Methods for high-precision determinations of radiative-leptonic decay form factors using lattice QCD

Speaker: Christopher Kane

In collaboration with: Davide Giusti, Christoph Lehner, Stefan Meinel, Amarjit Soni

Jefferson Lab Theory Seminar

Date: September 11, 2023

1University of Arizona
2University of Regensburg
3Brookhaven National Lab
Main Take-Away Points

**Main Take-Away Point 1:** Radiative leptonic decays are interesting in the regions of small and large photon energies

**Main Take-Away Point 2:** We have developed methods to achieve high precision for small computational cost


**Main Take-Away Point 3:** Working on physical calculation of $D_s \rightarrow \gamma\ell\nu_\ell$, out soon
**Table of Contents**

1. Introduction and Motivation

2. Extracting the hadronic tensor with Euclidean correlation functions

3. Methods study

4. $D_s \rightarrow \ell \nu \gamma$ preliminary form factor results
Outline for section 1

1 Introduction and Motivation

2 Extracting the hadronic tensor with Euclidean correlation functions

3 Methods study

4 $D_s \to \ell\nu\gamma$ preliminary form factor results
Radiative leptonic decays of pseudoscalar mesons

Flavor changing charged current (FCCC)

\[ H^+ \rightarrow \gamma \ell^+ \nu_\ell , \quad H^- \rightarrow \gamma \ell^- \bar{\nu}_\ell \]

Schematic diagram for $D_s^+ \rightarrow \gamma e^+ \nu_e$
Radiative leptonic decays of pseudoscalar mesons

Flavor changing charged current (FCCC)

\[ H^+ \to \gamma \ell^+ \nu_\ell, \quad H^- \to \gamma \ell^- \bar{\nu}_\ell \]

Schematic diagram for \( D_s^+ \to \gamma e^+ \nu_e \)

Flavor changing neutral current (FCNC)

\[ H^0 \to \gamma \ell^+ \ell^- \]

Schematic diagram for \( B_s^0 \to \gamma \ell^+ \ell^- \)
Radiative leptonic decays of pseudoscalar mesons

Flavor changing charged current (FCCC)

\[ H^+ \rightarrow \gamma \ell^+ \nu_\ell, \quad H^- \rightarrow \gamma \ell^- \bar{\nu}_\ell \]

Schematic diagram for \( D_s^+ \rightarrow \gamma e^+ \nu_e \)

Flavor changing neutral current (FCNC)

\[ H^0 \rightarrow \gamma \ell^+ \ell^- \]

Schematic diagram for \( B_s^0 \rightarrow \gamma \ell^+ \ell^- \)

Knowledge of **structure dependent** QCD form factors are of interest for both small and large photon energies
Regions of small photon energies
Precision determinations of CKM matrix elements

Determinations of CKM matrix elements $V_{q_1q_2}$ require meson decay constants $f_H$

$$\Gamma(H \to \ell \nu) = \frac{G_F^2}{8\pi} |V_{q_1q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_H^2}\right)^2 m_H f_H^2,$$

$$\langle 0 | A_\mu | H \rangle = ip_\mu f_H$$
Precision determinations of CKM matrix elements

Determinations of CKM matrix elements $V_{q_1 q_2}$ require meson decay constants $f_H$

$$\Gamma(H \rightarrow \ell\nu) = \frac{G_F^2}{8\pi} |V_{q_1 q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_H^2}\right)^2 m_H f_H^2,$$

$$\langle 0|A_\mu|H \rangle = ip_\mu f_H$$

Measure $\Gamma(H \rightarrow \ell\nu)$ experimentally

Calculate $f_H$ with lattice QCD

\bigg\{\begin{align*}
\text{determine } & |V_{q_1 q_2}|^2
\end{align*}\bigg\}
Precision determinations of CKM matrix elements

Determinations of CKM matrix elements $V_{q_1 q_2}$ require meson decay constants $f_H$

$$\Gamma(H \to \ell\nu) = \frac{G_F^2}{8\pi} |V_{q_1 q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_H^2}\right)^2 m_H f_H^2,$$

$$\langle 0|A_\mu|H\rangle = ip_\mu f_H$$

Measure $\Gamma(H \to \ell\nu)$ experimentally

Calculate $f_H$ with lattice QCD

\{ determine $|V_{q_1 q_2}|^2$

- Sub-percent precision for $f_H$ require $O(\alpha_{em})$ electromagnetic corrections $H \to \ell\nu(\gamma)$

- Radiative leptonic decay rate $H \to \gamma\ell\nu$ required to subtract IR divergences in $H \to \ell\nu(\gamma)$
  → by the Bloch-Nordsieck mechanism [Bloch, Nordsieck, PRD 1937]
Precision determinations of CKM matrix elements

Determinations of CKM matrix elements $V_{q_1q_2}$ require meson decay constants $f_H$

$$\Gamma(H \to \ell\nu) = \frac{G_F^2}{8\pi} |V_{q_1q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_H^2}\right)^2 m_H f_H^2,$$

$$\langle 0| A_\mu |H\rangle = ip_\mu f_H$$

Measure $\Gamma(H \to \ell\nu)$ experimentally 
Calculate $f_H$ with lattice QCD 
\{ determine $|V_{q_1q_2}|^2$

- Sub-percent precision for $f_H$ require $O(\alpha_{em})$ electromagnetic corrections $H \to \ell\nu(\gamma)$
- Radiative leptonic decay rate $H \to \gamma\ell\nu$ required to subtract IR divergences in $H \to \ell\nu(\gamma)$ 
  \rightarrow by the Bloch-Nordsieck mechanism [Bloch, Nordsieck, PRD 1937]

Approx. $\pi^-, K^-$ as point-like

- $\pi^- \to \mu^- \nu_\mu \gamma$ and $K^- \to \mu^- \bar{\nu}_\mu \gamma$

Precision determinations of CKM matrix elements

Determinations of CKM matrix elements $V_{q_1 q_2}$ require meson decay constants $f_H$

$$\Gamma(H \rightarrow \ell \nu) = \frac{G_F^2}{8\pi} |V_{q_1 q_2}|^2 m_\ell^2 \left(1 - \frac{m_\ell^2}{m_H^2}\right)^2 \frac{m_H f_H^2}{m_H}, \quad \langle 0 | A_\mu | H \rangle = i p_\mu f_H$$

Measure $\Gamma(H \rightarrow \ell \nu)$ experimentally

Calculate $f_H$ with lattice QCD

\begin{itemize}
  \item Sub-percent precision for $f_H$ require $O(\alpha_{em})$ electromagnetic corrections $H \rightarrow \ell \nu(\gamma)$
  \item Radiative leptonic decay rate $H \rightarrow \gamma \ell \nu$ required to subtract IR divergences in $H \rightarrow \ell \nu(\gamma)$
    \rightarrow by the Bloch-Nordsieck mechanism \[\text{[Bloch, Nordsieck, PRD 1937]}\]
\end{itemize}

Approx. $\pi^-, K^-$ as point-like

\begin{itemize}
  \item $\pi^- \rightarrow \mu^- \bar{\nu}_\mu \gamma$ and $K^- \rightarrow \mu^- \bar{\nu}_\mu \gamma$
\end{itemize}

Structure dependent form factors required

\begin{itemize}
  \item $\pi^- \rightarrow e^- \bar{\nu}_e \gamma$ and $K^- \rightarrow e^- \bar{\nu}_e \gamma$
\end{itemize}

Regions of large photon energies
**FCNC processes** $B_s^0 \to \ell^+\ell^-\gamma$ and $B^0 \to \ell^+\ell^-\gamma$

- Hard photon removes helicity suppression $(m_\ell/m_B)^2$
- This process sensitive to all operators in the $b \to s\ell^+\ell^-$ weak effective Hamiltonian including $O_9$, where slight tension with SM prediction exists

FCNC processes $B_s^0 \rightarrow \ell^+\ell^-\gamma$ and $B^0 \rightarrow \ell^+\ell^-\gamma$

- Hard photon removes helicity suppression $(m_\ell/m_B)^2$
- This process sensitive to all operators in the $b \rightarrow s\ell^+\ell^-$ weak effective Hamiltonian including $O_9$, where slight tension with SM prediction exists


- $\mathcal{B}(B^0 \rightarrow \ell^+\ell^-\gamma) < \mathcal{O}(10^{-7})$ for $\ell = e, \mu$ [BABAR: PRD 2008 / arXiv:0706.2870]
- $\mathcal{B}(B_s^0 \rightarrow \mu^+\mu^-\gamma) < 2.0 \times 10^{-9}$ for $m_{\mu\mu} > 4.9$ GeV [LHCb: PRD 2022 / arXiv:2108.09283]
**FCCC process** $B^- \rightarrow \gamma \ell^- \bar{\nu}$

- Hard photon removes helicity suppression $(m_\ell/m_B)^2$
- For large $E_\gamma^{(0)}$, simplest decay that probes the inverse moment of the B meson lightcone distribution amplitude

$$\frac{1}{\lambda_B} = \int_0^\infty d\omega \frac{\Phi_{B+}(\omega)}{\omega}$$

- $\lambda_B$ important input in QCD factorization approach to exclusive B decays, currently not well known

Hard photon removes helicity suppression \((m_\ell/m_B)^2\)

