Angular momentum and density

Energy-momentum tensor and AM operators

AM light-front density

Transition angular momentum $B \rightarrow B'$

Definition through light-front density

$N \rightarrow \Delta$ transition AM - isovector

Dynamical properties

Separation of spin and orbital AM

Peripheral AM density $N \rightarrow N$ from chiral dynamics

Based on

J.-Y. Kim, H.-Y. Won, J. Goity, C. Weiss,
Phys. Lett. B 844, 138083 (2023) [INSPIRE]

C. Granados, C. Weiss,
**AM: Operators**

*Invariance of action → conserved local currents → charges*

Space-time translations → EM tensor \( T^{\mu\nu}(x) \) → total momentum \( P^i = \int d^3x \, T^{0i}(x) \)

Rotations → AM tensor \( J^{\mu\alpha\beta}(x) \) → total AM \( J^i = \frac{1}{2} \epsilon^{ijk} \int d^3x \, T^{jk}(x) \)

**AM tensor**

\[
J^{\mu\alpha\beta} = J_q^{\mu\alpha\beta} + J_g^{\mu\alpha\beta}
\]

\[
J_q^{\alpha\beta}(x) = x^\alpha T_q^{\mu\beta}(x) - x^\beta T_q^{\mu\alpha}(x)
\]

Quark/gluon AM from EMT

\[
T_q^{\alpha\beta}(x) = \sum_f \bar{\psi}_f(x) \gamma^{[\alpha i} \nabla^{\beta]} \psi_f(x)
\]

Quark EMT (← GPDs)

Here: Use symmetric part of quark EMT → total quark AM \( J_q \)

Later: Non-symmetric EMT → separate orbital and spin quark AM \( L_q, S_q \)

Definitions see e.g. Lorce, Mantovani, Pasquini 2017

Only total EMT is conserved, not individual quark/gluon/flavor contributions
AM: EMT matrix elements

\[ \langle N' | T^{\alpha\beta} | N \rangle \]

Invariant form factors
\[ A, B, D, \bar{C}(t) \]

Light-front components
\[ T^{++}, T^{+i}, T^{ij}(\Delta_T) \]
\((\Delta^+ = 0 \text{ frame})\)

3D components
\[ T^{00}, T^{0k}, T^{kl}(\Delta) \]
\((\Delta^0 = 0 \text{ frame})\)

AM can be evaluated in any representation

e.g. GPDs → \[ A + B \rightarrow J \]

Ji 1996

GPDs → \[ [A + B](t) \rightarrow T^{0k}(r) \text{ density} \]

Polyakov 2003

Each representation has its uses/advantages

Here: Use representation by LF components in \( \Delta^+ = 0 \text{ frame} \)
\[ \rightarrow \text{Generalization} \langle B' | \ldots | B \rangle \]
AM: Transverse density

\[ p'^+ = p^+, \quad \Delta^+ = 0, \quad \Delta_T \neq 0 \]

\[ T^{+i}(\Delta_T | \sigma', \sigma) = \langle p', \sigma' | T^{+i}(0) | p, \sigma \rangle \]

\[ T^{+i}(b | \sigma', \sigma) = \int \frac{d^2 \Delta_T}{(2\pi)^2} e^{-i \Delta_T b} \langle p', \sigma' | T^{+i}(0) | p, \sigma \rangle \]

\[ 2S^z(\sigma', \sigma) J_q = \frac{1}{2p^+} \int d^2 b \left[ b \times T^{+T}(b | \sigma', \sigma) \right]^z \]

\[ J_q + J_g = \frac{1}{2} \]

Nucleon spin states: Light-front helicity states, prepared by LF boost from rest frame

\[ \Delta^+ = 0 \] frame

EMT transition matrix element

Transverse coordinate representation

AM transverse density and integral

Spin sum rule (quarks + gluons)
Advantages of LF formulation

LF density: Boost-invariant/covariant, frame independent, appropriate for relativistic systems

Mechanical interpretation: \( \mathbf{r} \times \mathbf{p} \) in transverse plane

Relation to invariant form factors:

\[
\left[ \mathbf{b} \times \mathbf{T}^+T(b \mid \sigma', \sigma) \right]^z = 2S^z(\sigma', \sigma) \frac{b}{2} \frac{d}{db} \left[ \rho_A(b) + \rho_B(b) \right]
\]

Can be generalized to transitions \( B \rightarrow B' \)

