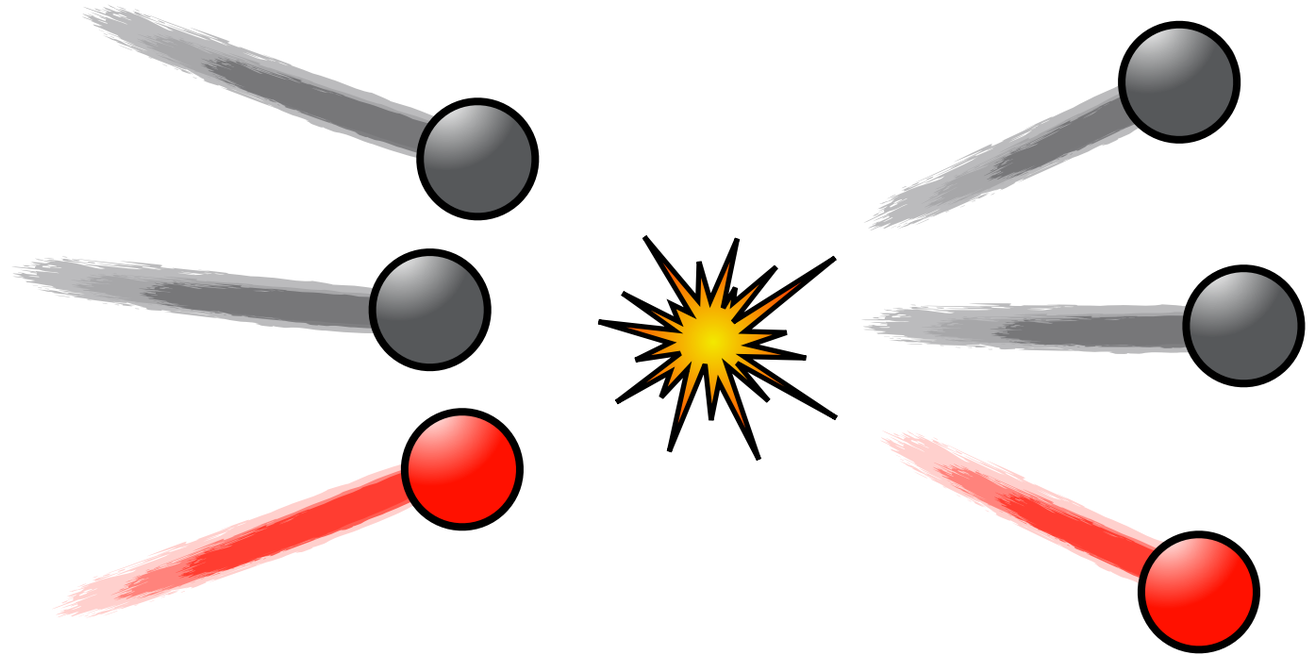
Three-Body Nuclear Phenomena from QCD

JSA Postdoctoral Award Talk

Andrew W. Jackura

2021 Jefferson Lab Users Organization Annual Meeting
Wednesday 23rd, June 2021



OLD DOMINION UNIVERSITY

Jefferson Lab
Thomas Jefferson National Accelerator Facility

JSA Postdoctoral Award

I am extremely honored to be awarded the JSA Postdoctoral Award

Many thanks to:

- The Jefferson Science Associates
- The Jefferson Lab User Organization
- My collaborators, supervisors, and mentors

Special thanks to:

