Two-photon exchange in lepton-proton scattering.

Low-energy nucleon structure
Outline

Experimental motivation

1) electron-proton scattering:
   near-forward and dispersive calculations

2) muon-proton scattering:
   near-forward and dispersive calculations

3) $^1S$ HFS in muonic hydrogen. $^2\chi$ in atomic physics
   uncertain corrections, relation to electronic hydrogen
Muon discrepancies: new physics?

anomalous magnetic moment

Lamb shift

formulation of QED as first successful QFT

J. Schwinger, F. Dyson, R. Feynman, S. Tomonaga (1947-1949)

anomalous magnetic moment

μH Lamb shift

proton size discrepancy

initially 5-6σ
eH, ep vs μH

3.6 σ

theory vs exp.

- hadronic uncertainty is dominant in theory
QCD running coupling

- asymptotic freedom at high energies

\[ \alpha_s(Q^2) \]

\[ \alpha_s(M_Z) = 0.1181 \pm 0.0013 \]

- EFTs, lattice QCD and phenomenology at low energies

Reasonably stable world average value of \( \alpha_s(M_Z^2) \), as well as a clear signature and proof of the energy dependence of \( \alpha_s \), in full agreement with the QCD prediction of asymptotic freedom. This is demonstrated in Fig. 9.3, where results of \( \alpha_s(Q^2) \) obtained at discrete energy scales \( Q \), now also including those based just on NLO QCD, are summarized.

Thanks to the results from the Tevatron and from the LHC, the energy scales at which \( \alpha_s \) is determined now extend up to more than 1 TeV.

\[ \alpha_s(Q^2) = 0.1181 \pm 0.0013 \]

**Figure 9.3:** Summary of measurements of \( \alpha_s \) as a function of the energy scale \( Q \).

The respective degree of QCD perturbation theory used in the extraction of \( \alpha_s \) is indicated in brackets (NLO: next-to-leading order; NNLO: next-to-next-to leading order; res. NNLO: NNLO matched with resummed next-to-leading logs; N^3LO: next-to-NNLO).

9.5. Acknowledgments


We note, however, that in many such studies, like those based on exclusive states of jet multiplicities, the relevant energy scale of the measurement is not uniquely defined. For instance, in studies of the ratio of 3- to 2-jet cross sections at the LHC, the relevant scale was taken to be the average of the transverse momenta of the two leading jets [379], but could alternatively have been chosen to be the transverse momentum of the 3rd jet.
Muon discrepancies: new physics?

anomalous magnetic moment

Lamb shift

Polykarp Kusch (1947)

formulation of QED as first successful QFT

J. Schwinger, F. Dyson, R. Feynman, S. Tomonaga (1947-1949)

W. Lamb and R. Retherford (1947)

anomalous magnetic moment

μH Lamb shift

proton size discrepancy

initially 5-6σ
eH, ep vs μH

- hadronic uncertainty is dominant in theory
Tool to explore the proton structure

Photon-proton vertex

\[ \Gamma^\mu(Q^2) = \gamma^\mu F_D(Q^2) + \frac{i\sigma^{\mu\nu} q_\nu}{2M} F_P(Q^2) \]

Dirac and Pauli form factors

Lepton energy

\[ Q^2 = -(k - k')^2 \]

Momentum transfer

1\(\gamma\) amplitude

\[ T = \frac{e^2}{Q^2} (\bar{u}(k', h') \gamma_\mu u(k, h)) \cdot (\bar{N}(p', \lambda') \Gamma^\mu(Q^2) N(p, \lambda)) \]
Form factors measurement

Sachs electric and magnetic form factors

\[ G_E = F_D - \tau F_P \quad G_M = F_D + F_P \]

Rosenbluth separation

\[ \frac{d\sigma^{\text{unpol}}}{d\Omega} \sim G_M^2(Q^2) + \frac{\varepsilon}{\tau} G_E^2(Q^2) \]

\[ \tau = \frac{Q^2}{4M^2} \quad \frac{d\sigma^{\text{unpol}}}{d\Omega} \]

\( \tau, \varepsilon \) kinematical variables

\[ \varepsilon \leftrightarrow \theta_{\text{lab}} \quad G_M^2(Q^2) \]

- Rosenbluth slope is sensitive to corrections beyond 1\( \chi \)

Qattan et al. (2005)
Form factors measurement

Sachs electric and magnetic form factors

\[ G_E = F_D - \tau F_P \quad G_M = F_D + F_P \]

polarization transfer

\[ \overrightarrow{e} + \overrightarrow{p} \rightarrow \overrightarrow{e} + \overrightarrow{p} \]

realized in 2000

\[ P_T \sim G_E(Q^2)G_M(Q^2) \]

\[ P_L \sim G_M^2(Q^2) \]
Proton form factors puzzle

Polarization transfer
JLab (Hall A, C) vs. Rosenbluth separation
SLAC, JLab (Hall A, C)

\[ \frac{\mu_p G_{Ep}}{G_{Mp}} \]

\[ Q^2 (\text{GeV}^2) \]

V. Punjabi et al. (2015)
Proton form factors puzzle

Polarization transfer
JLab (Hall A, C) vs.

