Fragmentation functions: definitions & sum rules

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Based on recent work with John Collins:
2309.03346 [hep-ph]

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Fragmentation functions

- Definition: $$d_{(0),h/j}(z) \equiv \int \mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}\, d_{(0),h/j}(z,\boldsymbol{p}_\mathrm{T})$$
$$= \frac{\mathrm{Tr}_D}{4}\sum_X z^{1-2\epsilon}\int \frac{\mathrm{d}x^+}{2\pi} e^{ik^-x^+}\gamma^-\langle 0|\psi_j^{(0)}(x/2)\,|h,X,\mathrm{out}\rangle\langle h,X,\mathrm{out}|\overline{\psi}_j^{(0)}(-x/2)\,|0\rangle$$

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- In QCD: Include Wilson lines and color trace.

Sum rules:

• Momentum	$\sum_h \int_0^1 \mathrm{d}z\; z\, d_{h/j}(z) = 1$
• Charge	$\sum_h \mathcal{Q}_h \int_0^1 \mathrm{d}z\; d_{h/j}(z) = \mathcal{Q}_j$

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• Multiplicity	$\sum_h \int_0^1 \mathrm{d}z\; d_{h/j}(z) = \langle N\rangle$
• Extensions to multihadron FFs & other correlation functions	$\sum_h \int \mathrm{d}z_1\mathrm{d}z_2\, d_{h_1h_2/j}(z_1,z_2) = \langle N(N-1)\rangle$ *Pitonyak et al, arXiv:2305.11995*
• TMD FFs, etc…	

General derivations

- Definition (bare)

$$d_{(0),h/j}(z,\boldsymbol{p}_{\mathrm{T}}) \sim \sum_X \langle \underbrace{\mathrm{quark}}_{\text{Off-shell quark}}|\overbrace{h, X, \mathrm{out}}^{\text{Count particles of type } h}\rangle\langle h, X, \mathrm{out}|\mathrm{quark}'\rangle$$

$$\sum_X |h, X, \mathrm{out}\rangle\langle h, X, \mathrm{out}| \equiv \sum_X a^\dagger_{h,p,\mathrm{out}}|X, \mathrm{out}\rangle\langle X, \mathrm{out}|a_{h,p,\mathrm{out}}$$

- Operators for conserved currents (e.g. momentum)

$$\mathcal{P}^\mu = \sum_h \int_0^\infty \frac{\mathrm{d}p^-}{2p^-} \int \frac{\mathrm{d}^{2-2\epsilon}\boldsymbol{p}_{\mathrm{T}}}{(2\pi)^{3-2\epsilon}}\, a^\dagger_{h,p,\mathrm{out}} p^\mu a_{h,p,\mathrm{out}} = a^\dagger_{h,p,\mathrm{out}} a_{h,p,\mathrm{out}}$$

- Sum rules follow from unitarity of asymptotic states

$$\sum_X |X, \mathrm{out}\rangle\langle X, \mathrm{out}| = \widehat{1}$$

- Preserved by standard renormalization

- Straightforward in **nongauge** theories

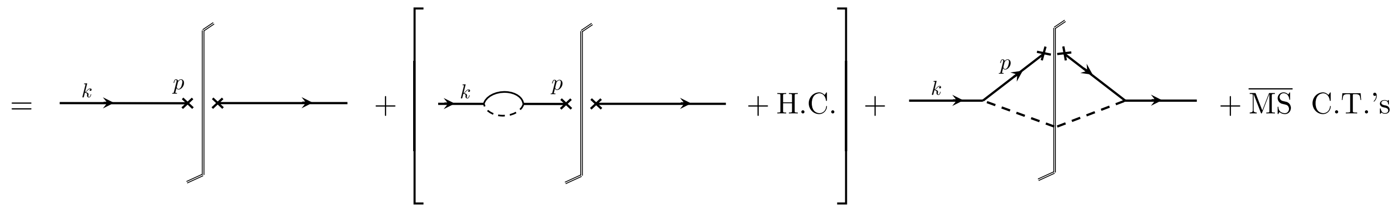
Examples in $\overline{MS}$

Dirac "quark" *Scalar "pion"*

- Scalar Yukawa theory: $\mathcal{L} = i\bar{\psi}\not{\partial}\psi + \frac{1}{2}(\partial\phi)^2 - m_q\bar{\psi}\psi - \frac{m_\pi^2}{2}\phi^2 - \lambda\,\bar{\psi}\psi\phi$