For large \(E_\gamma^{(0)}\), simplest decay that probes the inverse moment of the B meson lightcone distribution amplitude

\[
\frac{1}{\lambda_B} = \int_0^\infty d\omega \frac{\Phi_{B+}(\omega)}{\omega}
\]

\(\lambda_B\) important input in QCD factorization approach to exclusive B decays, currently not well known


Belle: \(\mathcal{B}(B^+ \rightarrow \ell^+ \nu \gamma) < 3.0 \times 10^{-6} \) \((E_\gamma^{(0)} > 1 \text{ GeV})\) [arXiv:1810.12976/PRD 2018]
Experimental status of radiative leptonic decays

- $K^- \rightarrow e^- \bar{\nu} \gamma$, $K^- \rightarrow \mu^- \bar{\nu} \gamma$, $\pi^- \rightarrow e^- \bar{\nu} \gamma$, $\pi^- \rightarrow \mu^- \bar{\nu} \gamma$
- $\rightarrow K^-$, $\pi^-$ partial branching fractions, photon-energy spectra, and angular distributions known from multiple experiments. [See PDG review by M. Bychkov and G. D’Ambrosio, 2018]
Experimental status of radiative leptonic decays

- $K^- \rightarrow e^- \bar{\nu} \gamma$, $K^- \rightarrow \mu^- \bar{\nu} \gamma$, $\pi^- \rightarrow e^- \bar{\nu} \gamma$, $\pi^- \rightarrow \mu^- \bar{\nu} \gamma$ 
  $\rightarrow K^-$, $\pi^-$ partial branching fractions, photon-energy spectra, and angular distributions known from multiple experiments. [See PDG review by M. Bychkov and G. D’Ambrosio, 2018]

- $D_s^+ \rightarrow e^+ \nu \gamma$: $\mathcal{B}(E^0_\gamma > 10\ MeV ) < 1.3 \times 10^{-4}$ [BESIII: PRD 2017 / arXiv:1702.05837]

- $D^+ \rightarrow e^+ \nu \gamma$: $\mathcal{B}(E^0_\gamma > 10\ MeV ) < 3.0 \times 10^{-5}$ [BESIII: PRD 2019 / arXiv:1902.03351]
Experimental status of radiative leptonic decays

- $K^- \rightarrow e^- \bar{\nu}\gamma, K^- \rightarrow \mu^- \bar{\nu}\gamma, \pi^- \rightarrow e^- \bar{\nu}\gamma, \pi^- \rightarrow \mu^- \bar{\nu}\gamma$
- $\rightarrow K^-, \pi^-$ partial branching fractions, photon-energy spectra, and angular distributions known from multiple experiments. [See PDG review by M. Bychkov and G. D’Ambrosio, 2018]

- $D_s^+ \rightarrow e^+ \nu\gamma$: $\mathcal{B}(E_\gamma^{(0)} > 10 \text{ MeV }) < 1.3 \times 10^{-4}$ [BESIII: PRD 2017 / arXiv:1702.05837]
- $D^+ \rightarrow e^+ \nu\gamma$: $\mathcal{B}(E_\gamma^{(0)} > 10 \text{ MeV }) < 3.0 \times 10^{-5}$ [BESIII: PRD 2019 / arXiv:1902.03351]

- $\mathcal{B}(B^+ \rightarrow \ell^+ \nu\gamma) < 3.0 \times 10^{-6} \ (E_\gamma^{(0)} > 1 \text{ GeV})$ [Belle: PRD 2018 / arXiv:1810.12976]
- Belle II expected to measure $\mathcal{B}(B^+ \rightarrow \ell^+ \nu\gamma)$ with 3.6% statistical uncertainty [Belle: PRD 2018 / arXiv:1810.12976]
Experimental status of radiative leptonic decays

- $K^− → e^− \bar{\nu} \gamma$, $K^− → \mu^− \bar{\nu} \gamma$, $\pi^− → e^− \bar{\nu} \gamma$, $\pi^− → \mu^− \bar{\nu} \gamma$
  → $K^−$, $\pi^−$ partial branching fractions, photon-energy spectra, and angular distributions known from multiple experiments. [See PDG review by M. Bychkov and G. D’Ambrosio, 2018]

- $D_s^+ → e^+ \nu \gamma$: $\mathcal{B}(E_\gamma^{(0)} > 10 \text{ MeV}) < 1.3 \times 10^{-4}$ [BESIII: PRD 2017 / arXiv:1702.05837]
- $D^+ → e^+ \nu \gamma$: $\mathcal{B}(E_\gamma^{(0)} > 10 \text{ MeV}) < 3.0 \times 10^{-5}$ [BESIII: PRD 2019 / arXiv:1902.03351]
- $\mathcal{B}(B^+ → \ell^+ \nu \gamma) < 3.0 \times 10^{-6}$ ($E_\gamma^{(0)} > 1 \text{ GeV}$) [Belle: PRD 2018 / arXiv:1810.12976]
- Belle II expected to measure $\mathcal{B}(B^+ → \ell^+ \nu \gamma)$ with 3.6% statistical uncertainty [Belle: PRD 2018 / arXiv:1810.12976]
- $\mathcal{B}(B^0 → \ell^+ \ell^- \gamma) < \mathcal{O}(10^{-7})$ for $\ell = e, \mu$ [BABAR: PRD 2008 / arXiv:0706.2870]
- $\mathcal{B}(B^0_s → \mu^+ \mu^- \gamma) < 2.0 \times 10^{-9}$ for $m_{\mu\mu} > 4.9 \text{ GeV}$ [LHCb: PRD 2022 / arXiv:2108.09283]
Review of lattice calculations of radiative leptonic decays

Lattice 2019
present first results

CFK, et. al, [arXiv:1907.00279]

G. Martinelli, et. al, [PoS Latice2019]


CFK, et. al, [arXiv:2110.13196]


[Frezzotti et. al, PRD 2021, arXiv:2306.05904]
Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results

CFK, et. al, [arXiv:1907.00279]

G. Martinelli, et. al, [PoS Latice2019]

Lattice 2019 present first results


physical calculation for $\pi, K, D, D_s$ limited photon energy range for $D_{(s)}$
Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results

CFK, et. al, [arXiv:1907.00279]

Lattice 2019 present first results

G. Martinelli, et. al, [PoS Latice2019]

physical calculation for $\pi, K, D, D_s$
limited photon energy range for $D_{(s)}$


Comparison to experiment for $\pi, K$

Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results

CFK, et. al, [arXiv:1907.00279]

Lattice 2019 present first results

G. Martinelli, et. al, [PoS Lattice2019]

physical calculation for $\pi, K, D, D_s$
limited photon energy range for $D(s)$

Desiderio, et. al, [PRD 2021,
arXiv:2006.05358]

Comparison to experiment for $\pi, K$

Frezzotti, et. al, [PRD 2021,
arXiv:2012.02120]

今日

tension with $K \to \mu \nu \mu \gamma, \pi \to e \nu e \gamma$
Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results

CFK, et. al, [arXiv:1907.00279]

G. Martinelli, et. al, [PoS Lattice2019]

Lattice 2020

physical calculation for π, K, D, Ds
limited photon energy range for D(s)


Lattice 2021 update on methods study

CFK, et. al, [arXiv:2110.13196]

Comparison to experiment for π, K


tension with

K → μνμγ, π → eνeγ
Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results
CFK, et. al, [arXiv:1907.00279]

Lattice 2021 update on methods study
CFK, et. al, [arXiv:2110.13196]

Methods paper

2019
2020
2021
2022
2023

Lattice 2019 present first results
G. Martinelli, et. al, [PoS Lattice2019]

physical calculation for $\pi, K, D, D_s$
limited photon energy range for $D(s)$

Comparison to experiment for $\pi, K$

tension with
$K \rightarrow \mu\nu\gamma, \pi \rightarrow e\nu\gamma$
Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results

- G. Martinelli, et. al, [PoS Lattice2019]

Lattice 2019 present first results

- CFK, et. al, [arXiv:1907.00279]

Lattice 2020

- physical calculation for $\pi, K, D, D_s$
  - limited photon energy range for $D_s$

Lattice 2021 update on methods study

- CFK, et. al, [arXiv:2110.13196]

Methods paper

- [Frezzotti et. al, PRD 2021, arXiv:2306.05904]

Comparison to experiment for $\pi, K$

- $D_s \rightarrow \gamma \ell \nu$ physical calc all $E_\gamma$

- $K \rightarrow \mu \nu \gamma$, $\pi \rightarrow e \nu e \gamma$

- [Frezzotti et. al, PRD 2021, arXiv:2306.05904]
Review of lattice calculations of radiative leptonic decays

Lattice 2019 present first results

CFK, et. al, [arXiv:1907.00279]

Lattice 2019 present first results

G. Martinelli, et. al, [PoS Latice2019]

Lattice 2021 update on methods study

CFK, et. al, [arXiv:2110.13196]

Methods paper


Prelim $D_s \rightarrow \ell \nu \gamma$ results

Today

2019

2020

2021

2022

2023

physical calculation for $\pi, K, D, D_s$ limited photon energy range for $D(s)$


Comparison to experiment for $\pi, K$


$D_s \rightarrow \gamma \ell \nu$ physical calc all $E_\gamma$

tension with $K \rightarrow \mu \nu, \pi \rightarrow e \nu, \gamma$

2019

2020

2021

2022

2023

10/41
Outline for section 2

1. Introduction and Motivation

2. Extracting the hadronic tensor with Euclidean correlation functions

3. Methods study

4. $D_s \to \ell \nu \gamma$ preliminary form factor results
Decay amplitude

To calculate decay amplitude:

- use effective Hamiltonian for weak current
- use 1st order perturbation theory for QED piece

Decay amplitude given by

$$A = G_F V^{\alpha \beta} \sqrt{2} [\bar{e} \epsilon^\gamma_{\mu} \nu (1 - \gamma^5) \nu \cdot T_{\mu\nu} - ieQ_\ell f D_s \cdot \bar{\ell} \epsilon^\gamma_{\mu} (1 - \gamma^5) \nu]$$

QCD physics left to calculate is Hadronic tensor $T_{\mu\nu}$
Decay amplitude

To calculate decay amplitude:

- use effective Hamiltonian for weak current
- use 1st order perturbation theory for QED piece

Decay amplitude given by

$$A = \frac{G_F V_{cs}}{\sqrt{2}} \left[ e \epsilon^*_\mu \bar{\ell} \gamma_\nu (1 - \gamma_5) \nu \cdot T^{\mu\nu} - ieQ_\ell f_{D_s} \cdot \bar{\ell} \epsilon^*_\mu \gamma^\mu (1 - \gamma_5) \nu \right]$$

QCD physics left to calculate is Hadronic tensor $T_{\mu\nu}$
Hadronic Tensor and Form Factors

\[ J^\text{em}_\mu = \sum_q e_q \bar{q} \gamma^\mu q, \quad J^\text{weak}_\nu = \bar{q}_1 \gamma^\nu (1 - \gamma^5) q_2 \]

\[ T_{\mu\nu} = -i \int d^4x \, e^{ip\cdot x} \langle 0 | T \left( J^\text{em}_\mu(x) J^\text{weak}_\nu(0) \right) | H(\vec{p}_H) \rangle \]
Hadronic Tensor and Form Factors

\[ J_{\mu}^{em} = \sum q e_q \bar{q} \gamma_\mu q, \quad J_{\nu}^{weak} = \bar{q}_1 \gamma_\nu (1 - \gamma_5) q_2 \]

\[ T_{\mu\nu} = -i \int d^4x \ e^{ip \cdot x} \langle 0 | T \left( J_{\mu}^{em}(x) J_{\nu}^{weak}(0) \right) | H(\vec{p}_H) \rangle \]

\[ = \epsilon_{\mu\nu\tau\rho} p_\gamma^\tau v^\rho F_V + i \left[ -g_{\mu\nu} (v \cdot p_\gamma) + v_\mu (p_\gamma)_\nu \right] F_A - i \frac{v_\mu v_\nu}{(v \cdot p_\gamma)} m_H f_H + (p_\gamma)_\mu \text{-terms} \]
Hadronic Tensor and Form Factors

\[ J_{\mu}^{em} = \sum_q e_q \bar{q} \gamma_\mu q, \quad J_{\nu}^{weak} = \bar{q}_1 \gamma_\nu (1 - \gamma_5) q_2 \]

\[ T_{\mu\nu} = -i \int d^4x \, e^{ip \cdot x} \langle 0 | T \left( J_{\mu}^{em}(x) J_{\nu}^{weak}(0) \right) | H(p_H) \rangle \]

\[ = \epsilon_{\mu\nu\tau\rho} p_\gamma^\tau v_\rho F_V + i \left[ -g_{\mu\nu} (\nu \cdot p_\gamma) + \nu_\mu (p_\gamma)_\nu \right] F_A - i \frac{\nu_\mu \nu_\nu}{(\nu \cdot p_\gamma)} m_H f_H + (p_\gamma)_\mu \text{-terms} \]

\[ F_{A,SD} = F_A - \left( -Q_\ell f_H / E_\gamma^{(0)} \right), \quad E_\gamma^{(0)} = p_B \cdot p_\gamma / m_B \]
Hadronic Tensor and Form Factors

\[ J_{\mu}^{em} = \sum_q e_q \bar{q} \gamma_\mu q, \quad J_{\nu}^{weak} = \bar{q}_1 \gamma_\nu (1 - \gamma_5) q_2 \]

\[ T_{\mu\nu} = -i \int d^4 x \ e^{i p \cdot x} \langle 0 | T \left( J_{\mu}^{em}(x) J_{\nu}^{weak}(0) \right) | H(\vec{p}_H) \rangle \]

\[ = \epsilon_{\mu\nu\tau\rho} p_\gamma^\tau \nu^\rho F_V + i \left[ -g_{\mu\nu} (\nu \cdot p_\gamma) + \nu_\mu (p_\gamma)_\nu \right] F_A - i \frac{\nu_\mu \nu_\nu}{(\nu \cdot p_\gamma)} m_H f_H + (p_\gamma)_\mu \text{-terms} \]

\[ F_{A,SD} = F_A - (-Q_\ell f_H / E_{\gamma}^{(0)}), \quad E_{\gamma}^{(0)} = p_B \cdot p_\gamma / m_B \]

Goal: Calculate \( F_V \) and \( F_{A,SD} \) as a function of \( E_{\gamma}^{(0)} \)
Euclidean correlation function

\[ C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y \ e^{-i\vec{p} \cdot \vec{x}} e^{i\vec{p}_H \cdot \vec{y}} \langle J^{em}_{\mu}(t_{em}, \vec{x}) J^{\text{weak}}_{\nu}(0) \phi^+_{H}(t_H, \vec{y}) \rangle \]

\[ \phi^+_{H} \sim \bar{Q}\gamma_5 u \]

(* all times are now Euclidean *)
Euclidean correlation function

\[ C_{3,\mu\nu}(t_{em}, t_{H}) = \int d^3 x \int d^3 y \, e^{-i\bar{p}_\gamma \cdot \bar{x}} e^{i\bar{p}_H \cdot \bar{y}} \langle J_{\mu}^{em}(t_{em}, \bar{x}) J_{\nu}^{weak}(0) \phi^\dagger_H(t_H, \bar{y}) \rangle \]

\[ \phi^\dagger_H \sim \bar{Q} \gamma_5 u \]

Define time-integrated correlation functions for each time ordering

(* all times are now Euclidean *)
Euclidean correlation function

\[ C_{3,\mu\nu}(t_{em}, t_H) = \int d^3x \int d^3y \ e^{-i\vec{p}\cdot\vec{x}} e^{i\vec{p}_H\cdot\vec{y}} \langle J_{\mu}^{em}(t_{em}, \vec{x}) J_{\nu}^{weak}(0) \phi_{H}^\dagger(t_H, \vec{y}) \rangle \]

\[ \phi_{H}^\dagger \sim \bar{Q}\gamma_5 u \]

Define time-integrated correlation functions for each time ordering

\[ I^{<}_{\mu\nu}(T, t_H) = \int_{-T}^{0} dt_{em} e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H) \]

\[ I^{>}_{\mu\nu}(T, t_H) = \int_{0}^{T} dt_{em} e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H) \]
Euclidean correlation function

\[
C_{3,\mu\nu}(t_{em}, t_H) = \int d^3 x \int d^3 y \ e^{-i \vec{p} \cdot \vec{x}} e^{i \vec{p} \cdot \vec{y}} \langle J_{\mu}^{em}(t_{em}, \vec{x}) J_{\nu}^{weak}(0) \phi_H^\dagger(t_H, \vec{y}) \rangle
\]

\[
\phi_H^\dagger \sim \bar{Q} \gamma_5 u
\]

Define time-integrated correlation functions for each time ordering

\[
I_{\mu\nu}^{<}(T, t_H) = \int_{-T}^{0} \ d t_{em} e^{E_{\gamma} t_{em}} \ C_{3,\mu\nu}(t_{em}, t_H)
\]

\[
I_{\mu\nu}^{>}(T, t_H) = \int_{0}^{T} \ d t_{em} e^{E_{\gamma} t_{em}} \ C_{3,\mu\nu}(t_{em}, t_H)
\]

Show relation between \( I_{\mu\nu}(T, t_H) \) and \( T_{\mu\nu} \)

→ compare spectral decompositions of both time orderings of \( I_{\mu\nu} \) and \( T_{\mu\nu} \)
Euclidean spectral decomposition of $I_{\mu\nu}$

Time ordering: $t_{em} > 0$

$$T_{\mu\nu}^> = \sum_n \frac{\langle 0 | J_{\mu}^{em}(0) | n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma) | J_{\nu}^{weak}(0) | H(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma} (E_\gamma - E_{n,\vec{p}_\gamma})}$$
Euclidean spectral decomposition of $I_{\mu\nu}^>$

Time ordering: $t_{em} > 0$

$$T_{\mu\nu}^> = \sum_n \frac{\langle 0 | J_{\mu}^{em}(0) | n(\bar{p}_{\gamma}) \rangle \langle n(\bar{p}_{\gamma}) | J_{\nu}^{weak}(0) | H(\bar{p}_{H}) \rangle}{2E_n,\bar{p}_{\gamma} (E_{\gamma} - E_{n,\bar{p}_{\gamma}})}$$

$$I_{\mu\nu}^>(t_H, T) = \int_0^T dt_{em} \ e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$
Euclidean spectral decomposition of $I_{\mu\nu}^>$