Rosenbluth separation
SLAC, JLab (Hall A, C)

possible explanation
two-photon exchange

\[ \frac{\mu_p G_{Ep}}{G_{Mp}} \text{ vs. } Q^2 \text{ (GeV}^2) \]

V. Punjabi et al. (2015)
Proton form factors puzzle

Polarization transfer
JLab (Hall A, C)

vs.

Rosenbluth separation
SLAC, JLab (Hall A, C)

possible explanation
two-photon exchange

new $2\gamma$ measurements

- discrepancy motivates study of $2\gamma$
$2\chi$ from experiment

\[ d\sigma \sim \begin{array}{c}
\begin{array}{c}
l \quad \gamma \\
p \quad \rightarrow \quad p' \\
l' \quad \rightarrow \quad p
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
l \quad \gamma \\
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\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
l \quad \gamma \quad \gamma \\
p \quad \rightarrow \quad p' \\
l' \quad \rightarrow \quad p
\end{array}
\end{array} + \ldots = \]

\[2Re\begin{array}{c}
\begin{array}{c}
l \quad \gamma \\
p \quad \rightarrow \quad p' \\
l' \quad \rightarrow \quad p
\end{array}
\end{array} \times \begin{array}{c}
\begin{array}{c}
l \quad \gamma \quad \gamma \\
p \quad \rightarrow \quad p' \\
l' \quad \rightarrow \quad p
\end{array}
\end{array}\]

\[ R = \frac{\sigma(e^+p)}{\sigma(e^-p)}\]
VEPP-3 and CLAS data

$\omega = 1.6 \text{ GeV}$

VEPP-3@Novosibirsk: I. A. Rachel et al. (2015)
VEPP-3 and CLAS data

\[ \omega = 1.6 \text{ GeV} \]

\[ \varepsilon = 0.88 \]

- 2\sigma effect within 2-3\sigma
- reasonable agreement theory vs. experiment
- in agreement with phenomenology
- radiative corrections are important
Proton form factors puzzle

- problem is not completely solved !!!

- interest in high-$Q^2$ measurements of $R$
Proton radius

electric charge radius

\[ <r_E^2> \equiv -6 \frac{dG_E(Q^2)}{dQ^2} \bigg|_{Q^2=0} \]

- ep elastic scattering

\[ r_E = 0.879 \pm 0.008 \text{ fm} \]
Proton radius

electric charge radius

\[ < r_E^2 > \equiv -6 \left. \left( \frac{dG_E(Q^2)}{dQ^2} \right) \right|_{Q^2=0} \]

- ep elastic scattering

\[ r_E = 0.879 \pm 0.008 \text{ fm} \]

- atomic spectroscopy

\[ \Delta E_{nS} \sim m_r^3 < r_E^2 > \]

H, D spectroscopy

\[ r_E = 0.8758 \pm 0.0077 \text{ fm} \]

CODATA 2010

\[ \mu H \text{ Lamb shift} \]

\[ r_E = 0.8409 \pm 0.0004 \text{ fm} \]

CREMA (2010, 2013)
Proton radius puzzle

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5.6\(\sigma\) difference!
Proton radius puzzle

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CREMA (2010, 2013)

eH 2S-4P (Garching, 2017)

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Proton radius puzzle

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\( r_E = 0.879 \pm 0.008 \) fm ?

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CODATA 2010

eH 1S-3S (LKB, Paris, 2018)

5.6σ difference!

eH 2S-4P (Garching, 2017)

eH 2S-2P (York U., 2019)

μH, μD Lamb shift

PRad (JLab, 2019)

CREMA (2010, 2013)
Lamb shift and hyperfine splitting in H

\[ \text{2S}_{1/2} \rightarrow \text{2P}_{3/2} \]

- 1S HFS in \( \mu \text{H} \) with 1 ppm accuracy at PSI, J-PARC, RIKEN-RAL

A. Antognini et al. (2013)

R. Pohl et al. (2016)
μH Lamb shift and $2\chi$

$2\chi$ hadronic correction

$$\Delta E_{2P-2S}^{2\gamma} = 33 \pm 2 \, \mu eV$$


- important to reduce ambiguities of $2\chi$

2P-2S transition in μH

- discrepancy: 310 μeV
- μH uncertainty: 2.5 μeV
Elastic electron-proton scattering
and two-photon exchange
Scattering experiments and $2\chi$