Quark-in-quark fragmentation function

$d_{q_j/j}(z;\mu)$



$$= \delta(1-z) - a_\lambda(\mu)\delta(1-z)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right] + a_\lambda(\mu)(1-z)\left[\ln\left(\frac{1}{(1-z)^2+z}\right) + \ln\frac{\mu^2}{m^2} - 1 + \frac{(1+z)^2}{(1-z)^2+z}\right]$$

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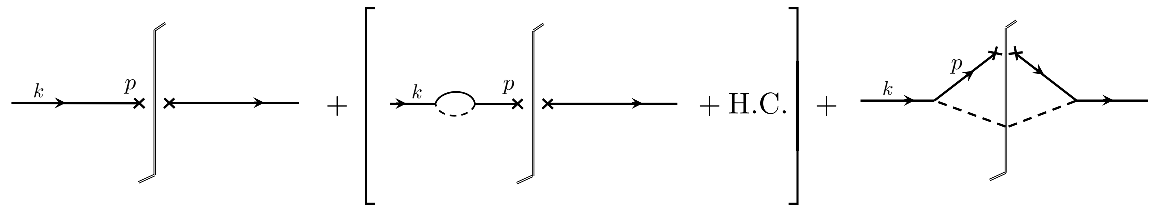
$=$ 

$+ \overline{\mathrm{MS}}$ C.T.'s

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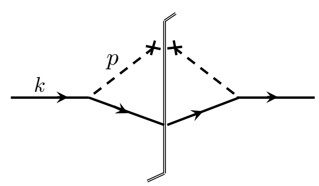
Pion-in-quark fragmentation function

$d_{\pi/j}(z;\mu) =$ 

$+ \overline{\mathrm{MS}}$ C.T.

$$= a_\lambda(\mu)z\left[\ln\left(\frac{1}{(1-z)+z^2}\right) + \ln\frac{\mu^2}{m^2} - 1 + \frac{(2-z)^2}{(1-z)+z^2}\right]$$

Examples in $\overline{MS}$

- Moments

$$\int_0^1 \mathrm{d}z\, d_{q_j/j}(z;\mu) = 1 - a_\lambda(\mu)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right] + a_\lambda(\mu)\left[\pi\frac{\sqrt{3}}{2} - 2 + \ln\frac{\mu}{m}\right] = 1\,,$$

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$$\int_0^1 \mathrm{d}z\, z d_{q_j/j}(z;\mu) = 1 - a_\lambda(\mu)\left[-\frac{13}{9} + \frac{\pi}{\sqrt{3}} + \frac{1}{3}\ln\frac{\mu^2}{m^2}\right]\,,$$

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- Check sum rules:

✓ $$\sum_h \int_0^1 \mathrm{d}z\ z\, d_{h/j}(z) = 1$$

✓ $$\sum_h \mathcal{Q}_h \int_0^1 \mathrm{d}z\ d_{h/j}(z) = \mathcal{Q}_j$$

✓ $$\sum_h \int_0^1 \mathrm{d}z\ d_{h/j}(z) = \langle N \rangle$$

A paradox in the definitions?

- Are FFs zero?

$$\langle \text{quark}|\text{quark}'\rangle = \sum_X \langle \text{quark}|\underbrace{X, \text{out}\rangle}_{T \to \infty}\langle X, \text{out}|\text{quark}'\rangle = 0$$