Adam Szczepaniak,

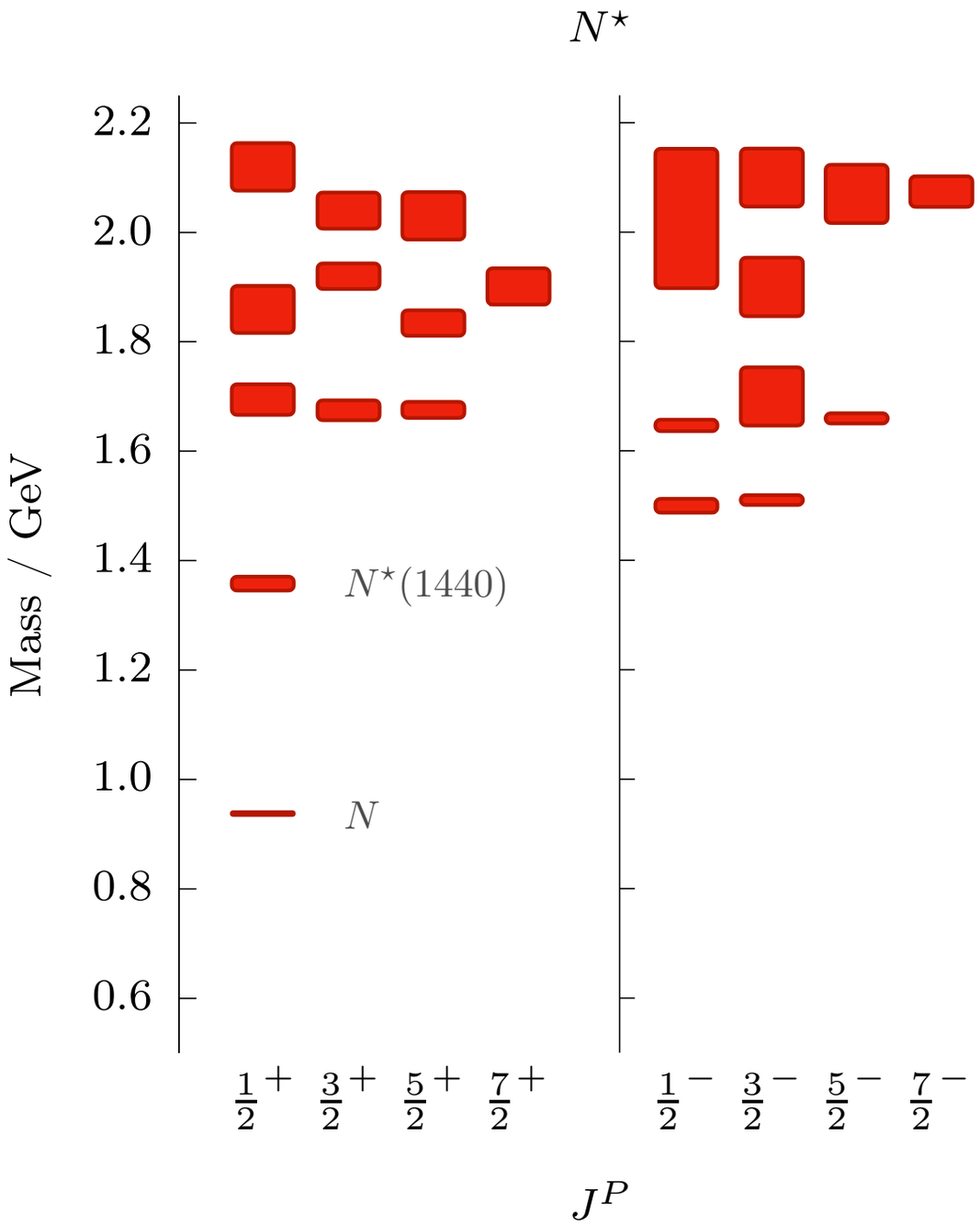
Raúl Briceño,

and Robert Edwards

Nuclear Physics, Lattice QCD, and Three-body Interactions

Goal: Compute hadronic/nuclear properties from first principles QCD

- *e.g.* ***Excited nucleon spectrum***



PDG listings

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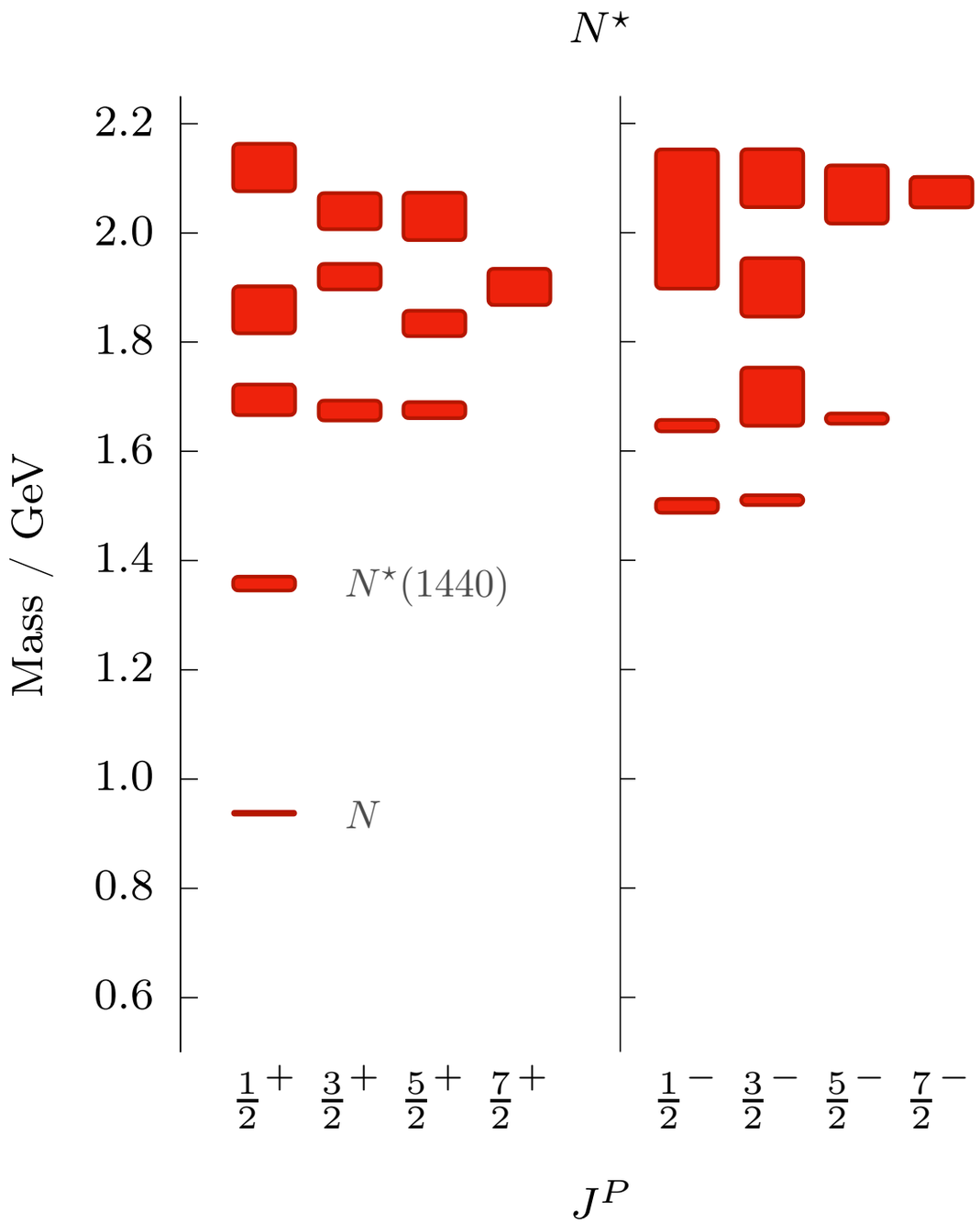
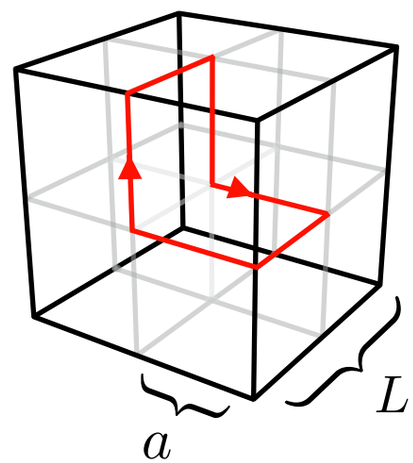


Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral
  - *Euclidean spacetime,* $t \to -i\tau$
  - *Finite volume,* L
  - *Discrete spacetime,* a
  - *Heavier than physical quark mass,* $m > m_{\text{phys.}}$



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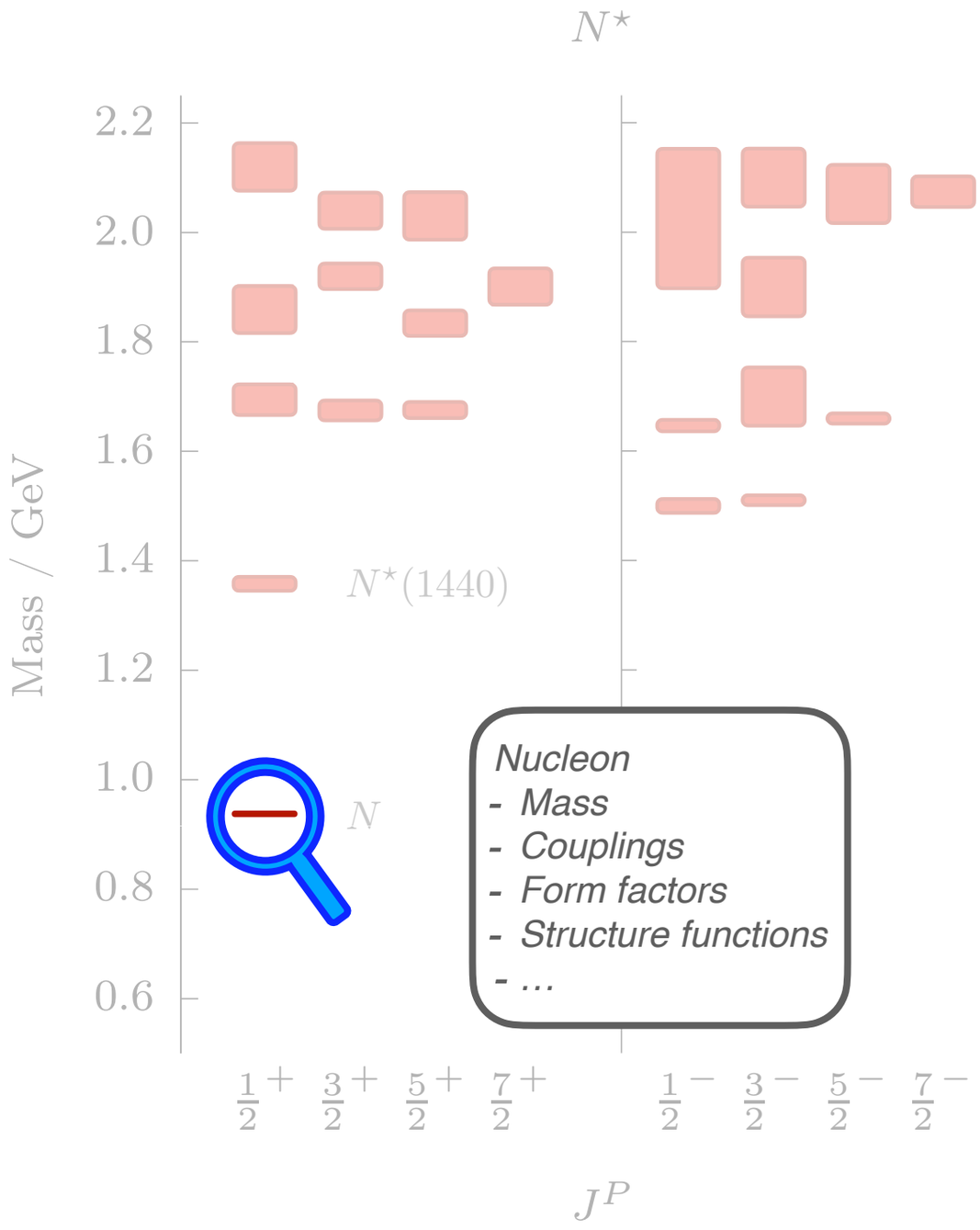