- $2\chi$ is not among standard radiative corrections

$$\sigma^{\exp} \equiv \sigma_{1\gamma}(1 + \delta_{\text{rad}} + \delta_{\text{soft}} + \delta_{2\gamma})$$

- soft-photon contribution is included

- hard-photon contribution modelled by Feshbach correction

- charge radius insensitive to $2\chi$ model

- magnetic radius depends on $2\chi$ model
Elastic lepton-proton scattering and $2\gamma$

momentum transfer

$$Q^2 = -(k - k')^2$$

crossing-symmetric variable

$$\nu = \frac{(k, p + p')}{2}$$

- leading $2\gamma$ contribution: interference term

$$\delta_{2\gamma} = \frac{2 \sum_{\text{spin}} T^{1\gamma} \bar{\gamma} T^{2\gamma}}{\sum_{\text{spin}} |T^{1\gamma}|^2}$$

- $2\gamma$ correction to cross section is given by amplitudes real parts
Elastic lepton-proton scattering and $2\chi$

$$K = \frac{k + k'}{2}$$

$$P = \frac{p + p'}{2}$$

- electron-proton scattering: 3 structure amplitudes

$$T_{\text{non-flip}}^{\text{non-flip}} = \frac{e^2}{Q^2} \bar{l}\gamma_\mu l \cdot \bar{N} \left( G_M(\nu, Q^2)\gamma^\mu - F_2(\nu, Q^2) \frac{P^\mu}{M} + F_3(\nu, Q^2) \hat{K} P^\mu \right) N$$


- real parts of all $2\chi$ amplitudes extracted at one point $Q^2 = 2.5 \text{ GeV}^2$

J. Guttmann, N. Kivel, M. Meziane and M. Vanderhaeghen (2011)
Dm. Borisyuk and A. Kobushkin (2011)
I. A. Qattan (2017-2018)

- $2\chi$ correction to cross section is given by amplitudes real parts
non-forward scattering at low momentum transfer

photoproduction vertex or Compton tensor

assumption about the vertex
non-forward scattering
proton state

\[ \Gamma^\mu(Q^2) = \gamma^\mu F_D(Q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2M} F_P(Q^2) \]


assumption about the vertex
violation of unitarity for resonances!
non-forward scattering
inelastic states

works at small scattering angles

forward doubly-virtual Compton tensor

box diagram

unpolarized proton structure

proton + inelastic = total

M. E. Christy, P. E. Bosted (2010)
Low-$Q^2$ inelastic $2\chi$ correction (e-p)

- $2\chi$ blob: near-forward virtual Compton scattering

Feshbach inelastic elastic

\[ \text{ep: } \delta_{2\gamma} \sim a \sqrt{Q^2} + b Q^2 \ln Q^2 + c Q^2 \ln^2 Q^2 \]

unpolarized proton structure

\[ \delta_{2\gamma} = \int d\nu_{\gamma} dQ^2 (w_1(\nu_{\gamma}, Q^2) \cdot F_1(\nu_{\gamma}, Q^2) + w_2(\nu_{\gamma}, Q^2) \cdot F_2(\nu_{\gamma}, Q^2)) \]

- $2\chi$ at large $\varepsilon$ agrees with empirical fit
At low momentum transfer, non-forward scattering leads to dispersion relations. The photoproduction vertex or Compton tensor is based on on-shell information.

Box diagram assumption about the vertex.

Dispersion relations based on on-shell information.
Fixed-$Q^2$ dispersion relation framework

on-shell $1\chi$ amplitudes

experimental data

unitarity

$2\chi$ imaginary parts

$\Re \mathcal{F}(\nu) = \frac{2\nu}{\pi} \mathcal{P} \int_{\nu_{\text{min}}}^{\infty} \frac{\Im \mathcal{F}(\nu' + i0)}{\nu'^2 - \nu^2} d\nu'$

disp. rel.