Asymptotic hadronic states

$$\langle \text{quark}|\text{hadron}\rangle = 0$$

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Asymptotic hadronic states

$$\langle \text{quark}|\text{hadron}\rangle = 0$$

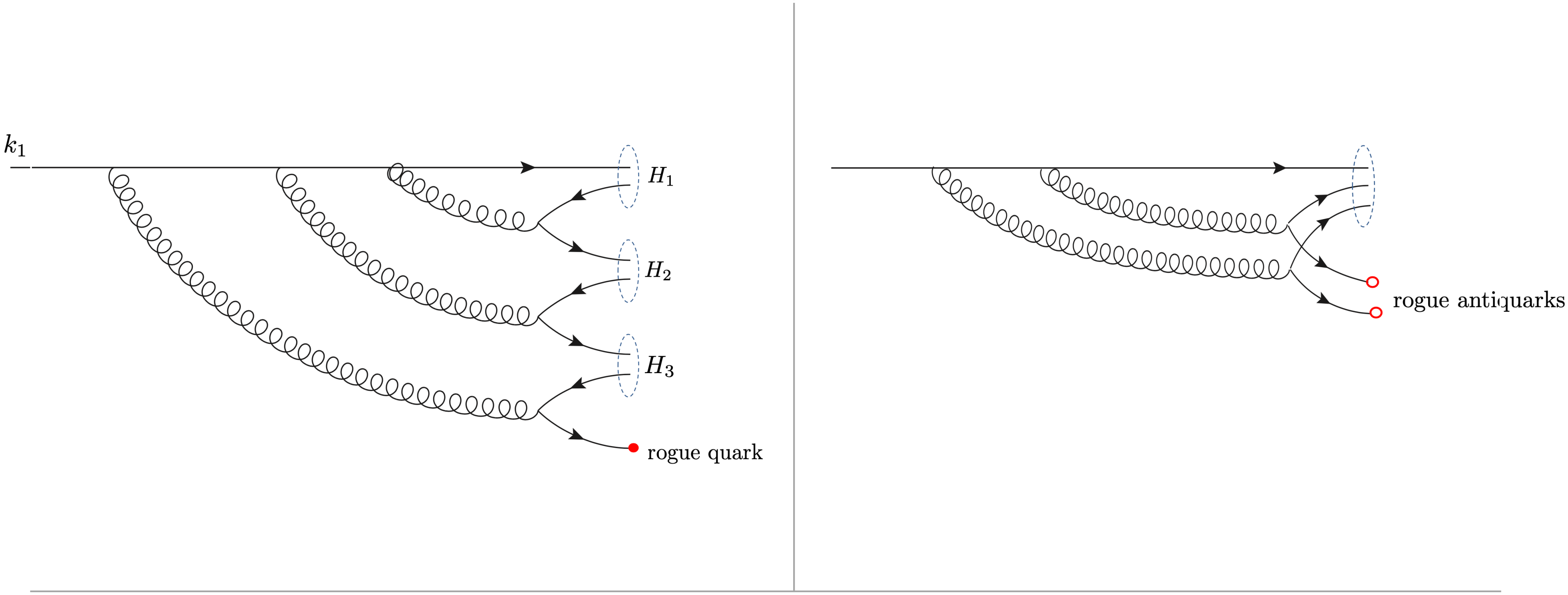
- Non-gauge theories: Fields directly correspond to asymptotic physical states
- In QCD/QED: Local fields are not gauge invariant – they do not create unambiguously physical particle states
  - $\overline{\psi}(y)|0\rangle \to \overline{\psi}(y) WL[\infty, y; n]|0\rangle$
  - Wilson line is a source of color charge
  - Asymptotic states must include quark – Wilson line bound state