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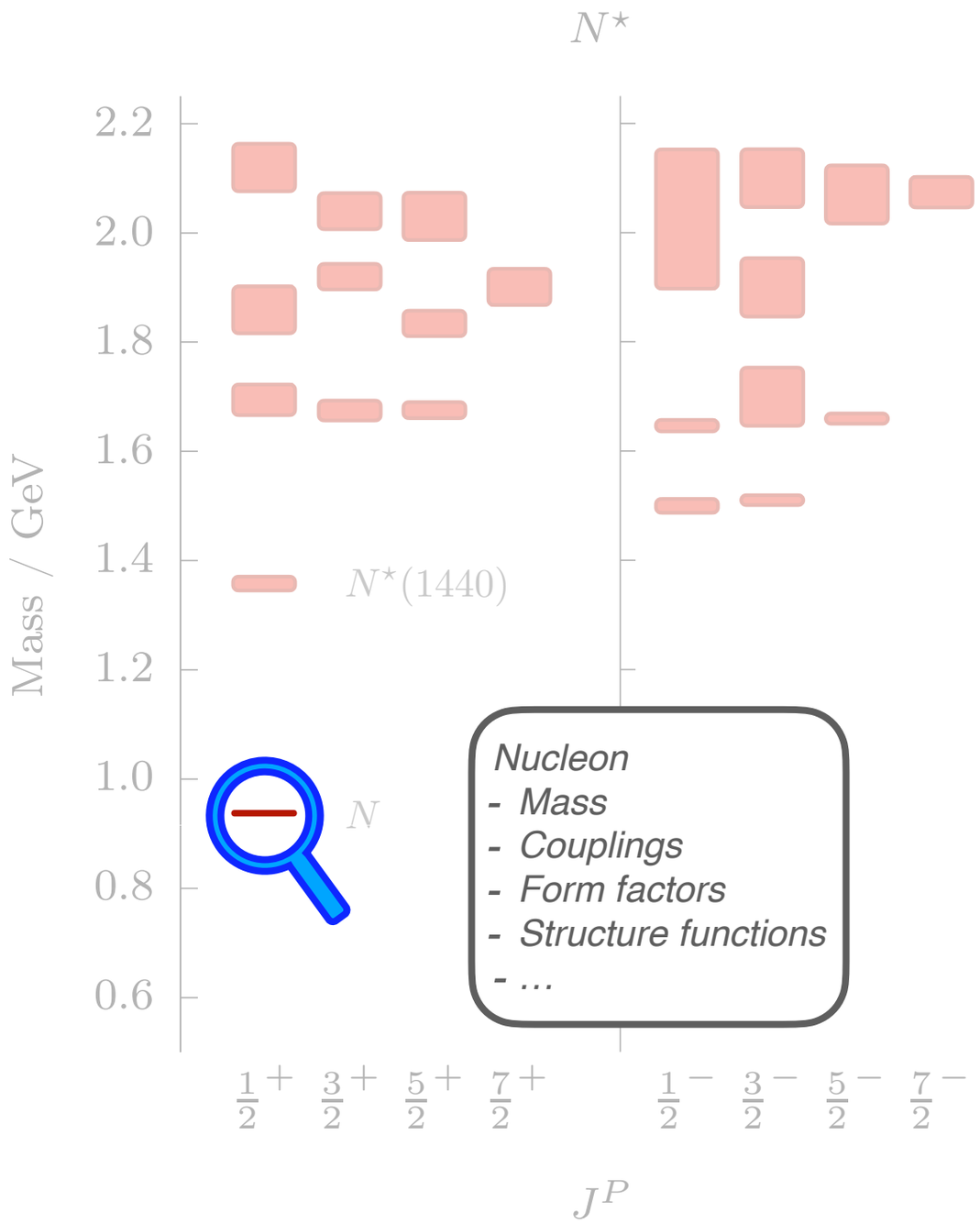

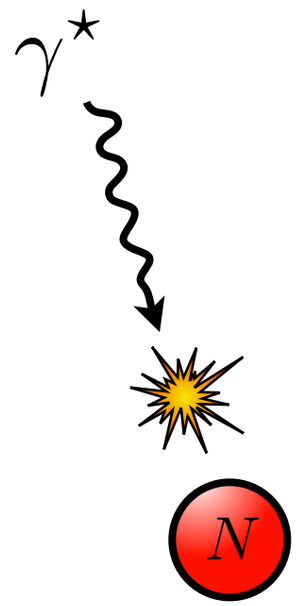

@JLab
- *Clas12*
- *GlueX*

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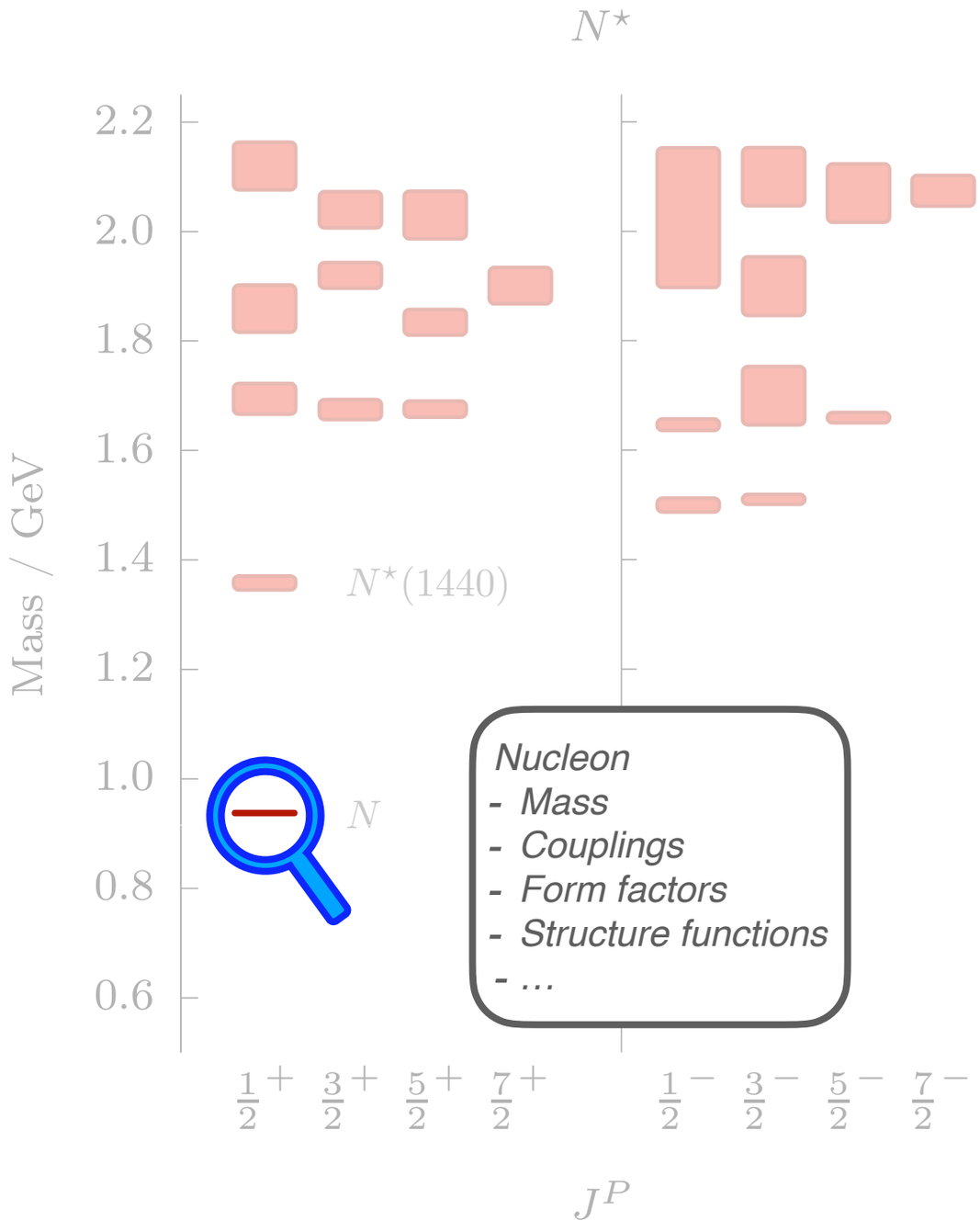