$2\chi$ real parts

elastic and $\pi N$

cross section correction

$\delta_{\text{TPE}}(Q^2, \epsilon=0)$
Fixed-$Q^2$ dispersion relation framework

on-shell $1\chi$ amplitudes

![Graph showing experimental data, on-shell $1\chi$ amplitudes, and 2$\chi$ prediction]

2$\chi$ prediction

- Experimental data
- Elastic and $\pi N$
- Cross section correction

Fixed-$Q^2$ dispersion relation framework

On-shell $1\chi$ amplitudes

- Fixed-$Q^2$ dispersion relation framework

$\Im F(\nu) = \frac{2\nu}{\pi} \mathcal{P} \int_{\nu_{\text{min}}}^{\infty} \frac{\Im F(\nu' + i0)}{\nu'^2 - \nu^2} d\nu'$

$\Re F(\nu)$

2$\chi$ imaginary parts

Unitarity

2$\chi$ real parts

O. T. and M. Vanderhaeghen (2014-17)
\( \pi N \) in dispersive framework (e-p)

- \( \pi N \) is dominant inelastic 2\( \gamma \)

\[ Q^2 = 0.05 \text{ GeV}^2 \]

- dispersion relations agree with near-forward at large \( \varepsilon \)

\( Q^2 = 0.005 \text{ GeV}^2 \)

O. T., B. Pasquini and M. Vanderhaeghen (2017)
Comparison with data

\[ R_{2\gamma} = \frac{\sigma(e^+ p)}{\sigma(e^- p)} \approx 1 - 2\delta_{2\gamma} \]

- near-forward \(2\gamma\) agree with data
- multi-particle \(2\gamma\), e.g. \(\pi\pi N\), is important
Comparison with data

- dispersion relations agree with CLAS data

Comparison with data

VEPP-3 (2015)

CLAS (2016)

O. T., B. Pasquini and M. Vanderhaeghen (2017)
non-forward scattering

photoproduction vertex or Compton tensor

valid at small scattering

based on on-shell information
Our best $2\chi$ knowledge

- small $Q^2$: near-forward at large $\varepsilon$, all inelastic states
- $Q^2 \lesssim 1$ GeV$^2$: elastic+$\pi$N within dispersion relations
- intermediate range: interpolation

$Q^2 = 0.1$ GeV$^2$
Applications to nucleon form factors

BBBA: fit used in neutrino physics

PRad data or $\mu$H charge radius?

$z$ expansion fit

$$z(Q^2) = \frac{\sqrt{t_{cut} + Q^2} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} + Q^2} + \sqrt{t_{cut} - t_0}}$$

with 4-5 independent coefficients

$$G(Q^2) = \sum_{k=0}^{k_{max}} a_k z(Q^2)^k$$


- first model-independent fits presenting covariance matrix
- $2\chi$ provides nontrivial hadronic radiative correction
- proton charge radius as a constraint
Applications to nucleon form factors

$z$ expansion fit

$$z(Q^2) = \frac{\sqrt{t_{cut} + Q^2} - \sqrt{t_{cut} - t_0}}{\sqrt{t_{cut} + Q^2} + \sqrt{t_{cut} - t_0}}$$

with 4-5 independent coefficients

$$G(Q^2) = \sum_{k=0}^{k_{max}} a_k z(Q^2)^k$$

BBBA: fit used in neutrino physics

PRad data or $\mu$H charge radius?


- inclusion of PRad data is consistent with $\mu$H constant
- significant difference in magnetic form factor propagates to observables
Elastic muon-proton scattering
and two-photon exchange
Elastic muon-proton scattering

- charge radius extractions:

<table>
<thead>
<tr>
<th>eH, eD spectroscopy</th>
<th>ep scattering</th>
</tr>
</thead>
<tbody>
<tr>
<td>μH, μD spectroscopy</td>
<td>μp scattering</td>
</tr>
</tbody>
</table>

- μp elastic scattering is planned by MUSE@PSI(ongoing)
  measure with both electron/muon charges

- three nominal beam energies: 115, 153, 210 MeV, $Q^2 < 0.1$ GeV$^2$

- 2γ correction in MUSE?
Elastic lepton-proton scattering and $2\chi$

- electron-proton scattering: 3 structure amplitudes

$$\begin{align*}
K &= \frac{k + k'}{2} \\
E_{\text{non-flip}} &= \frac{e^2}{Q^2} \bar{l} \gamma_\mu l \cdot \vec{N} \left( G_M(\nu, Q^2) \gamma^\mu - F_2(\nu, Q^2) \frac{P^\mu}{M} + F_3(\nu, Q^2) \frac{\hat{K} P^\mu}{M^2} \right) N
\end{align*}$$


- $2\chi$ correction to cross section is given by amplitudes real parts
Elastic lepton-proton scattering and $2\chi$

$$K = \frac{k + k'}{2}$$

$$P = \frac{p + p'}{2}$$

- electron-proton scattering: 3 structure amplitudes

$$T^{\text{non-flip}} = \frac{e^2}{Q^2} \bar{l} \gamma_{\mu} l \cdot \bar{N} \left( G_M(\nu, Q^2) \gamma^\mu - F_2(\nu, Q^2) \frac{P^\mu}{M} + F_3(\nu, Q^2) \frac{\hat{K} P^\mu}{M^2} \right) N$$