$$\underline{\mathcal{E}} \to \mathcal{E} \otimes \underline{\mathcal{B}}$$

Normal hadronic Fock space — $\mathcal{E}$

Space of quark-Wilson bound states — $\mathcal{B}$

Visualizing the problem

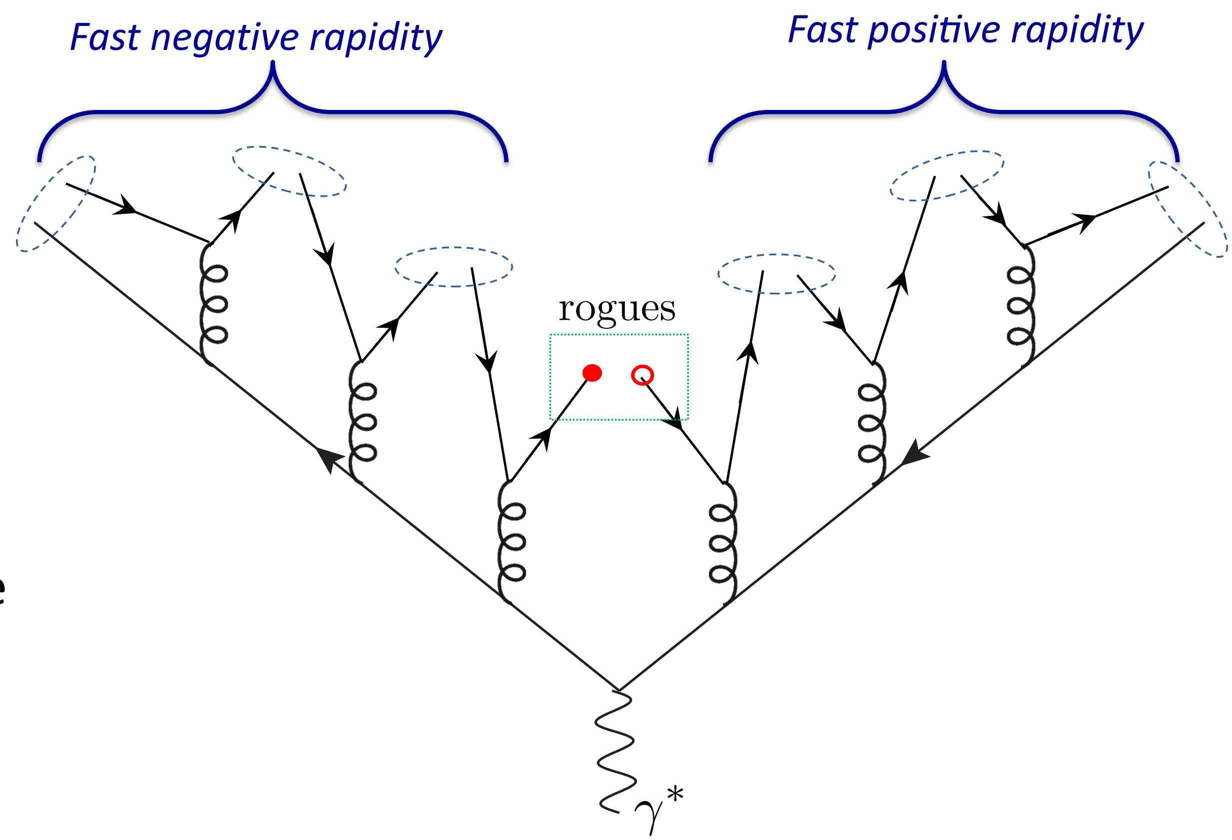


- At least one “rogue” (anti)quark is aways left over

What occurs in a factorization derivation?

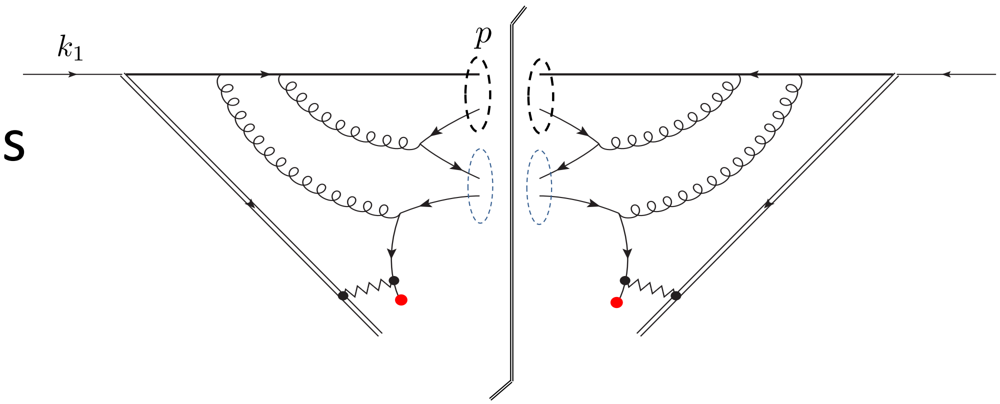
$$e^+e^- \rightarrow h_1 + h_2 + X$$

- Must match FFs onto full process in region of unclassifiable ≈ 0 rapidity hadrons
- Split unclassifiable hadron(s) & insert zero rapidity Wilson lines
- Slow hadrons lie outside the region relevant to the factorization theorem



Deficit fragmentation functions

- Proposal: Take the operator definitions of "deficit fragmentation functions" seriously



- Momentum operator becomes

$$\mathcal{P}^\mu = \mathcal{P}_H^\mu + \mathcal{P}_B^\mu$$

$$\mathcal{P}_H^\mu \equiv \sum_{h\in H} \int_0^\infty \frac{\mathrm{d}p^-}{2p^-} \int \frac{\mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}}{(2\pi)^{3-2\epsilon}} a^\dagger_{h,p,\mathrm{out}} p^\mu a_{h,p,\mathrm{out}} \,,$$

$$\mathcal{P}_B^\mu \equiv \sum_{b\in B} \int_0^\infty \frac{\mathrm{d}p^-}{2p^-} \int \frac{\mathrm{d}^{2-2\epsilon}\boldsymbol{p}_\mathrm{T}}{(2\pi)^{3-2\epsilon}} a^\dagger_{b,p,\mathrm{out}} p^\mu a_{b,p,\mathrm{out}} \,.$$