Time ordering: $t_{em} > 0$

$$T_{\mu\nu}^> = \sum_n \frac{\langle 0| J_{\mu}^{em}(0) | n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma) | J_{\nu}^{weak}(0) | H(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})}$$

$$I_{\mu\nu}^>(t_H, T) = \int_0^T dt_{em} \ e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$

$$= \sum_m e^{E_m t_H} \frac{\langle m(\vec{p}_H) | \phi_H^+(0) | 0 \rangle}{2E_{m,\vec{p}_H}}$$

$$\times \sum_n \frac{\langle 0| J_{\mu}^{em}(0) | n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma) | J_{\nu}^{weak}(0) | m(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})} \left[ 1 - e^{(E_\gamma - E_{n,\vec{p}_\gamma}) T} \right]$$

Require $E_\gamma - E_{n,\vec{p}_\gamma} < 0$

Because the states $| n(\vec{p}_\gamma) \rangle$ have mass, $\sqrt{m_n^2 + \vec{p}_\gamma^2} > |\vec{p}_\gamma|$ is automatically satisfied
Euclidean spectral decomposition of $I^>_{\mu\nu}$

Time ordering: $t_{em} > 0$

$$T^>_{\mu\nu} = \sum_n \frac{\langle 0|J^em_{\mu}(0)|n(\vec{p}_\gamma)\rangle\langle n(\vec{p}_\gamma)|J^{weak}_\nu(0)|H(\vec{p}_H)\rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})}$$

$$I^>_{\mu\nu}(t_H, T) = \int_0^T dt_{em} e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$

$$= \sum_m e^{E_m t_H} \frac{\langle m(\vec{p}_H)|\phi^+_H(0)|0\rangle}{2E_{m,\vec{p}_H}}$$

$$\times \sum_n \frac{\langle 0|J^em_{\mu}(0)|n(\vec{p}_\gamma)\rangle\langle n(\vec{p}_\gamma)|J^{weak}_\nu(0)|m(\vec{p}_H)\rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})} \left[ 1 - e^{(E_\gamma - E_{n,\vec{p}_\gamma}) T} \right]$$

$t_H \to -\infty$ to achieve ground state saturation

$T \to \infty$ to remove unwanted exponentials that come with intermediate states
Euclidean spectral decomposition of $I^{>}_{\mu\nu}$

Time ordering: $t_{em} > 0$

\[
T^{>}_{\mu\nu} = \sum_n \frac{\langle 0| J^{em}_{\mu}(0)| n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma)| J^{weak}_{\nu}(0)| H(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})}
\]

\[
I^{>}_{\mu\nu}(t_H, T) = \int_0^T dt_{em} \ e^{E_{\gamma} t_{em}} C_{\mu\nu}(t_{em}, t_H)
\]

\[
= \sum_m e^{E_m t_H} \frac{\langle m(\vec{p}_H)| \phi_H^\dagger(0)| 0 \rangle}{2E_{m,\vec{p}_H}}
\]

\[
\times \sum_n \frac{\langle 0| J^{em}_{\mu}(0)| n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma)| J^{weak}_{\nu}(0)| m(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})} \left[ 1 - e^{(E_\gamma - E_{n,\vec{p}_\gamma}) T} \right]
\]

- Require $E_\gamma - E_{n,\vec{p}_\gamma} < 0$
- Because the states $|n(\vec{p}_\gamma)\rangle$ have mass, $\sqrt{m_n^2 + \vec{p}_\gamma^2} > |\vec{p}_\gamma|$ is automatically satisfied

\[ t_H \to -\infty \text{ to achieve ground state saturation} \]
Euclidean spectral decomposition of $I^{>}_{\mu\nu}$

Time ordering: $t_{em} > 0$

$$I^{>}_{\mu\nu}(t_H, T) = \int_0^T dt_{em} \ e^{E_\gamma t_{em}} C_{\mu\nu}(t_{em}, t_H)$$

$$= \sum_m e^{E_m t_H} \frac{\langle 0 | J_{em}^\mu(0) | n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma) | J_{\nu}^{weak}(0) | H(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})}$$

$$\times \sum_n \frac{\langle 0 | J_{\mu}^{em}(0) | n(\vec{p}_\gamma) \rangle \langle n(\vec{p}_\gamma) | J_{\nu}^{weak}(0) | m(\vec{p}_H) \rangle}{2E_{n,\vec{p}_\gamma}(E_\gamma - E_{n,\vec{p}_\gamma})} \left[ 1 - e^{(E_\gamma - E_{n,\vec{p}_\gamma}) T} \right]$$

- Require $E_\gamma - E_{n,\vec{p}_\gamma} < 0$

- Because the states $| n(\vec{p}_\gamma) \rangle$ have mass, $\sqrt{m_n^2 + \vec{p}_\gamma^2} > |\vec{p}_\gamma|$ is automatically satisfied
Final relation

For $\mathbf{p}_\gamma \neq \mathbf{0}$,

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\bar{p}_H)\phi_H^\dagger|0\rangle} \int_{-T}^{T} dt_{em} e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H) I_{\mu\nu}(T, t_H)$$
Outline for section 3

1 Introduction and Motivation

2 Extracting the hadronic tensor with Euclidean correlation functions

3 Methods study

4 $D_s \rightarrow \ell \nu \gamma$ preliminary form factor results
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{p}_H)|\phi_H^\dagger|0 \rangle} \int_{-T}^{T} dt_{em} e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1. 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

2. 4d sequential propagator through $J_{em}$ → calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free

Limitation of 4d method: cannot resolve time orderings

$= \Rightarrow 4d \succ \prec \text{method}$: perform two sequential solves to resolve $t_{em} < 0$ and $t_{em} > 0$
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\bar{\phi}_H)|\phi_H^\dagger|0\rangle} \int_{-T}^{T} dt_{em} e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger \to$ calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

Limitation of 4d method: cannot resolve time orderings $= \Rightarrow$ 4d $>, <$ method: perform two sequential solves to resolve $t_{em} < 0$ and $t_{em} > 0$
Calculating $I_{\mu\nu}(T, t_H)$

\[
T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\bar{\rho}_H)|\phi_H^\dagger|0\rangle} \int_{-T}^{T} dt_{em} \ e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H)
\]

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

Limitation of 4d method: cannot resolve time orderings $\Rightarrow$ 4d $>,<$ method: perform two sequential solves to resolve $t_{em} < 0$ and $t_{em} > 0$
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\bar{\rho}_H)|\phi_H^\dagger|0\rangle} \int_{-T}^{T} dt_{em} \, e^{E_H t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

Limitation of 4d method: cannot resolve time orderings $= \Rightarrow$ 4d $>,<$ method: perform two sequential solves to resolve $t_{em} < 0$ and $t_{em} > 0$
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T\to\infty} \lim_{t_H\to-\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{p}_H)|\phi_H^\dagger|0\rangle} \int_{-T}^{T} dt_{em} \, e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

2: 4d sequential propagator through $J_{\mu}^{em}$ → calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\bar{\rho}_H) | \phi_H^\dagger | 0 \rangle} \int_{-T}^{T} dt_{em} \, e^{E_H t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger \rightarrow$ calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

2: 4d sequential propagator through $J_{\mu}^{em} \rightarrow$ calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free

Limitation of 4d method: cannot resolve time orderings $\Rightarrow$ 4d $>,<$ method: perform two sequential solves to resolve $t_{em} < 0$ and $t_{em} > 0$
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} -2E_He^{-E_Ht_H} \langle H(\bar{\rho}_H)|\phi_H^\dagger|0\rangle \int_{-T}^{T} dt_{em} \ e^{E_\gamma t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger \rightarrow$ calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

2: 4d sequential propagator through $J_{\mu}^{em} \rightarrow$ calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free

Limitation of 4d method: cannot resolve time orderings
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\tilde{\rho}_H)|\phi_H^\dagger|0 \rangle} \int_{-T}^{T} dt_{em} \, e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

2: 4d sequential propagator through $J_{\mu}^{em} \rightarrow$ calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free

Limitation of 4d method: cannot resolve time orderings
Calculating $I_{\mu\nu}(T, t_H)$

$$T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\tilde{\rho}_H)|\phi_H^\dagger|0\rangle} \int_{-T}^{T} dt_{em} e^{E_H t_{em}} C_{3,\mu\nu}(t_{em}, t_H)$$

Two methods to calculate $I_{\mu\nu}(T, t_H)$:

1: 3d (timeslice) sequential propagator through $\phi_H^\dagger$ → calculate $C_{3,\mu\nu}(t_{em}, t_H)$ on lattice, fixed $t_H$ get all $t_{em}$ for free

2: 4d sequential propagator through $J_{\mu}^{em}$ → calculate $I_{\mu\nu}(T, t_H)$ on lattice, fixed $T$ get all $t_H$ for free

Limitation of 4d method: cannot resolve time orderings

$\Rightarrow$ **4d^>;< method**: perform two sequential solves to resolve $t_{em} < 0$ and $t_{em} > 0$
Past lattice studies

\[ T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_H e^{-E_H t_H}}{\langle H(\vec{p}_H)|\phi_H^\dagger|0 \rangle} \int_{-T}^{T} dt_{em} e^{E_{\gamma} t_{em}} C_{3,\mu\nu}(t_{em}, t_H) \]