LF quantization: Transverse motion Galilean - nonrelativistic. Transverse localization does not depend on mass of state (cf. 3D Breit frame densities)
Transition AM: Definition

\[ m' \neq m \quad B = \{ S, S_3, I, I_3 \}, \quad B' = \{ S', S'_3, I', I'_3 \} \]

\[ p'^+ = p^+, \quad \Delta^+ = 0, \quad \Delta_T \neq 0, \quad \Delta^- = \frac{m'^2 - m^2}{p^+} \neq 0 \]

\[ T^{+i}(\Delta_T | B', B) = \langle p', B' | T^{+i}(0) | p, B \rangle \]

\[ 2S^z(B', B) J_{B \rightarrow B'} = \frac{1}{2p^+} \int d^2b \left[ b \times T^{+T}(b | B', B) \right]^z \]

Baryon states, spin/isospin quantum numbers

\[ \Delta^+ = 0 \] frame

Transition matrix element
isoscalar/isovector operator

Transition AM \( B \rightarrow B' \)

Kinematic spin dependence factored out:
Reduced matrix element

Transitions as allowed by quantum numbers of states:
Isoscalar/isovector component of quark operator

Kim, Won, Goity, Weiss 2023
**Transition AM:** \( N \rightarrow \Delta \)

\((T^V)^{\alpha \beta} \equiv T^{\alpha \beta}_u - T^{\alpha \beta}_d\)

"Isovector quark EMT" : New operator, not conserved, not related to symmetry

\[\langle B', p | (T^V)^{\alpha \beta} | B, p \rangle \rightarrow J_{B \rightarrow B'}\]

**Analysis using 1/Nc expansion**

1/\(N_c\) expansion of 3D multipole form factors of EMT in \(\Delta^0 = 0\) frame

Light-front components from “matching” in frame where \(\Delta^+ = 0\) and \(\Delta^0 = 0\)

LO relations: \(J^V_{p \rightarrow p} = \frac{1}{\sqrt{2}} J^V_{p \rightarrow \Delta^+}\)

Numerical estimates using Lattice QCD results for \(J^V_{p \rightarrow p} \equiv J^{u-d}\) in proton

<table>
<thead>
<tr>
<th>Lattice QCD</th>
<th>(J^S_{p \rightarrow p})</th>
<th>(J^S_{\Delta^+ \rightarrow \Delta^+})</th>
<th>(J^V_{p \rightarrow p})</th>
<th>(J^V_{p \rightarrow \Delta^+})</th>
<th>(J^V_{\Delta^+ \rightarrow \Delta^+})</th>
</tr>
</thead>
<tbody>
<tr>
<td>[9] (\mu^2 = 4) GeV(^2)</td>
<td>0.33*</td>
<td>0.33</td>
<td>0.41*</td>
<td>0.58</td>
<td>0.08</td>
</tr>
<tr>
<td>[10] (\mu^2 = 4) GeV(^2)</td>
<td>0.21*</td>
<td>0.21</td>
<td>0.22*</td>
<td>0.30</td>
<td>0.04</td>
</tr>
<tr>
<td>[11] (\mu^2 = 4) GeV(^2)</td>
<td>0.24*</td>
<td>0.24*</td>
<td>0.23*</td>
<td>0.33</td>
<td>0.05</td>
</tr>
<tr>
<td>[12] (\mu^2 = 1) GeV(^2)</td>
<td>–</td>
<td>–</td>
<td>0.23*</td>
<td>0.33</td>
<td>0.05</td>
</tr>
<tr>
<td>[13] (\mu^2 = 4) GeV(^2)</td>
<td>–</td>
<td>–</td>
<td>0.17*</td>
<td>0.24</td>
<td>0.03</td>
</tr>
</tbody>
</table>


Kim, Won, Goity, Weiss 2023
Dynamics: Orbital and spin AM

\[ T_{q}^{\alpha\beta} = T_{q}^{\{\alpha\beta\}} + T_{q}^{[\alpha\beta]} \]

\[ \bar{\psi}(x) \gamma^{[\alpha i} \nabla^{\beta]} \psi(x) = -2\epsilon^{\alpha\beta\mu\nu} \partial_{\mu} [\bar{\psi}(x)\gamma_{\nu}\gamma_{5}\psi(x)] \]

\[ J_{q}^{\mu\alpha\beta} = L_{q}^{\mu\alpha\beta} + S_{q}^{\mu\alpha\beta} \]

\[ L_{q}^{\mu\alpha\beta}(x) = x^{\alpha}T_{q}^{\mu\beta}(x) - x^{\beta}T_{q}^{\mu\alpha}(x) \]

\[ S^{\alpha\beta\mu} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} \bar{\psi}(x)\gamma_{\nu}\gamma_{5}\psi(x) \]