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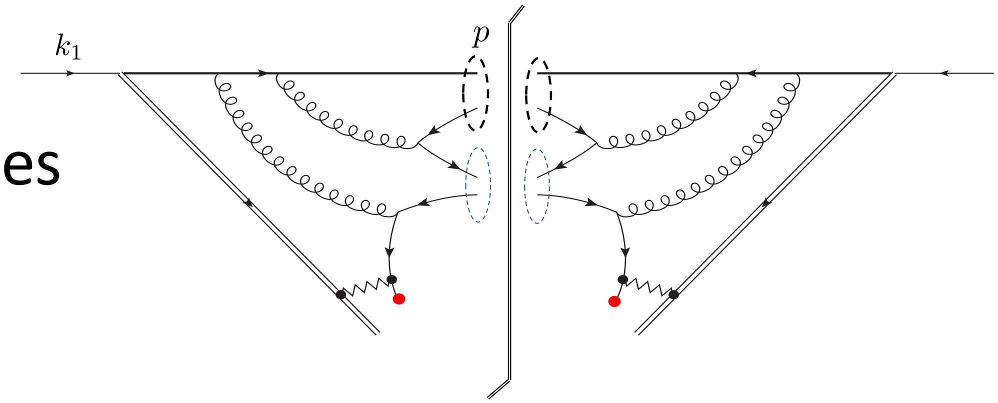

- Momentum sum rule becomes

$$\sum_{h\in H}\int_0^1 \mathrm{d}z\; z\, d_{h/j}(z) = 1 - \sum_{b\in B}\underline{\int_0^1 \mathrm{d}z\; z\, d_{b/j}(z)}$$

- Other sum rules

$$\sum_{h\in H}\mathcal{Q}_h\int_0^1 \mathrm{d}z\; d_{h/j}(z) = \mathcal{Q}_j - \sum_{b\in B}\mathcal{Q}_b\underline{\int_0^1 \mathrm{d}z\; d_{b/j}(z)}$$

$$\sum_{h\in H}\int_0^1 \mathrm{d}z\; d_{h/j}(z) = \langle N\rangle - \sum_{b\in B}\underline{\int_0^1 \mathrm{d}z\; d_{b/j}(z)}$$

To consider

- Momentum sum rule is preserved if deficit ff is $\propto \delta(z)$. Other sum rules are not. Are deficit ffs *exactly* localized at z = 0?
- Can calculating deficit fragmentation functions nonperturbatively lead to insights about hadronization?

See, e.g., J.C. Collins, "Do fragmentation functions in factorization theorems correctly treat non-perturbative effects?"

Other considerations

- Parton model derivation of momentum sum rule
  - Definition of inclusive cross section

$$\sum_h \int \mathrm{d}^3\mathbf{p}_h \frac{\mathrm{d}\sigma^h}{\mathrm{d}x\,\mathrm{d}Q^2\,\mathrm{d}^3\mathbf{p}_h} = \langle N\rangle \frac{\mathrm{d}\sigma}{\mathrm{d}x\,\mathrm{d}Q^2} \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z\, F_{1,h}(x,z,Q^2) = \langle N\rangle F_1(x,Q^2)$$

$$\Longrightarrow \quad \sum_h \int \mathrm{d}z\, zF_{1,h}(x,z,Q^2) = F_1(x,Q^2)$$

  - Parton model $F_{1,h}(x,z,Q^2) = H_1 f(x) d_h(z)\,,\ F_1(x,Q^2) = H_1 f(x)$

$$\sum_h \int \mathrm{d}z\, zF_{1,h}(x,z,Q^2) = H_1 f(x) \left(\sum_h \int \mathrm{d}z\, z d_h(z)\right) = H_1 f(x) \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z\, z d_h(z) = 1$$

Other considerations

- Experimentalists and theorists means something different by "inclusive!"

- What does $1 = \sum_X |X\rangle\langle X|$ really mean?

  – Not included in (many) experimental SIDIS measurements: $$eN \to e + N + \pi \qquad eN \to e + \rho + X \quad ??$$

  – But included in DIS measurements

Other considerations

- What if elastic pions are subtracted?

$$\cancel{eN \to e + N + \pi}$$

Other considerations

- Parton model derivation of momentum sum rule
  - Definition of inclusive cross section

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$$\Longrightarrow \sum_h \int \mathrm{d}z \, zF_{1,h}(x,z,Q^2) \neq F_1(x,Q^2)$$

No elastic pions *Elastic pions*

  - Parton model $F_{1,h}(x,z,Q^2) = H_1 f(x) d_h(z)\,,\ F_1(x,Q^2) = H_1 f(x)$

$$\sum_h \int \mathrm{d}z \, zF_{1,h}(x,z,Q^2) = H_1 f(x) \left(\sum_h \int \mathrm{d}z \, z d_h(z) \right) \neq H_1 f(x) \quad \Longrightarrow \quad \sum_h \int \mathrm{d}z \, z d_h(z) = 1$$