- [1] we presented results at Lattice 2019 using 3d method
  - fitting to a constant looking for plateaus in \( T \) and \( t_H \)

- [2, 3] use 4d method to perform realistic physical calculation
  - set \( T = N_T/2 \) and fit to constant in \( t_H \) where data has plateaued

- [4] perform a methods study comparing 3d and 4d methods
  - data does not always plateau in \( T \) and \( t_H \)
  - develop fit methods to extrapolate to \( T \to \infty \) and \( t_H \to -\infty \)

[1] [CFK, Lehner, Meinel, Soni, arXiv:1907.00279]
[3] [Frezzotti, Gagliardi, Lubicz, Martinelli, Mazzetti, Sachrajda, Sanfilippo, Simula, Tantalo, PRD 2021, arXiv:2306.05904]
Comparison of 3d and 4d methods

Show fit methods to take $\lim_{T \to \infty}$ and $\lim_{t_H \to -\infty}$
- fitting only 4d$^{>,<}$ method data
- fitting only 3d method data
- performing global fits to both 3d and 4d$^{>,<}$ method data

Goal: find methods with best control over $\lim_{T \to \infty}$ and $\lim_{t_H \to -\infty}$ limits for cheapest cost
Simulation parameters for 3d/4d method comparison

- $N_f = 2 + 1$ DWF, RBC/UKQCD gauge ensemble

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$(L/a)^3 \times (T/a)$</th>
<th>$L_5/a$</th>
<th>$a^{-1}(\text{GeV})$</th>
<th>$am_l$</th>
<th>$am_s$</th>
<th>$\approx M_\pi(\text{MeV})$</th>
<th>$N_{\text{conf}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24I</td>
<td>$24^3 \times 64$</td>
<td>16</td>
<td>1.785</td>
<td>0.005</td>
<td>0.04</td>
<td>340</td>
<td>25</td>
</tr>
</tbody>
</table>

- Use local currents with mostly non-perturbative renormalization
- charm valence quarks $\rightarrow$ Möbius domain-wall with “stout” smearing
- u/d/s valence quarks $\rightarrow$ same DWF action as sea quarks
- Neglect disconnected diagrams
- Use all-mode averaging with 1 exact and 16 sloppy solves per configuration
- $\mathbb{Z}_2$ random wall sources
Parameters for $D_s \rightarrow \gamma \ell \nu$ runs

Meson and photon momenta:

<table>
<thead>
<tr>
<th>Method</th>
<th>Source</th>
<th>Meson Momentum</th>
<th>Photon Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td>3d</td>
<td>$\mathbb{Z}_2$-wall</td>
<td>$\vec{p}_{D_s} = (0, 0, 0)$</td>
<td>$</td>
</tr>
<tr>
<td>4d</td>
<td>$\mathbb{Z}_2$-wall</td>
<td>$\vec{p}_{D_s,z} \in 2\pi/L{-1, 0, 1, 2}$</td>
<td>$\vec{p}_{\gamma,z} = 2\pi/L$</td>
</tr>
</tbody>
</table>

4d $> <$ method:

- 3 values of integration range $T/a \in \{6, 9, 12\}$

3d method:

- 3 values of source-sink separation $t_H/a \in \{-6, -9, -12\}$

Fit form factors $F(t_H, T)$ directly instead of time-integrated correlation function $I_{\mu\nu}(t_H, T)$
Fit form: $4d^{>;<}$ method

Include terms to fit

(1) unwanted exponential from first intermediate state
(2) first excited state
Fit form: 4d⁺⁻ method

Include terms to fit

1. unwanted exponential from first intermediate state
2. first excited state

Time ordering $t_{em} > 0$:

$$F^>(t_H, T) = F^> + B_F^> e^{(E_{\gamma} - E^>) T} + C_F^> e^{\Delta E t_H}$$

- fit parameters
Fit form: 4d\rangle:\langle method

Include terms to fit
(1) unwanted exponential from first intermediate state
(2) first excited state

Time ordering $t_{em} > 0$:

$$F^\rangle(t_H, T) = F^\rangle + B_F^\rangle e^{(E_\gamma - E^\rangle)T} + C_F^\rangle e^{\Delta E t_H}$$

- fit parameters
**Fit form: 4d_>;< method**

Include terms to fit

1. unwanted exponential from first intermediate state
2. first excited state

Time ordering $t_{em} > 0$:

$$F^>(t_H, T) = F^> + B_F^> e^{(E\gamma - E^>)T} + C_F^> e^{\Delta E t_H}$$

*fit parameters*
4d>,< method: Fits to $F_V$ for $t_{em} > 0$ time ordering

$$F^>(t_H, T) = F^> + B^>_F e^{(E_\gamma - E^>) T} + C^>_F e^{\Delta E t_H}$$
Fit form: 3d method

Include terms to fit

(1) unwanted exponential from first intermediate state
(2) first excited state
Fit form: 3d method

Include terms to fit
(1) unwanted exponential from first intermediate state
(2) first excited state

Time ordering $t_{em} < 0$:

$$F^<(t_H, T) = F^< + B_F^<(1 + B_{F,exc}^< e^{\Delta E(T + t_H)}) e^{-(E_{\gamma} - E_H + E^<)T} + C_F^< e^{\Delta E t_H}$$

- fit parameters
Fit form: 3d method

Include terms to fit

(1) unwanted exponential from first intermediate state
(2) first excited state

Time ordering $t_{em} < 0$:

$$F^<(t_H, T) = F^< + B_F^<(1 + B_F^{exc} e^{\Delta E(T+t_H)}) e^{-(E_\gamma - E_H + E^<)T} + C_F^< e^{\Delta E t_H}$$

- fit parameters
Fit form: 3d method

Include terms to fit

1. unwanted exponential from first intermediate state
2. first excited state

Time ordering \( t_{em} < 0 \):

\[
F^<(t_H, T) = F^< + B_F^< (1 + B_{F, \text{exc}}^\Delta E(T+t_H)) e^{-(E_\gamma-E_H+E^<)T} + C_F^\Delta E t_H
\]

- fit parameters
Fit form: 3d method

Include terms to fit

1. unwanted exponential from first intermediate state
2. first excited state

Time ordering $t_{em} < 0$:

$$F^<(t_H, T) = F^< + B^<_F (1 + B^<_F,\text{exc} e^{\Delta E(T+t_H)}) e^{-(E\gamma - E_H + E^<)T} + C^<_F e^{\Delta E t_H}$$

- fit parameters

To help stabilize the fits

→ Determine $\Delta E$ from the pseudoscalar two-point correlation function
→ use result as Gaussian prior in form factor fits
3d method: Fits to $F_V$ for $t_{em} < 0$ time ordering

$$F_V^<(t_{Ds}, T) = F_V^< + B_F^<(1 + B_{F, exc} e^{\Delta E(T+t_H)}) e^{-(E_{\gamma} - E_{H} + E^<)T} + C_F^< e^{\Delta E t_H}$$
$F_V$ as function of $E_{\gamma}^{(0)}$ using 3d and 4d methods

$x_{\gamma} = 2E_{\gamma}^{(0)}/m_{D_s}$

$0 \leq x_{\gamma} \leq 1 - \frac{m_b^2}{m_{D_s}^2}$
$F_V$ as function of $E_\gamma^{(0)}$ using 3d and 4d methods

$x_\gamma = \frac{2E_\gamma^{(0)}}{m_{D_s}}$

$0 \leq x_\gamma \leq 1 - \frac{m_\ell^2}{m_{D_s}^2}$

Summary:
$F_V$ as function of $E_\gamma^{(0)}$ using 3d and 4d methods

\[ x_\gamma = \frac{2E_\gamma^{(0)}}{m_{D_s}} \]

\[ 0 \leq x_\gamma \leq 1 - \frac{m_\gamma^2}{m_{D_s}^2} \]

Summary:
- 3d and 4d $>,<$ methods offer good control over systematics
$F_V$ as function of $E_\gamma^{(0)}$ using 3d and 4d methods

\[ x_\gamma = 2E_\gamma^{(0)}/m_{D_s} \]
\[ 0 \leq x_\gamma \leq 1 - \frac{m_\gamma^2}{m_{D_s}^2} \]

Summary:
- 3d and 4d $>,<$ methods offer good control over systematics
- Combined fits offer small improvement relative to individual
$F_V$ as function of $E_\gamma^{(0)}$ using 3d and 4d methods

\[ x_\gamma = 2E_\gamma^{(0)}/m_{D_s} \]
\[ 0 \leq x_\gamma \leq 1 - \frac{m_\ell^2}{m_{D_s}^2} \]

Summary:
- 3d and 4d $>,<$ methods offer good control over systematics
- Combined fits offer small improvement relative to individual

Compare computational cost of 3d and 4d $>,<$ methods
Number of propagator solves per configuration

<table>
<thead>
<tr>
<th>Source</th>
<th>3d method</th>
<th>4d $&gt;,&lt;$ method</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>$2(1 + N_{tH} N_{PH})$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{Z}_2$ wall</td>
<td>$2(1 + N_{tH} N_{PH} + N_{PH} N_{P\gamma})$</td>
<td></td>
</tr>
</tbody>
</table>
Number of propagator solves per configuration

<table>
<thead>
<tr>
<th>Source</th>
<th>3d</th>
<th>4d(&gt;),(&lt;) method</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>(2(1 + N_{tH}N_{PH}))</td>
<td>(2(1 + 2 \times 4N_TN_{P\gamma}))</td>
</tr>
<tr>
<td>(\mathbb{Z}_2) wall</td>
<td>(2(1 + N_{tH}N_{PH} + N_{PH}N_{P\gamma}))</td>
<td>(2(1 + 2 \times 4N_TN_{P\gamma} + N_{P\gamma}N_{PH}))</td>
</tr>
</tbody>
</table>
Number of propagator solves per configuration

<table>
<thead>
<tr>
<th>Source</th>
<th>3d</th>
<th>4d&gt;,&lt;</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>(2(1 + N_{tH} N_{PH}))</td>
<td>(2(1 + 2 \times 4 N_T N_{P\gamma}))</td>
</tr>
<tr>
<td>(\mathbb{Z}_2) wall</td>
<td>(2(1 + N_{tH} N_{PH} + N_{PH} N_{P\gamma}))</td>
<td>(2(1 + 2 \times 4 N_T N_{P\gamma} + N_{P\gamma} N_{PH}))</td>
</tr>
</tbody>
</table>

The 4d\(>,<\) method generally more expensive than 3d method if cover full \(\gamma\) range.