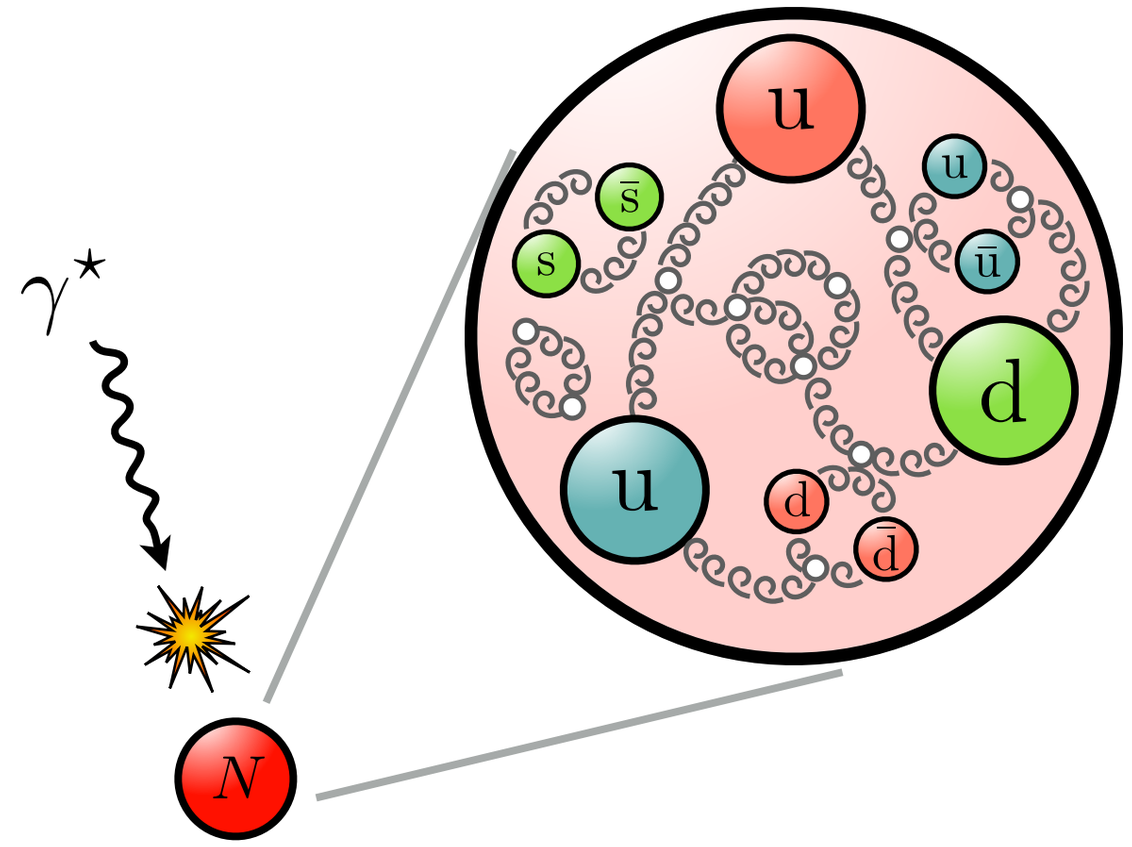

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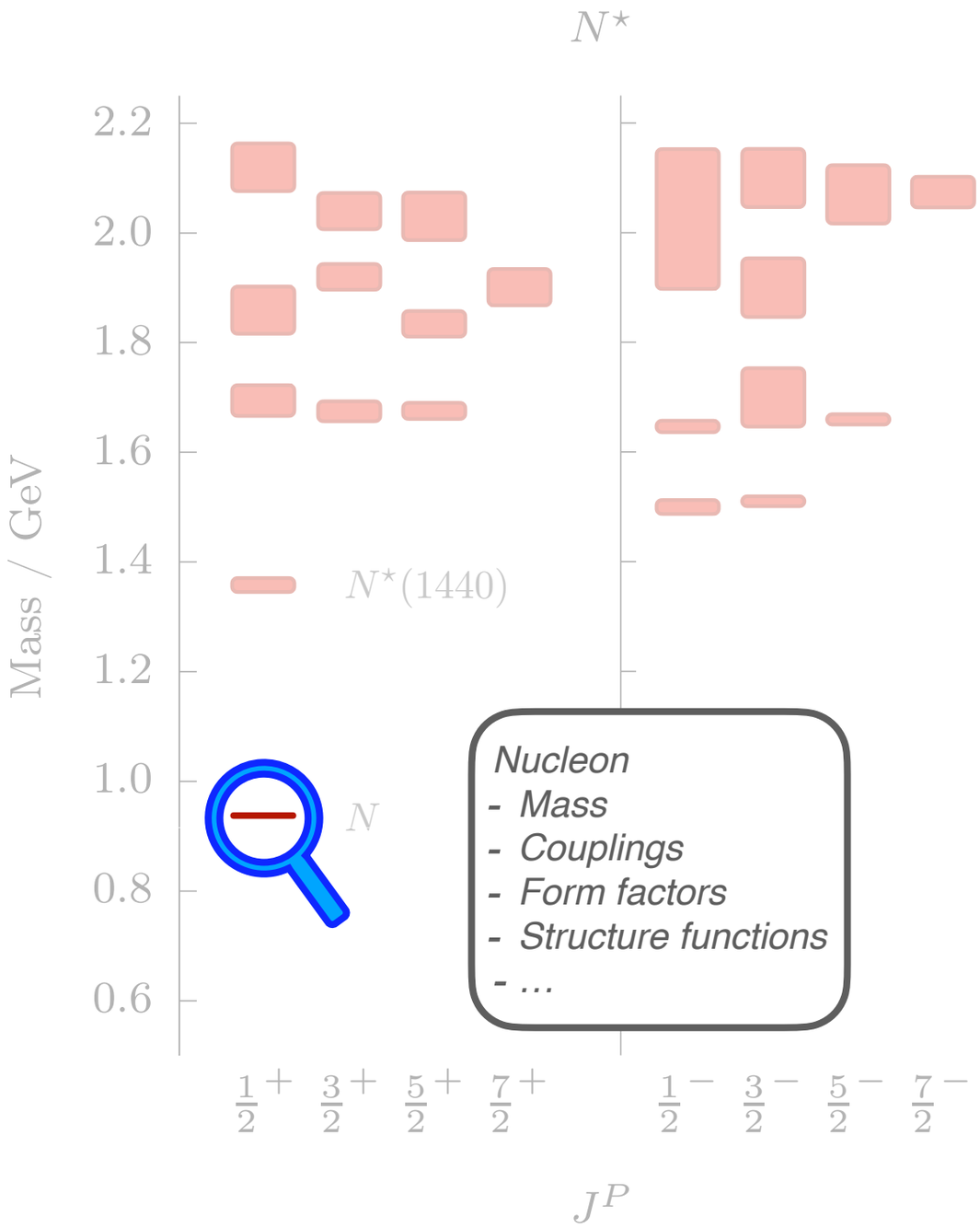
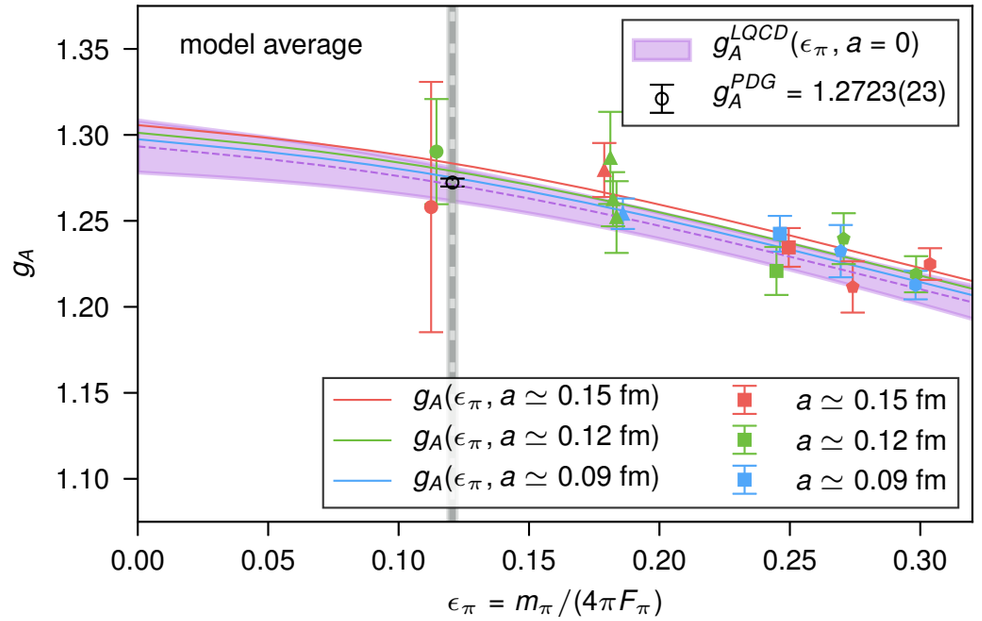
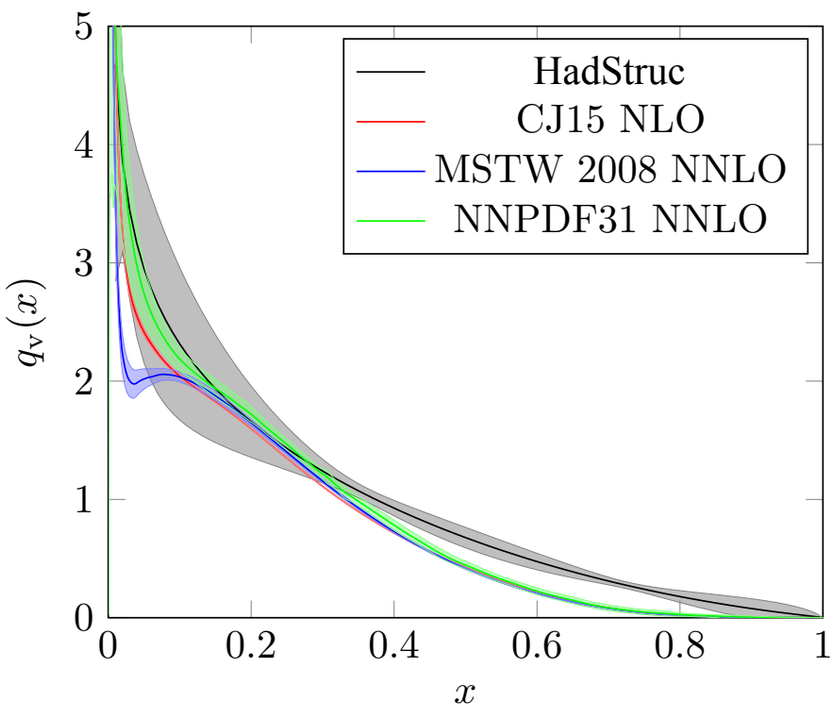