- muon-proton scattering: add helicity-flip amplitudes

$$T^{\text{flip}} = \frac{e^2}{Q^2} m \bar{l} \gamma_{\mu} l \cdot \bar{N} \left( F_4(\nu, Q^2) + F_5(\nu, Q^2) \frac{\hat{K}}{M} \right) N + \frac{e^2}{Q^2} m \bar{F}_6(\nu, Q^2) \bar{l} \gamma_5 l \cdot \bar{N} \gamma_5 N$$


- $2\chi$ correction to cross section is given by amplitudes real parts
Dispersion relation approach?

- proton state contribution to $\gamma\gamma$:

- problematic amplitude: $F_4$

unsubtracted approach violates low-$Q^2$ behavior

$\delta_{2\gamma} \rightarrow 0$

O. T. and M. Vanderhaeghen (2018)

- dispersion relation approach requires a subtraction
MUSE@PSI (2018-19) estimates ($\mu$-$p$)

- proton box diagram model + inelastic $2\chi$

- expected muon over electron ratio

  small inelastic $2\chi$

  small $2\chi$ uncertainty

- MUSE can test $r_E$ in one charge channel

K. Mesick talk (PAVI 2014), MUSE TDR (2016)
- elastic $\mu p$ scattering at SPS with 100 GeV beam

- measure $G_E^2 + \tau G_M^2$ at forward angles

- data taking in 2022
COMPASS proton radius experiment

- elastic $\mu p$ scattering at SPS with 100 GeV beam

- measure $G_E^2 + \tau G_M^2$ at forward angles

- data taking in 2022

2$\gamma$ corrections?

- Feshbach correction (+ recoil)

$$\delta_{2\gamma} = \frac{\alpha \pi Q}{2\omega} \left( 1 + \frac{m}{M} \right) \quad \rightarrow \quad 2\text{-}3 \text{ orders below MUSE}$$

- inelastic states: kinematically enhanced

- sub per mille level of 2$\gamma$ in COMPASS kinematics
COMPASS proton radius experiment

- elastic $\mu p$ scattering at SPS with 100 GeV beam

- measure $G_E^2 + \tau G_M^2$ at forward angles

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**2$\chi$ corrections?**

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\[ \delta_{2\chi} = \frac{\alpha \pi Q}{2\omega} \left( 1 + \frac{m}{M} \right) \rightarrow 2-3 \text{ orders below MUSE} \]

- inelastic states: kinematically enhanced

- sub per mille level of $2\chi$ in COMPASS kinematics
Lepton-pair production

- universality test by lepton-pair photoproduction:

\[ \gamma p \rightarrow l^+ l^- p @ \text{MAMI} \]

- \( \omega = 0.5 - 1.5 \text{ GeV} \)
- \( Q^2 = 0.0018 - 0.042 \text{ GeV}^2 \)

\[ \frac{\sigma(e^+ e^-) + \sigma(\mu^+ \mu^-)}{\sigma(e^+ e^-)} \]

below and above muon threshold

normalisation and proton structure errors are suppressed


radiative corrections are presented in analytic form

M. Heller, O. T., M. Vanderhaeghen and Sh. Wu (2018-2019)

- one-loop QED is calculated
- \( 2\chi \) vanishes averaging over lepton angles
Hyperfine splitting in ordinary and muonic hydrogen
2\(\gamma\) correction to \(\mu\)H HFS

\[
\delta E_{\text{HFS}}^{2\gamma} = \Delta_{\text{HFS}} E_F
\]

\[
E_F = \frac{8\alpha^4}{3} \frac{M^2 m^2}{(M + m)^3} \frac{\mu_P}{n^3}
\]

before 2017

- reduction of uncertainty is needed
Hyperfine splitting correction

- traditional decomposition of $2\gamma$:

$$\Delta_{\text{HFS}} = \Delta_Z + \Delta^R + \Delta^{\text{pol}}$$

- leading correction:

$$\Delta_Z = \frac{8\alpha m_r}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left( \frac{G_M(Q^2) G_E(Q^2)}{\mu_P} - 1 \right)$$

- uncertainty budget:

  $> 100$ ppm  $< 10$ ppm  $100$ ppm
Zemach correction in $\mu$H

- Zemach correction expanding form factors:

$$\Delta Z = \frac{8\alpha m_r}{\pi} \int_{Q_0}^{\infty} \frac{dQ}{Q^2} \left( \frac{G_M(Q^2) G_E(Q^2)}{\mu P} - 1 \right) + \frac{4\alpha m_r Q_0}{3\pi} \left(-r_E^2 - r_M^2 + \frac{r_E^2 r_M^2}{18} Q_0^2\right)$$