The 3d method offers good control over systematics for cheapest cost.
Number of propagator solves per configuration

<table>
<thead>
<tr>
<th>Source</th>
<th>3d</th>
<th>4d $&gt;\gamma&lt;$</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>$2(1 + N_{tH} N_{pH})$</td>
<td>$2(1 + 2 \times 4 N_{T \gamma} N_{p_{\gamma}})$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$ wall</td>
<td>$2(1 + N_{tH} N_{pH} + N_{pH} N_{p_{\gamma}})$</td>
<td>$2(1 + 2 \times 4 N_{T} N_{p_{\gamma}} + N_{p_{\gamma}} N_{p_{H}})$</td>
</tr>
</tbody>
</table>
Number of propagator solves per configuration

<table>
<thead>
<tr>
<th>Source</th>
<th>3d Method</th>
<th>4d Method</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>$2(1 + N_{th} N_{ph})$</td>
<td>$2(1 + 2 \times 4 N_T N_{p\gamma})$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$ wall</td>
<td>$2(1 + N_{th} N_{ph} + N_{ph} N_{p\gamma})$</td>
<td>$2(1 + 2 \times 4 N_T N_{p\gamma} + N_{p\gamma} N_{ph})$</td>
</tr>
</tbody>
</table>

- 4d $>,<$ method generally more expensive than 3d method if cover full $E_\gamma^{(0)}$ range
Number of propagator solves per configuration

<table>
<thead>
<tr>
<th>Source</th>
<th>3d method</th>
<th>4d $&gt;,&lt;$ method</th>
</tr>
</thead>
<tbody>
<tr>
<td>point</td>
<td>$2(1 + N_{tH} N_{PH})$</td>
<td>$2(1 + 2 \times 4 N_T N_{P\gamma})$</td>
</tr>
<tr>
<td>$\mathbb{Z}_2$ wall</td>
<td>$2(1 + N_{tH} N_{PH} + N_{PH} N_{P\gamma})$</td>
<td>$2(1 + 2 \times 4 N_T N_{P\gamma} + N_{P\gamma} N_{PH})$</td>
</tr>
</tbody>
</table>

- 4d $>,<$ method generally more expensive than 3d method if cover full $E_\gamma^{(0)}$ range

3d method offers good control over systematics for cheapest cost
Improved estimators using 3d method
Simulation parameters for final 3d method dataset

- \( N_f = 2 + 1 \) DWF, 3 RBC/UKQCD gauge ensembles

<table>
<thead>
<tr>
<th>ensemble</th>
<th>((L/a)^3 \times (T/a))</th>
<th>(L_5/a)</th>
<th>(\approx a^{-1})(GeV)</th>
<th>(a m_t)</th>
<th>(a m_s)</th>
<th>(\approx M_\pi)(MeV)</th>
<th>(N_{\text{conf}})</th>
</tr>
</thead>
<tbody>
<tr>
<td>24I</td>
<td>(24^3 \times 64)</td>
<td>16</td>
<td>1.785</td>
<td>0.005</td>
<td>0.04</td>
<td>340</td>
<td>25</td>
</tr>
<tr>
<td>32I</td>
<td>(32^3 \times 64)</td>
<td>16</td>
<td>2.383</td>
<td>0.004</td>
<td>0.03</td>
<td>304</td>
<td>26</td>
</tr>
<tr>
<td>48I</td>
<td>(48^3 \times 96)</td>
<td>24</td>
<td>1.730</td>
<td>0.00078</td>
<td>0.0362</td>
<td>139</td>
<td>7</td>
</tr>
</tbody>
</table>

- Use local currents with mostly non-perturbative renormalization
- charm valence quarks → Möbius domain-wall with “stout” smearing
- u/d/s valence quarks → same DWF action as sea quarks
- Neglect disconnected diagrams
- Use all-mode averaging 4 exact and 64 sloppy solves per config
Simulation parameters for final 3d method dataset

- $N_f = 2 + 1$ DWF, 3 RBC/UKQCD gauge ensembles

<table>
<thead>
<tr>
<th>ensemble</th>
<th>$(L/a)^3 \times (T/a)$</th>
<th>$L_5/a$</th>
<th>$a^{-1}$(GeV)</th>
<th>$am_l$</th>
<th>$am_s$</th>
<th>$M_\pi$(MeV)</th>
<th>$N_{conf}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>24I</td>
<td>$24^3 \times 64$</td>
<td>16</td>
<td>1.785</td>
<td>0.005</td>
<td>0.04</td>
<td>340</td>
<td>25</td>
</tr>
<tr>
<td>32I</td>
<td>$32^3 \times 64$</td>
<td>16</td>
<td>2.383</td>
<td>0.004</td>
<td>0.03</td>
<td>304</td>
<td>26</td>
</tr>
<tr>
<td>48I</td>
<td>$48^3 \times 96$</td>
<td>24</td>
<td>1.730</td>
<td>0.00078</td>
<td>0.0362</td>
<td>139</td>
<td>7</td>
</tr>
</tbody>
</table>

- Use local currents with mostly non-perturbative renormalization
- charm valence quarks $\rightarrow$ Möbius domain-wall with “stout” smearing
- $u/d/s$ valence quarks $\rightarrow$ same DWF action as sea quarks
- Neglect disconnected diagrams
- Use all-mode averaging 4 exact and 64 sloppy solves per config

Final improved estimators using 3d method:

- $\mathbb{Z}_2$ random wall sources and point sources
- Two datasets: $J_{\nu}^{\text{weak}}(0)$ or $J_{\mu}^{\text{em}}(0)$
- For point-sources use translational invariance to fix em/weak operator at origin
  $\rightarrow$ use “infinite-volume approximation” to generate data at arbitrary photon momenta
  (only exponentially small FVEs introduced)
Alternate correlation function

- Fix em current at origin: \( J_{\mu}^{em}(0) \)

\[
C_{3,\mu\nu}^{EM}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y \ e^{-i\vec{p}_H \cdot \vec{x}} e^{i\vec{p}_H \cdot \vec{y}} \langle J_{\mu}^{em}(0) J_{\nu}^{weak}(t_W, \vec{x}) \phi_H^\dagger(t_H, \vec{y}) \rangle
\]
Alternate correlation function

- Fix em current at origin: \( J_{\mu}^{em}(0) \)

\[
C_{3,\mu\nu}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y \ e^{-i\vec{p}_H \cdot \vec{x}} e^{i\vec{p}_H \cdot \vec{y}} \langle J_{\mu}^{em}(0) J_{\nu}^{weak}(t_W, \vec{x}) \phi_H(t_H, \vec{y}) \rangle
\]

- \( e^{E_H t_W} \) and \( e^{-i\vec{p}_H \cdot \vec{x}} \) shift weak current relative to other operators
Alternate correlation function

- Fix em current at origin: $J_{\mu}^{\text{em}}(0)$

$$C_{3,\mu\nu}^{\text{EM}}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y \ e^{-i\vec{p}_H \cdot \vec{x}} e^{i\vec{p}_H \cdot \vec{y}} \langle J_{\mu}^{\text{em}}(0) J_{\nu}^{\text{weak}}(t_W, \vec{x}) \phi_{H}^{\dagger}(t_H, \vec{y}) \rangle$$

- $e^{E_H t_W}$ and $e^{-i\vec{p}_H \cdot \vec{x}}$ shift weak current relative to other operators

- using point sources, can reuse sequential propagators to get for free
Alternate correlation function

- Fix em current at origin: $J_{\mu}^{\text{em}}(0)$

$$C_{3,\mu\nu}^{\text{EM}}(t_W, t_H) = e^{E_H t_W} \int d^3x \int d^3y \ e^{-ip_H \cdot \vec{x}} e^{ip_H \cdot \vec{y}} \langle J_{\mu}^{\text{em}}(0) J_{\nu}^{\text{weak}}(t_W, \vec{x}) \phi_H^\dagger(t_H, \vec{y}) \rangle$$

- $e^{E_H t_W}$ and $e^{-i\vec{p}_H \cdot \vec{x}}$ shift weak current relative to other operators