C. Chang et al.
Nature **558**, (2018) 7708



B. Joó et al. [HadStruc]
Phys. Rev. Lett. **125**, (2020) 232003

PDG listings

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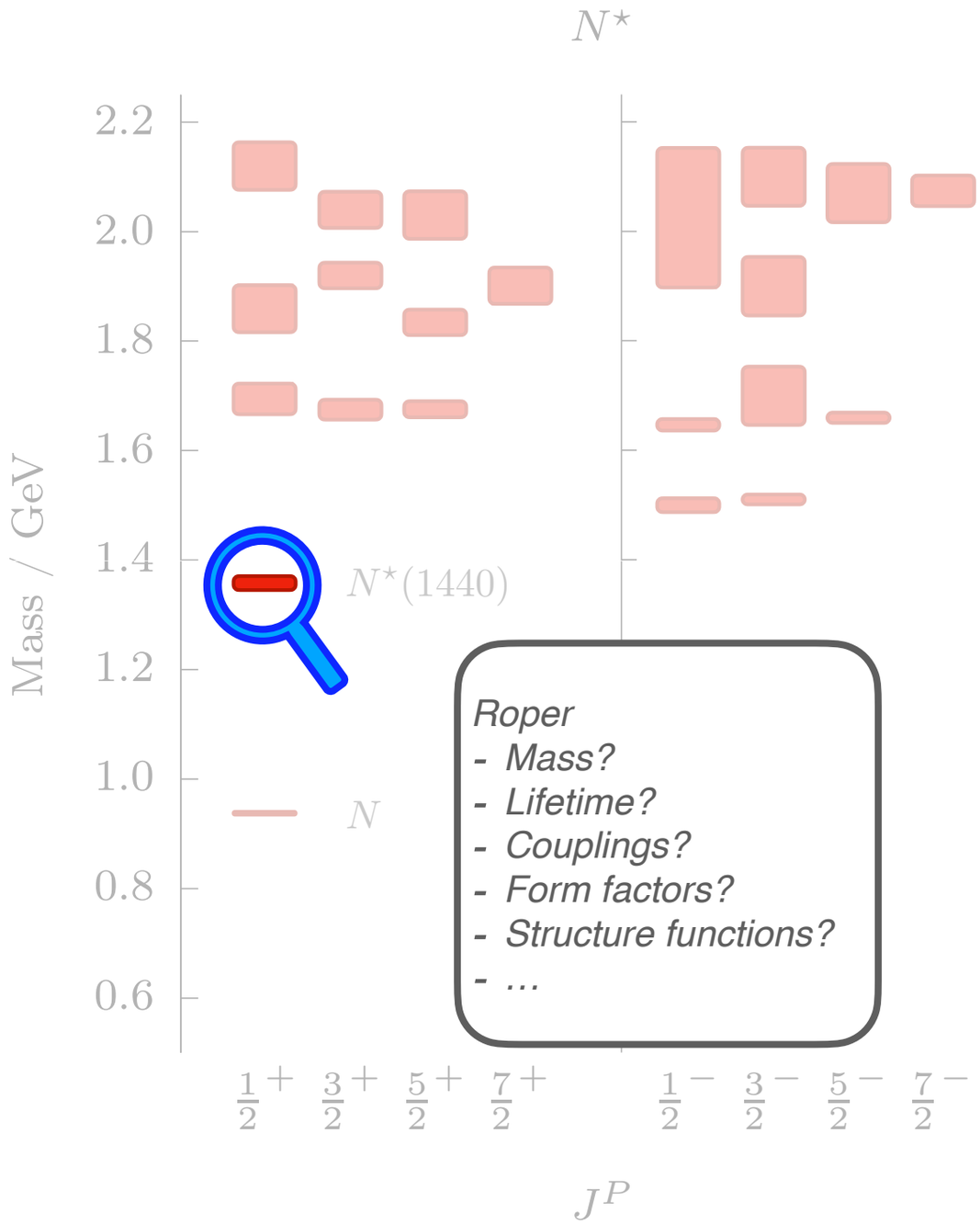