- dependence on splitting: consistency check

- 95 ppm change for $\mu$H and ep radii with $Q_0 = 0.2$ GeV

- 1.5-2 times more precise

- magnetic radius is equally important
$2\gamma$ correction in eH 1S HFS

- measurements of 1S HFS in eH (21 cm line):

$$\nu_{\text{HFS}}(\text{H}) = 1420.4057517667(9) \text{ MHz}$$

Pioneer plaque, from Wikipedia
2$\chi$ correction in eH 1S HFS

- measurements of 1S HFS in eH (21 cm line):

$$\nu_{\text{HFS}}(\text{H}) = 1420.4057517667(9) \text{ MHz}$$

- accuracy $10^{-12}$ - precise extraction of $2\chi$

- dispersive evaluation and phenomenological extractions agree

higher order corrections

M. I. Eides et al. (2008)
A. P. Martynenko et al.

error

$\alpha \Delta_{\text{HFS}}$

- using $R_E$ from ep
- using $R_E$ from $\mu$H

Carlson et al.

$\Delta^{\text{pol}}$, Faustov et al.
$\Delta^{Z+R}$, Bodwin et al.
1S HFS measurement
2$\gamma$ correction in eH 1S HFS

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- accuracy $10^{-12}$ - precise extraction of 2$\gamma$

- dispersive evaluation and phenomenological extractions agree
Hyperfine splitting correction

- traditional decomposition of $2\gamma$:

$$\Delta_{\text{HFS}} = \Delta_Z + \Delta^R + \Delta^{\text{pol}}$$

- leading correction:

$$\Delta_Z = \frac{8\alpha m_r}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left( \frac{G_M(Q^2) G_E(Q^2)}{\mu_P} - 1 \right)$$

- polarizability:

$$\Delta^{\text{pol}} \sim m_r$$

- uncertainty budget:

$> 100 \text{ ppm}$  $< 10 \text{ ppm}$  $100 \text{ ppm}$
Hyperfine splitting correction

- traditional decomposition of $2\gamma$:

\[
\Delta_{\text{HFS}} = \Delta_Z + \Delta^R + \Delta^{\text{pol}}
\]

- leading correction:

\[
\Delta_Z = \frac{8\alpha m_r}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left( \frac{G_M (Q^2) G_E (Q^2)}{\mu_P} - 1 \right) \quad \Delta^{\text{pol}} \sim m_r
\]

- uncertainty budget:

  > 100 ppm  
  < 10 ppm  
  100 ppm

- $\mu H$ splitting from eH:

\[
\Delta_{\text{HFS}} (\mu H) = \frac{m_r (m_\mu)}{m_r (m_e)} \Delta_{\text{HFS}} (eH) + \Delta^{\text{th}}_{\text{HFS}} (m_\mu) - \frac{m_r (m_\mu)}{m_r (m_e)} \Delta^{\text{th}}_{\text{HFS}} (m_e)
\]

- Zemach correction vanishes and polarizability term is almost 0
2\(\chi\) correction in \(\mu H\) from eH HFS

- error of 2\(\gamma\) is significantly reduced
2\(\chi\) correction in \(\mu\)H from eH HFS

100-200 ppm → 49 ppm

- precise S-level HFS prediction

\[ E_{1S}^{HFS} = 182.625 \pm 0.012 \text{ meV} \]

1S-2S splitting in hydrogen
1S-2S transition in hydrogen and 2\(\chi\)

- measurements of 1S-2S transition in eH with \(4\times10^{-15}\) accuracy:

\[
\nu_{1S-2S}(H) = 2466061413187018(11) \text{ Hz}
\]

- more precise than recent Lamb shift measurement (error: 3.2 kHz)

\[
\nu_{nS} = -\frac{R_\infty}{n^2} + \frac{L_{1S}(r_E)}{n^3}
\]

- main input to determine Rydberg constant

---

A. Matveev et al. (2013)

N. Bezginov et al. (2019)
1S-2S transition in hydrogen and $2\chi$

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$$\nu_{1S-2S}(H) = 2466061413187018(11) \text{ Hz}$$

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$$\nu_{nS}^{2\gamma} = \nu_{\text{Born}} + \nu_{\text{subt}} + \nu_{\text{inel}}$$