- using point sources, can reuse sequential propagators to get for free

- define analogous time-integrated correlation function $I_{\mu\nu}^{\text{EM}}(T, t_H)$ and $I_{\mu\nu}^{\text{EM}}(T, t_H)$
Alternate correlation function

- Fix em current at origin: $J_{μ}^{em}(0)$

$$C_{3,μν}^{EM}(t_W, t_H) = e^{E_H t_W} \int d^3 x \int d^3 y \ e^{-i\vec{p}_H \cdot \vec{x}} e^{i\vec{p}_H \cdot \vec{y}} \langle J_{μ}^{em}(0) J_{ν}^{weak}(t_W, \vec{x}) \phi_H(t_H, \vec{y}) \rangle$$

- $e^{E_H t_W}$ and $e^{-i\vec{p}_H \cdot \vec{x}}$ shift weak current relative to other operators
- using point sources, can reuse sequential propagators to get for free
- define analogous time-integrated correlation function $I_{μν}^{<,EM}(T, t_H)$ and $I_{μν}^{>,EM}(T, t_H)$

Spectral decompositions show us

$$I_{μν}^{<,EM}(T, t_H) = I_{μν}^{>,EM}(T, t_H) + \text{excited state effects}$$

$$I_{μν}^{>,EM}(T, t_H) = I_{μν}^{<,EM}(T, t_H) + \text{excited state effects}$$

Perform combined fits to take advantage of this relation
Alternate correlation function

Can integrate past $T = -t_H$ using alternate correlation function
Alternate correlation function: vector form factor

\[ F_{V} = \begin{cases} 
0.0 & x_{\gamma} = 0.0 \\
0.2 & x_{\gamma} = 0.2 \\
0.4 & x_{\gamma} = 0.4 \\
0.6 & x_{\gamma} = 0.6 \\
0.8 & x_{\gamma} = 0.8 \\
1.0 & x_{\gamma} = 1.0 \\
1.2 & x_{\gamma} = 1.2 
\end{cases} \]
Non-perturbative subtraction of IR divergent lattice artifacts

\[ F_{A,SD} = F_A - (-Q^e \frac{f_H}{E_\gamma^{(0)}}) \]

Blue data: improved subtraction of point-like contribution

\( \sigma(a^2/x_\gamma) \) artifacts first observed in arXiv:2006.05358

We use

\[ \int d^3x \ d^3y (e^{-i \vec{p} \cdot \vec{x}} - 1) \langle J_i^{em}(x) J_i^A(0) \phi_H^+(y) \rangle \]

to subtract the pt-like contribution

Blue data: improved subtraction of point-like contribution
Taking ratios of correlation functions

\[ C_{3,\mu\nu}^{\text{improved}}(\vec{p}_\gamma, t) = C_{3,\mu\nu}^{\text{point}}(\vec{p}_\gamma, t) \frac{C_{3,\mu\nu}^{Z^2}(\vec{p}^*, t)}{C_{3,\mu\nu}^{\text{point}}(\vec{p}^*, t)}, \quad \vec{p}^* = \frac{2\pi}{L} n \]
Averaging over $\pm \vec{p}_\gamma$
Outline for section 4

1. Introduction and Motivation
2. Extracting the hadronic tensor with Euclidean correlation functions
3. Methods study
4. $D_s \rightarrow \ell\nu\gamma$ preliminary form factor results
We are testing various fit functions to provide a parameterization of the form-factor lattice data.
$D_s \rightarrow \ell \nu \ell \gamma$ preliminary results

similar cancellations observed in $D_s D_s^* \gamma$ couplings, corresponding to pole residues in $D_s \rightarrow \gamma \ell \nu \ell$ form factors \cite{1},\cite{2}

\[ F_V = F_V^{(c)} + F_V^{(s)} \]
$D_s \to \ell \nu \ell \gamma$ comparison

sign: different convention in FF decomp

Can be tamed using 3d method
$D_s \to \ell \nu \ell \gamma$ comparison

**Figure 1:**
- $F_{A,SD}$ comparison
- $F_V$ comparison

**Sign:** different convention in FF decompositions

**Signatures:**
- Preliminary data

---

**Legend:**
- Data points from different experiments

**Citations:**
- R. Frezzotti et al., arXiv:2306.05904
- Phys. Rev. D 103, 014502 (2021)
\( D_s \rightarrow \ell \nu \ell \gamma \) comparison

\[ \text{At large } x_\gamma, \text{ signal to noise decays exponentially with } T \]

(observed in [R. Frezzotti et. al, arXiv:2306.05904])

Can be tamed using 3d method
Future work

- Investigating different fit models to parameterize lattice form factors
- Improving statistics with new computing allocation from ACCESS
- Go to the $B$ using new RBC-UKQCD $a^{-1} \approx 3.5$ GeV and 4.5 GeV lattices
- Have data for $\pi, K, D$, analyze this and compare our results to physical calculation in
Radiative leptonic decays are physically interesting at large and small photon momentum
Summary

- Radiative leptonic decays are physically interesting at large and small photon momentum
- Two sources of systematic errors inherent in lattice QCD calc., need to take $T \to \infty$ and $t_H \to -\infty$

\[
T_{\mu\nu} = \lim_{T \to \infty} \lim_{t_H \to -\infty} \frac{-2E_{H}e^{-E_{H}t_{H}}}{\langle H(\bar{\phi}_{H})|\phi_{H}^{+}|0\rangle} I_{\mu\nu}(T, t_{H})
\]
Radiative leptonic decays are physically interesting at large and small photon momentum.

Two sources of systematic errors inherent in lattice QCD calc., need to take $T \to \infty$ and $t_H \to -\infty$.

Compared 3d sequential propagator and 4d sequential propagators
\[ \rightarrow \text{found 3d method to offer good control over systematic uncertainties for cheapest cost} \]
Summary

- Radiative leptonic decays are physically interesting at large and small photon momentum.
- Two sources of systematic errors inherent in lattice QCD calc., need to take $T \to \infty$ and $t_H \to -\infty$.
- Compared 3d sequential propagator and 4d sequential propagators. → found 3d method to offer good control over systematic uncertainties for cheapest cost.
- Implemented number of improvements to 3d method.
Radiative leptonic decays are physically interesting at large and small photon momentum.

Two sources of systematic errors inherent in lattice QCD calc., need to take $T \to \infty$ and $t_H \to -\infty$.

Compared 3d sequential propagator and 4d sequential propagators → found 3d method to offer good control over systematic uncertainties for cheapest cost.

Implemented number of improvements to 3d method.

Presented preliminary results for $D_s \to \gamma \ell \nu$ on three RBC/UKQCD ensembles using DWF for all flavors.
Main Take-Away Points

**Main Take-Away Point 1:** Radiative leptonic decays are interesting in the regions of small and large photon energies

**Main Take-Away Point 2:** We have developed methods to achieve high precision for small computational cost


**Main Take-Away Point 3:** Working on physical calculation of $D_s \rightarrow \gamma \ell \nu_\ell$, out soon
Backup slides
Time order visualization

\[ J_{\nu}^{\text{weak}}(0) \]

- \( t_H < t_{\text{em}} < 0 \)
- \( t_H < 0 < t_{\text{em}} \)
- \( t_{\text{em}} < t_H < 0 \)
- \( aN_T/2 - |t_H| \)
- \( \phi_H^+(t_H) \)
Minkowski spectral decomposition of $T_{\mu\nu}$

Time ordering $t_{em} < 0$:

$\hat{1} = |0\rangle\langle 0| + \sum_n \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_n(\vec{p})} |n(\vec{p})\rangle\langle n(\vec{p})|$

\[
T_{\mu\nu}^\lessgtr = -i \int_{-\infty}^0 dt_{em} \int d^3x \ e^{ip\gamma \cdot x} \langle 0| J_{\nu}^{\text{weak}}(0) \hat{1} J_{\mu}^{\text{em}}(x) |B^{-}(\vec{p}_B)\rangle
\]

\[
= -\sum_n \frac{1}{2E_n,\vec{p}_B-\vec{p}_\gamma} \frac{1}{E_\gamma + E_n,\vec{p}_B-\vec{p}_\gamma - E_B,\vec{p}_B - i\epsilon}
\]

\[
\times \langle 0| J_{\nu}^{\text{weak}}(0) |n(\vec{p}_B - \vec{p}_\gamma)\rangle \langle n(\vec{p}_B - \vec{p}_\gamma) |J_{\mu}^{\text{em}}(0) |B(\vec{p}_B)\rangle
\]

(In infinite volume, the sum over $n$ includes an integral over the continuous spectrum of multi-particle states.)
Euclidean spectral decomposition of $I_{\mu\nu}$

Time ordering $t_{em} < 0$: (for large negative $t_B$)

$$I_{\mu\nu}^<(t_B, T) = \int_{-T}^{0} dt_{em} \ e^{E_{\gamma} t} C_{3,\mu\nu}(t_{em}, t_B)$$