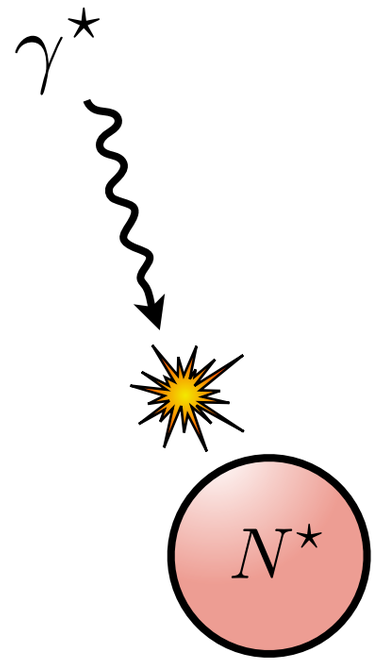

PDG listings

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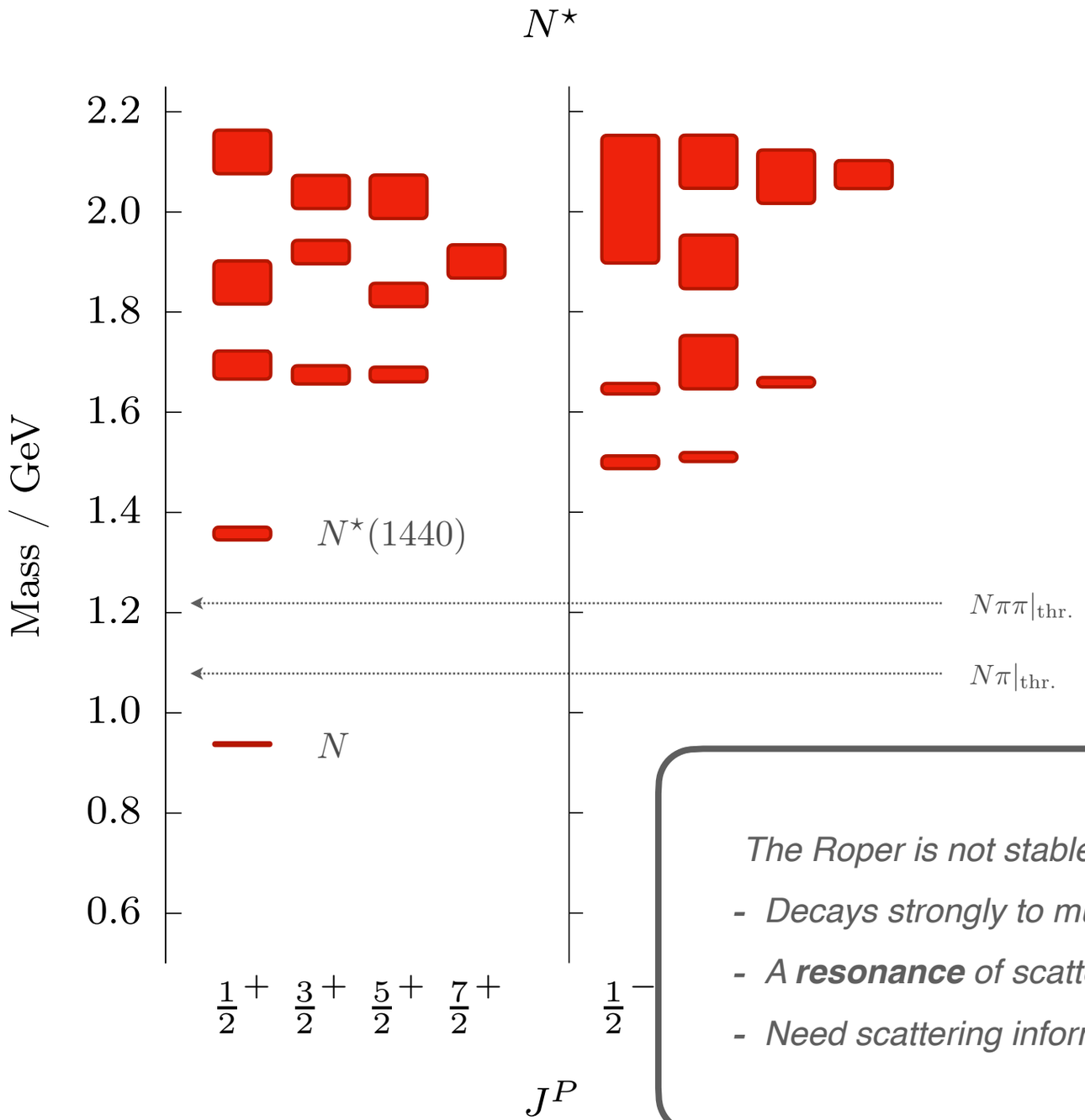
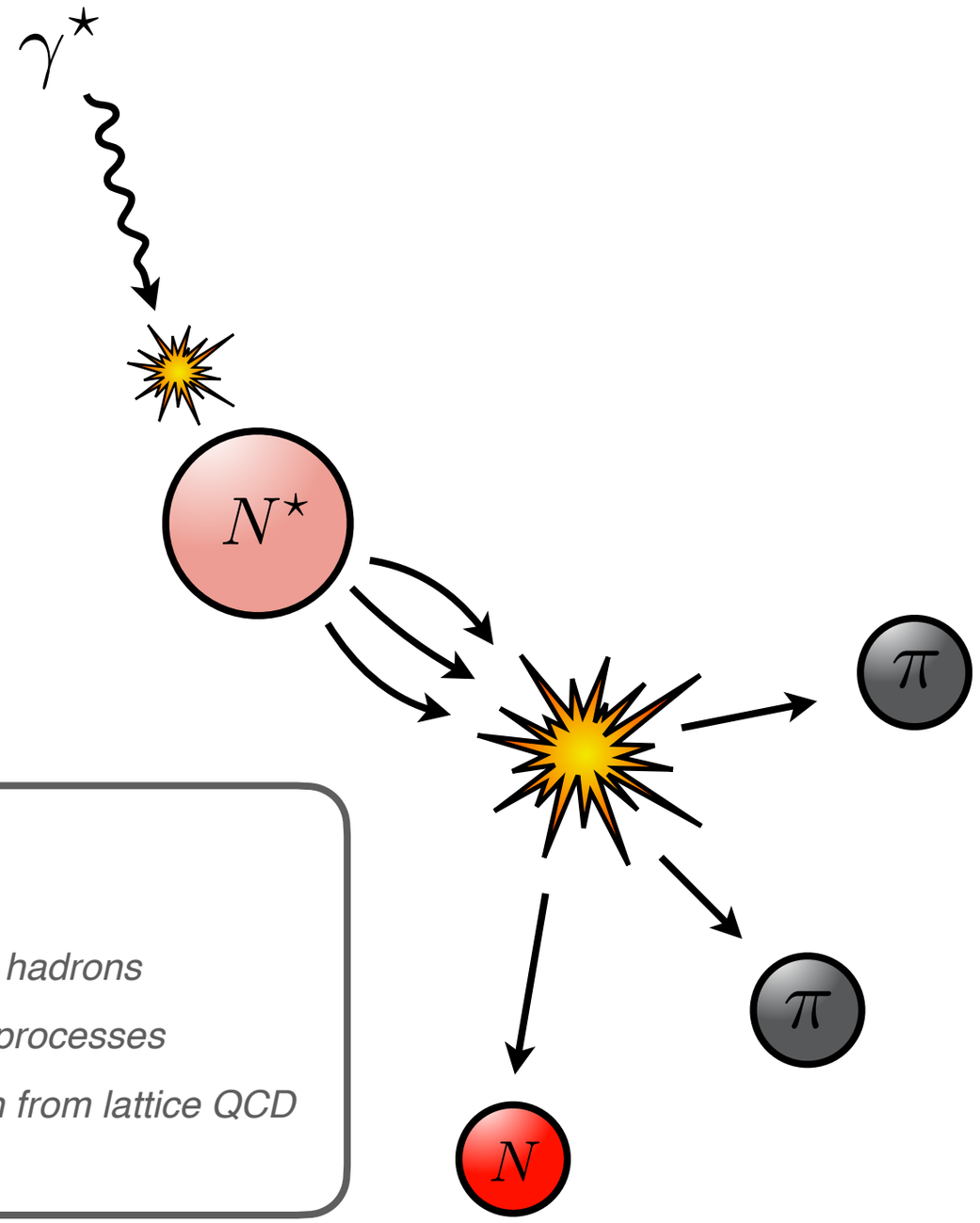