“Kinetic” EMT, non-symmetric

Separate orbital and spin AM operators

Antisymmetric part =

total derivative of axial current
(for each flavor)

Ji 1996. Discussion in Lorce, Mantovani, Pasquini 2017

Transition AM \( B \rightarrow B' \) can be extended to separate orbital and spin AM

AM densities depend on choice of EMT (kinetic, improved)
Charge \( J_{q} \) independent of definition

Explore orbital—spin separation in dynamical models
Peripheral AM density from chiral dynamics

Densities at $b = \mathcal{O}(M_{\pi}^{-1})$ governed by chiral dynamics

Computed using ChEFT: Systematic, model-independent

Pion EMT derived from Chiral Lagrangian + Noether Thm, uniquely determined

Peripheral densities from 2-pion cut of EMT matrix elements, evaluated using dispersion relation

**Peripheral AM density** $N \rightarrow N$

$L^z(b)$ leading — 2-pion cut

$S^z(b)$ suppressed — only 3-pion cut

AM density in nucleon’s chiral periphery is mainly orbital

Granados, Weiss, 2019
Peripheral AM density from chiral dynamics

$L^z(b)$ decays exponentially

$L^z(b) \sim e^{-2M_\pi b} \times \text{function}(M_\pi, m; b)$

$L^z(b)$ similar to charge density $\rho_1(b)$

Contributions of $N$ and $\Delta$ intermediate states have opposite sign; cancel in large-$N_c$ limit → correct $1/N_c$ scaling of peripheral density

Disclaimer: LO ChEFT result is not quantitatively realistic, relevant only at $b \sim$ few times $M_\pi^{-1}$. where densities are extremely small. Realistic results can be obtained with dispersive improvement

Scalar and vector form factors: Alarcon, Weiss, 2017+
Peripheral AM density from chiral dynamics

Light-front formulation of ChEFT process:
Sequence in LF time $x^+$

Transition $N \rightarrow \pi N, \pi \Delta$ described by chiral LF wave function:

$$\Psi_{N\rightarrow\pi N}(y, k_T | \sigma', \sigma) = \frac{\langle \pi N | \mathcal{L}_{\text{chiral}} | N \rangle}{M_{\pi N}^2 - m_N^2}$$

Peripheral density as LF wave function overlap (transverse coordinate representation, $r = b/\bar{y}$)

$$L^z(b) = \int \frac{dy}{y\bar{y}} \sum_{\sigma'} \Psi^*_{N\rightarrow\pi N}(y, r_T | \sigma', \sigma) \left[ r_T \times (-i) \frac{\partial}{\partial r_T} \right] \Psi_{N\rightarrow\pi N}(y, r_T | \sigma', \sigma) + (N \rightarrow \pi \Delta)$$

First-quantized representation

AM operator is quantum mechanical angular momentum

Granados, Weiss, 2019
Peripheral AM density from chiral dynamics

\[ \sigma = 1/2 \]

\[ \sigma = +1/2 \]

\[ L_z = 1,0 \]

\[ L_z = 2,1,0,-1 \]

\[ \Delta \]

\[ \Delta \]

\[ N \]

\[ N \]

\[ z \]

\[ z \]

\[ b \]

\[ \rightarrow \]

“Story” of peripheral AM

Original nucleon with spin \( \sigma = +1/2 \)

Transition to intermediate \( \pi N/\pi \Delta \) state with orbital AM \( L_z \) \( \leftarrow \) intermediate \( N/\Delta \) spin \( \sigma' \)

Peripheral AM given by \( L_z \), summed over all intermediate states

Light-front representation provides simple mechanical picture

Equivalent to result obtained from 2-pi cut in invariant EMT form factors

Based on ChEFT = “true” large-distance dynamics of QCD
EMT matrix elements can be characterized in several representations, each with distinct uses/advantages:

- **Invariant form factors** → Analytic properties
- **Light-front components in \( \Delta^+ = 0 \) frame** → Densities, mechanical interpretation, generalization to transitions \( B \to B' \)
- **3D components in \( \Delta = 0 \) frame** → Multipoles, \( 1/N_c \) expansion

AM definition as light-front density can be generalized to \( B \to B' \) transitions

\( N \to \Delta \) transition AM pure isovector, connected with \( J^{u-d} \) in nucleon in large-\( N_c \) limit

Peripheral AM density at \( b = \mathcal{O}(M_\pi^{-1}) \) can be computed in Chiral EFT
Peripheral \( N \to N \) isoscalar AM density mostly orbital, spin density suppressed