(* all times are now Euclidean *)

$$= \langle B(\vec{p}_B)|\phi_B^\dagger(0)|0\rangle \frac{1}{2E_{B,\vec{p}_B}} e^{E_{B} t_B}$$

$$\times \sum_n \frac{1}{2E_{n,\vec{p}_B-\vec{p}_\gamma}} \frac{\langle 0|J_{\nu}^{\text{weak}}(0)|n(\vec{p}_B - \vec{p}_\gamma)\rangle \langle n(\vec{p}_B - \vec{p}_\gamma)|J_{\mu}^{em}(0)|B(\vec{p}_B)\rangle}{E_{\gamma} + E_{n,\vec{p}_B-\vec{p}_\gamma} - E_{B,\vec{p}_B}}$$

$$\times \left[ 1 - e^{-(E_{\gamma} + E_{n,\vec{p}_B-\vec{p}_\gamma} - E_{B,\vec{p}_B}) T} \right]$$

Require $E_{\gamma} + E_{n,\vec{p}_B-\vec{p}_\gamma} - E_{B,\vec{p}_B} > 0$ to get rid of unwanted exponential

States $|n(\vec{p}_B - \vec{p}_\gamma)\rangle$ has same flavor quantum numbers as B meson

$\rightarrow E_{n,\vec{p}_B-\vec{p}_\gamma} \geq E_{B,\vec{p}_B-\vec{p}_\gamma} = \sqrt{m_B^2 + (\vec{p}_B - \vec{p}_\gamma)^2}$

For $\vec{p}_\gamma \neq 0$, $|\vec{p}_\gamma| + \sqrt{m_n^2 + (\vec{p}_B - \vec{p}_\gamma)^2} > \sqrt{m_B^2 + \vec{p}_B}$ is automatically satisfied
Comparison to experiment and lattice: [PRD 2021/arXiv:2012.02120]

Depends on $F^\pm(x_\gamma) = F_V(x_\gamma) \pm F_{A,SD}(x_\gamma)$, at $\mathcal{O}(\alpha_{\text{em}})$, three pieces

- Point-like (pt): universal, does not probe internal structure of meson
- Structure-dependent: $\text{SD} \sim \text{SD}^+((F^+)^2) + \text{SD}^-((F^-)^2)$
- Interference between (pt) and (SD): $\text{INT} \sim \text{INT}^-(F^+) + \text{INT}^-(F^-)$

**KLOE experiment**

- $K \to e\nu_e\gamma$: perform cuts so sensitive to mainly $\text{SD}^+$ $\implies (F^+)^2$ consistent
- $K \to \mu\nu_\mu\gamma$: perform cuts so sensitive to

**E787 experiment:**

- $K \to \mu\nu_\mu\gamma$: perform cuts so sensitive to mainly $\text{SD}^+$ $\implies (F^+)^2$, slight dependence on $\text{SD}^- + \text{INT}^-\text{ at small } x_\gamma$
- Tension between prediction for $F(x_\gamma)^+$ between KLOE and E787
Let $F^\pm(x_\gamma) = F_V(x_\gamma) \pm F_{A,SD}(x_\gamma)$, then

<table>
<thead>
<tr>
<th>Piece</th>
<th>QCD FF</th>
</tr>
</thead>
<tbody>
<tr>
<td>PT</td>
<td>none</td>
</tr>
<tr>
<td>SD$^+$</td>
<td>$(F^+)^2$</td>
</tr>
<tr>
<td>SD$^-$</td>
<td>$(F^-)^2$</td>
</tr>
<tr>
<td>INT$^+$</td>
<td>$F^+$</td>
</tr>
<tr>
<td>INT$^-$</td>
<td>$F^-$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Process</th>
<th>Sensitive to</th>
<th>Theory vs Exp.</th>
</tr>
</thead>
<tbody>
<tr>
<td>KLOE</td>
<td>$K \to e\nu_e\gamma$</td>
<td>SD$^+$</td>
<td>Agree</td>
</tr>
<tr>
<td>E787</td>
<td>$K \to \mu\nu_\mu\gamma$</td>
<td>SD$^+$, SD$^-$ + INT$^-$</td>
<td>Tension</td>
</tr>
<tr>
<td>ISTRA+</td>
<td>$K \to \mu\nu_\mu\gamma$</td>
<td>INT$^-$</td>
<td>Tension</td>
</tr>
<tr>
<td>OKA</td>
<td>$K \to \mu\nu_\mu\gamma$</td>
<td>INT$^-$</td>
<td>Tension</td>
</tr>
<tr>
<td>PIBETA</td>
<td>$\pi \to e\nu_e\gamma$</td>
<td>SD$^+$</td>
<td>Tension</td>
</tr>
</tbody>
</table>
Fit form: 4d method

Use fit ranges where data has plateaued in $t_H$, i.e. $t_H \to -\infty$

Include terms to fit

(1) unwanted exponential from first intermediate state

Limitation of 4d method $\rightarrow$ cannot resolve the two time orderings
$\rightarrow$ Fit sum of both time orderings $F_V(T, t_H) = F_V^<(T, t_H) + F_V^>(T, t_H)$

$$F(t_H, T) = F + B_F^< e^{-(E_\gamma - E_H + E^<)T} + B_F^> e^{(E_\gamma - E^>)T}$$

- fit parameters

Only have three values of $T$, fitting multiple exponentials not possible
$\rightarrow$ Use broad Gaussian prior on $E^>$ exclude unphysical values
Comparison of 4d and 4d\(>,<\)

\[ x_\gamma = 2E^{(0)}_\gamma / m_{Ds} \]

\[ 0 \leq x_\gamma \leq 1 - \frac{m^2}{m^2_{Ds}} \]

Summary:
- Cost 4d\(>,<\) method roughly twice 4d method
- 4d\(>,<\) resolves time orders, allows better control over \(T \to \infty\) limit
- 4d\(>,<\) method smaller uncertainty than 4d method
Infinite volume approximation

For point-sources use translational invariance to fix em/weak operator at origin
→ use “infinite-volume approximation” to generate data at arbitrary photon momenta
→ only exponentially small FVEs introduced
Infinite volume approximation

For point-sources use translational invariance to fix em/weak operator at origin
→ use “infinite-volume approximation” to generate data at arbitrary photon momenta
→ only exponentially small FVEs introduced

Strategy:
- Work in rest frame of $D_s$ meson
- Calculate correlation function for arbitrary values of photon momentum
- Propagator solves per config
→ $2(1 + N_{tH})$
Infinite volume approximation

For point-sources use translational invariance to fix em/weak operator at origin
→ use “infinite-volume approximation” to generate data at arbitrary photon momenta
→ only exponentially small FVEs introduced

Strategy:
- Work in rest frame of $D_s$ meson
- Calculate correlation function for arbitrary values of photon momentum
- Propagator solves per config
  → $2(1 + N_{tH})$

Photon momenta for 24l ensemble:

$$p_{\gamma,z} = 2\pi/L \times \{0.1, 0.2, 0.4, 0.6, 0.8, 1.0, 1.4, 1.8, 2.2, 2.4\}$$

$$C_{3,\mu\nu}(t_{em}, t_H) = \int d^3 x \int d^3 y \ e^{-i\vec{p}_\gamma \cdot \vec{x}} \langle J_{\mu}^{em}(t_{em}, \vec{x}) J_{\nu}^{weak}(0) \phi_H^\dagger(t_H, \vec{y}) \rangle$$
Infinite volume approximation

We assume there exist $c, d, \Lambda, \Lambda' \in \mathbb{R}^+$ and $L_0 \in \mathbb{N}$ for which

$$
\tilde{C}^L(q) \equiv \sum_{x=-L/2}^{L/2-1} C^L(x) e^{iqx}
$$

for all $x$ with $-L/2 \leq x \leq L/2$ and $L \geq L_0$ and

$$
|C^\infty(x)| \leq de^{-\Lambda'|x|}
$$

for all $x$ with $|x| > L/2$. We now define

$$
|C^\infty(x) - C^L(x)| \leq ce^{-\Lambda L} \quad \text{and} \quad \tilde{C}^\infty(q) \equiv \sum_{x=-\infty}^{\infty} C^\infty(x) e^{iqx}.
$$

Under the above assumptions, it then follows that there is a $\tilde{c} \in \mathbb{R}^+$ for which

$$
|\tilde{C}^\infty(q) - \tilde{C}^L(q)| \leq \tilde{c}e^{-\Lambda_0 L}
$$

for all $q \in [-\pi, \pi]$ and all $L \geq L_0$, with $\Lambda_0 \equiv \min(\Lambda, \Lambda'/2)$. 
Fits to 3d data for $F_V$ with improved methods
Fits to 3d data for $F_{A,SD}$ with improved methods
Simulation parameters

- 3 ETMC gauge ensembles with $N_f = 2 + 1 + 1$
- twisted mass fermions in sea?
- 3 lattice spacings $a \in \{0.089, 0.082, 0.062\}$ fm
- Lightest
- twisted boundary conditions to assign arbitrary $p_\gamma, p_{\text{Meson}}$

Physical calculation for $D_s \rightarrow \gamma \ell \nu_\ell$ for full kinematic range of $E_\gamma$ [Frezzotti, Gagliardi, Lubicz, Martinelli, Mazzetti, Sachrajda, Sanfilippo, Simula, Tantalo, PRD 2021, arXiv:2306.05904]
- 4 ETMC gauge ensembles with $N_f = 2 + 1 + 1$
- twisted mass fermions
- 4 lattice spacings $a \in [0.058, 0.09]$ fm
- physical pion mass
- twisted boundary conditions to assign arbitrary $p_\gamma, p_{D_s}$
- Osterwalder-Seiler fermions