Born

$G_E, G_M$

rE from $\mu H$: -39.9(6.8) Hz

average rE: -44.1(9.6) Hz

subtraction

$T_1$

O. T. and M. Vanderhaeghen (2015)

18.5(4.4) Hz

inelastic

$F_1, F_2$

-83.6(5.7) Hz

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inelastic

$F_1, F_2$

-83.6(5.7) Hz

- $2\gamma$ shifts 1S level by -109.2(12.0) Hz

- $2\gamma$ uncertainty is at the level of experimental precision

O. T. (2019)

A. Matveev et al. (2013)

N. Bezginov et al. (2019)

O. T. and M. Vanderhaeghen (2015)
Lamb shift from $2\chi$ on nucleons
2\hbar to Lamb shift on neutrons

- forward 2\gamma on nucleons contributes to Lamb shift

\[ \nu_{\text{nS}}^{2\gamma} = \nu_{\text{Born}} + \nu_{\text{subt}} + \nu_{\text{inel}} \]

Born:
\[ G_E, G_M \]

subtraction:
\[ T_1 \]

inelastic:
\[ F_1, F_2 \]

O. T. and M. Vanderhaeghen (2015)

- neutron magnetic form factor determines Born contribution

<table>
<thead>
<tr>
<th>( \nu_{1S}^{2\gamma} )</th>
<th>ep, Hz</th>
<th>en, Hz</th>
</tr>
</thead>
<tbody>
<tr>
<td>Born</td>
<td>-39.9(6.8)</td>
<td>9.4(0.6)</td>
</tr>
<tr>
<td>subtraction</td>
<td>18.5(4.4)</td>
<td>28.8(10.2)</td>
</tr>
<tr>
<td>inelastic</td>
<td>-83.6(5.7)</td>
<td>-81.7(8.4)</td>
</tr>
<tr>
<td>total</td>
<td>-109.2(12.0)</td>
<td>-43.5(13.2)</td>
</tr>
</tbody>
</table>

- 2\gamma on nucleons in electronic and muonic nuclei

O. T. (2019)

- neutral neutron vs charged proton differs in Born contribution
HFS from $2\chi$ on nucleons
2γ to HFS on neutrons

- traditional decomposition of 2γ:

\[ r_{2\gamma}^N = r_Z + r^R + r^{pol} \]

Zemach term
GE, GM

recoil correction
GE, GM

polarizability
FP, g1, g2

- leading correction:

\[ r_Z = -\frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \frac{G_M(Q^2) G_E(Q^2)}{\mu_n} \]

- Zemach radius expanding form factors:

\[ r_Z = -\frac{4}{\pi} \int_{Q_0}^\infty \frac{dQ}{Q^2} \frac{G_M(Q^2) G_E(Q^2)}{\mu_n} + \frac{2Q_0r_E^2}{3\pi} \left( 1 - \frac{r_M^2}{18} Q_0^2 \right) \]

extraction of radii by S. G. Karshenboim (2014)
calculation of rZ for proton by O. T. (2017)

- dependence on splitting value: consistency check

O. T. (2017)
- Zemach radius expanding form factors:

\[ r_Z = -\frac{4}{\pi} \int_{Q_0}^{\infty} dQ \frac{G_M(Q^2)}{Q^2} \frac{G_E(Q^2)}{\mu_n} + \frac{2Q_0r_E^2}{3\pi} \left( 1 - \frac{r_M^2}{18}Q_0^2 \right) \]

- dependence on splitting value: consistency check


- two form factor parametrizations are valid for \( r_Z \) calculation

extraction of radii by S. G. Karshenboim (2014)
calculation of \( r_Z \) for proton by O. T. (2017)
- polarizability for neutron is estimated from MAID solution assuming a similar to proton saturation as for GDH sum rule

<table>
<thead>
<tr>
<th>$\nu_{HFS}^{2\gamma}$</th>
<th>en, fm</th>
<th>$\mu n$, fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zemach</td>
<td>-0.0449(13)</td>
<td>-0.0449(13)</td>
</tr>
<tr>
<td>recoil</td>
<td>0.328(2)</td>
<td>0.0823(8)</td>
</tr>
<tr>
<td>polarizability</td>
<td>0.064(38)</td>
<td>0.065(39)</td>
</tr>
<tr>
<td>total</td>
<td>0.347(38)</td>
<td>0.102(39)</td>
</tr>
</tbody>
</table>

- proton effective radii are the same for $e$ and $\mu$:

<table>
<thead>
<tr>
<th>$\nu_{HFS}^{2\gamma}$</th>
<th>ep, fm</th>
<th>$\mu p$, fm</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zemach</td>
<td>1.055(13)</td>
<td>1.055(13)</td>
</tr>
<tr>
<td>recoil</td>
<td>-0.1411(14)</td>
<td>-0.1203(8)</td>
</tr>
<tr>
<td>polarizability</td>
<td>-0.051(13)</td>
<td>-0.052(13)</td>
</tr>
<tr>
<td>total</td>
<td>0.863(20)</td>
<td>0.883(19)</td>
</tr>
</tbody>
</table>