The Roper is not stable!

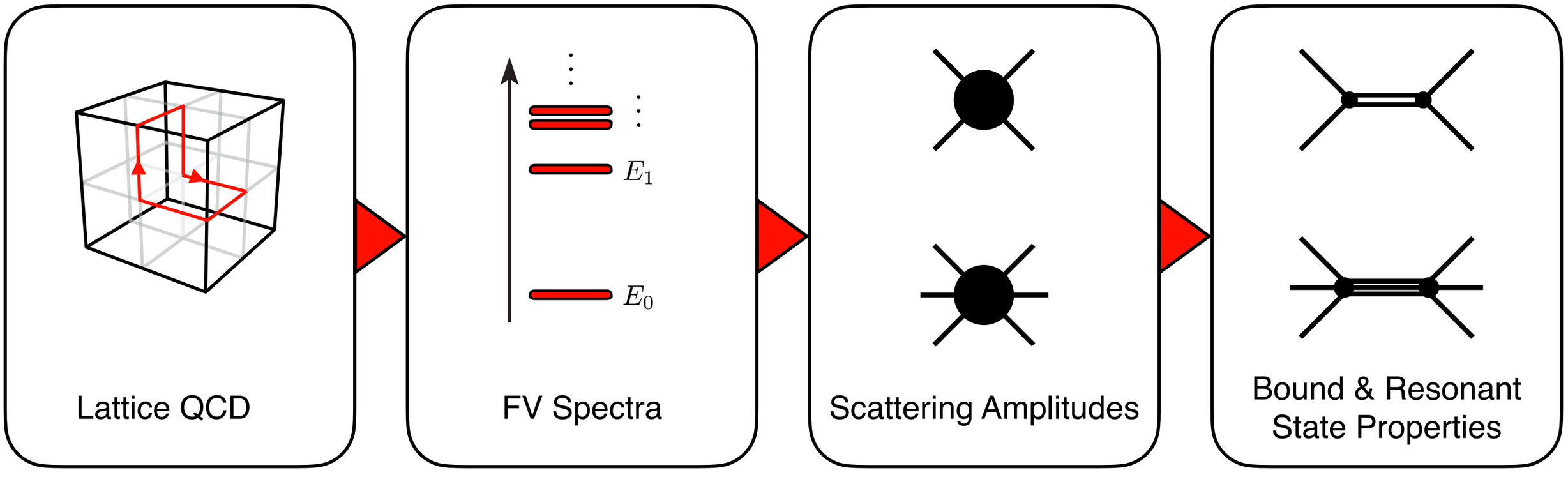
- *Decays strongly to multiple hadrons*
- *A **resonance** of scattering processes*
- *Need scattering information from lattice QCD*

PDG listings

Path to scattering physics from QCD

Use Lüscher methodology to connect lattice QCD observables to scattering amplitudes

