_{gq}(z) \ln z + C_F z,$$

$$N_{h/j}^D \equiv 2\pi z^2 \int_0^{\infty} dk_{\text{T}} k_{\text{T}} D_{\text{core},h/j}(z, z\mathbf{k}_{\text{T}}; Q_0^2).$$

Backup

$$\begin{aligned}
f_{\text{inpt},i/p}^c(x; \mu_{Q_0}) &= 2\pi \int_0^{\mu_{Q_0}} dk_{\text{T}} k_{\text{T}} f_{\text{inpt},i/p}(x, \mathbf{k}_{\text{T}}; \mu_{Q_0}, Q_0^2) = \\
& C_{i/p}^f f_{\text{core},i/p}^c(x; \mu_{Q_0}) + \frac{1}{2} A_{i/p}^{f,g}(x; \mu_{Q_0}) \ln \left(1 + \frac{\mu_{Q_0}^2}{m_{f_{g,p}}^2} \right) \\
& + \frac{1}{2} A_{i/p}^f(x; \mu_{Q_0}) \ln \left(1 + \frac{\mu_{Q_0}^2}{m_{f_{i,p}}^2} \right) + \frac{1}{4} B_{i/p}^f(x; \mu_{Q_0}) \left[\ln^2 \left(\frac{m_{f_{i,p}}^2}{Q_0^2} \right) - \ln^2 \left(\frac{\mu_{Q_0}^2 + m_{f_{i,p}}^2}{Q_0^2} \right) \right] \\
& = f_{i/p}(x; \mu_{Q_0}) + O \left(\alpha_s(\mu_0), \frac{m^2}{Q_0^2} \right),
\end{aligned}$$

$$\begin{aligned}
d_{\text{inpt},h/j}^c(z; \mu_{Q_0}) &= 2\pi z^2 \int_0^{\mu_{Q_0}} dk_{\text{T}} k_{\text{T}} D_{\text{inpt},h/j}(z, z\mathbf{k}_{\text{T}}; \mu_{Q_0}, Q_0^2) = \\
& C_{h/j}^D d_{\text{core},h/j}^c(z; \mu_{Q_0}) + \frac{1}{2} A_{h/j}^{D,g}(z; \mu_{Q_0}) \ln \left(1 + \frac{\mu_{Q_0}^2}{m_{D_{h,g}}^2} \right) \\
& + \frac{1}{2} A_{h/j}^D(z; \mu_{Q_0}) \ln \left(1 + \frac{\mu_{Q_0}^2}{m_{D_{h,j}}^2} \right) + \frac{1}{4} B_{h/j}^D(z; \mu_{Q_0}) \left[\ln^2 \left(\frac{m_{D_{h,j}}^2}{Q_0^2} \right) - \ln^2 \left(\frac{\mu_{Q_0}^2 + m_{D_{h,j}}^2}{Q_0^2} \right) \right] \\
& = d_{h/j}(z; \mu_{Q_0}) + O \left(\alpha_s(\mu_0), \frac{m^2}{Q_0^2} \right),
\end{aligned}$$