- lepton mass scale contributes to recoil correction

- effective radii $e$ vs $\mu$ are distinct for neutron

O. T. (2019)
Conclusions

- total $2\gamma$ estimate in ep and $\mu p$: proton+inelastic states

- dispersive framework in ep: elastic and $\pi N$
  input for nucleon form factor fits

- dispersive framework in $\mu p$: elastic; expect MUSE data

- precise $2\gamma$ in $\mu H$ from $1S$ HFS in eH

- $2\gamma$ correction to $1S$-$2S$ transition in eH, $2\gamma$ on neutrons
Conclusions

- total $2\gamma$ estimate in $ep$ and $\mu p$: proton+inelastic states
- dispersive framework in $ep$: elastic and $\pi N$
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- $2\gamma$ correction to $1S$-$2S$ transition in $eH$, $2\gamma$ on neutrons
Thanks for your attention !!!
Future projects

- ep elastic scattering at low $Q^2$:

  **ProRad@PRAE**
  \[
  \omega = 30 - 70 \text{ MeV}
  \]
  \[
  Q^2 = 10^{-5} - 10^{-4} \text{ GeV}^2
  \]

  **ep scattering@MAMI**
  \[
  \omega = 500 \text{ and } 720 \text{ MeV}
  \]
  \[
  Q^2 = 0.002 - 0.02 \text{ GeV}^2
  \]

  **Ultra-Low $Q^2$@Tohoku**
  \[
  \omega = 20 - 60 \text{ MeV}
  \]
  \[
  Q^2 = 0.0003 - 0.008 \text{ GeV}^2
  \]

  **ISR@MAMI**
  \[
  \omega = 0.2 - 0.5 \text{ GeV}
  \]
  \[
  Q^2 = 0.001 - 0.17 \text{ GeV}^2 \rightarrow 0.0002 \text{ GeV}^2
  \]

  **MAGIX@MESA**
  magnetic form factor

arXiv: 1905.11182
Future projects

- μp elastic scattering at low $Q^2$:

**MUSE@PSI**

\[
\omega = 115 - 210 \text{ MeV}
\]

\[
Q^2 = 0.0016 - 0.08 \text{ GeV}^2
\]

e and $\mu$

**COMPASS@CERN**

\[
\omega = 100 \text{ GeV}
\]

\[
Q^2 = 0.001 - 0.2 \text{ GeV}^2
\]

small structure effects

- universality test by lepton-pair photoproduction:

**$\gamma p \rightarrow l^+ l^- p@MAMI$**

\[
\omega = 0.5 - 1.5 \text{ GeV}
\]

\[
Q^2 = 0.0018 - 0.042 \text{ GeV}^2
\]

\[
\frac{\sigma(e^+ e^-) + \sigma(\mu^+ \mu^-)}{\sigma(e^+ e^-)}
\]

below and above muon threshold

normalisation and proton structure errors are suppressed
Lamb shift $2\chi$ correction. Forward VVCS

$2\chi$ blob - forward virtual Compton scattering

Optical theorem

$\text{Im } T_1 \sim F_1$  \hspace{1cm} $\text{Im } T_2 \sim F_2$  \hspace{1cm} $\text{Im } S_1 \sim g_1$  \hspace{1cm} $\text{Im } S_2 \sim g_2$

Fixed-$Q^2$ dispersion relations

Disp. rel. for amplitude $T_1$ requires subtraction function

Unsubtracted disp. rel. works for $T_2, S_1, S_2$
Empirical estimate of subtraction function

Subtract Regge behavior + DR + resonance region and DIS data


expected low-$Q^2$ behavior

$T_{1}^{\text{subtr}}(0, Q^2) = \beta(Q^2)Q^2$

$\beta(0) = \beta_M$ - magnetic polarizability
Empirical estimate of subtraction function

Subtract Regge behavior + DR + resonance region and DIS data


expected low-$Q^2$ behavior

$$T_{1}^{\text{subtr}}(0, Q^2) = \beta(Q^2)Q^2$$

$$\beta(0) = \beta_M$$ - magnetic polarizability

data-based result vs. theoretical predictions

Lamb shift correction

$$\Delta E_{2S}^{\text{subt}}(\mu H) \approx 2.3 \pm 1.3 \ \mu eV$$

slightly smaller than Birse et al.

$$\Delta E_{2S}^{\text{subt}}(\mu H) \approx 4.2 \pm 1.0 \ \mu eV$$