- Non-perturbative, model-independent, systematically improvable

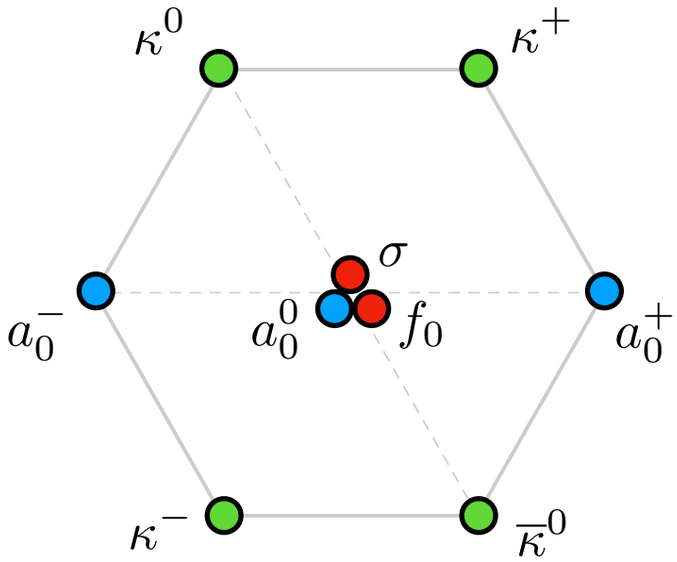


R. Briceño, J. Dudek, and R. Young
Rev.Mod.Phys. **90**, (2018) 025001

Path to scattering physics from QCD

Much success in two-body sector

- e.g., ***the Scalar nonet***



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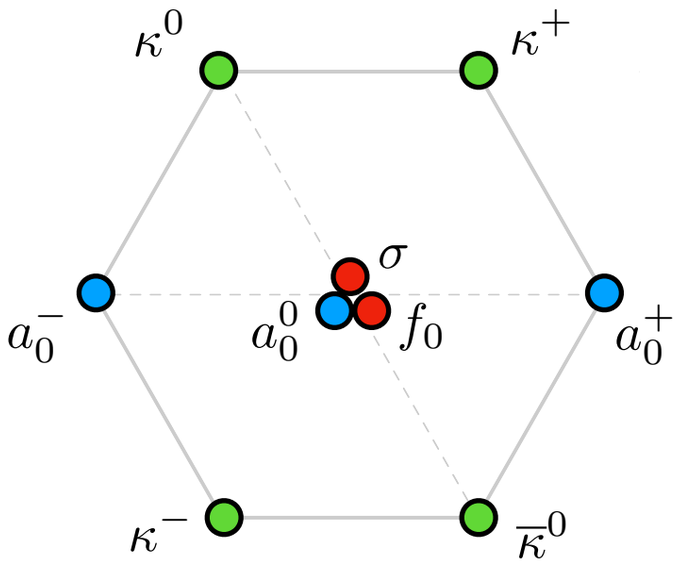
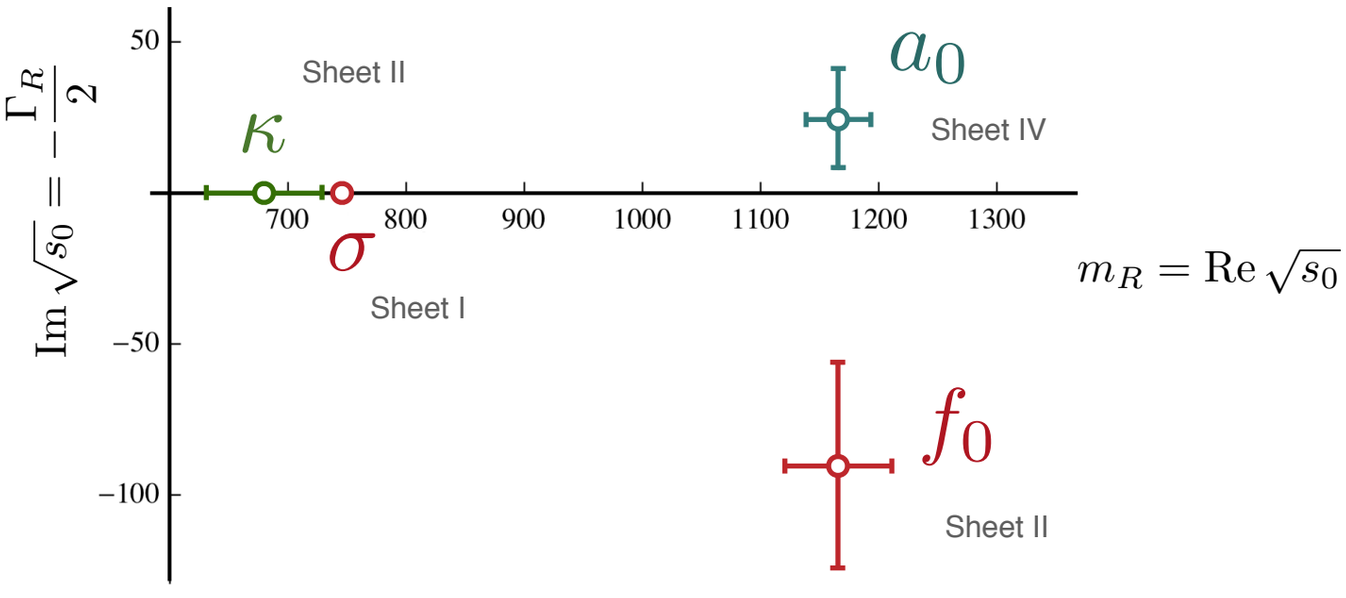
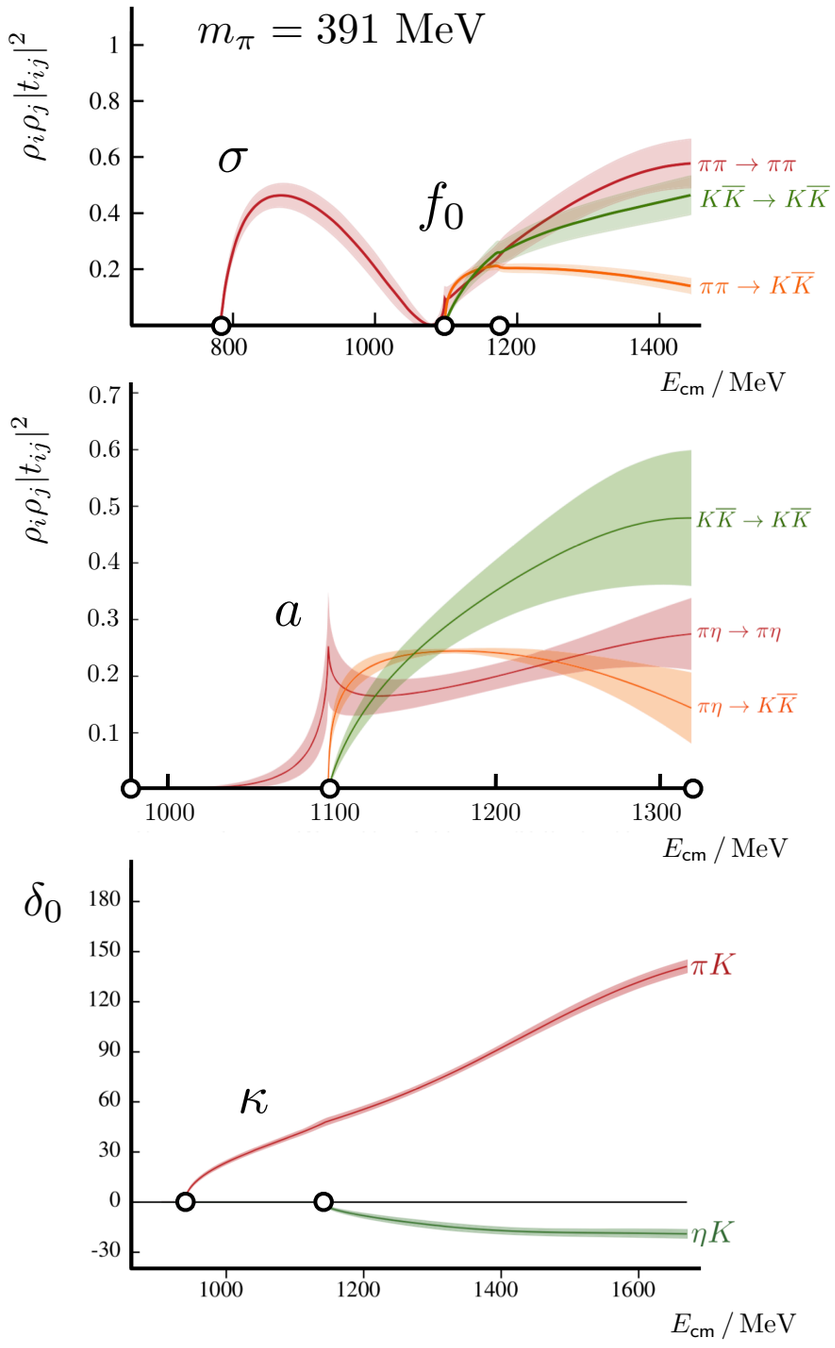


$m_\pi = 391$ MeV





R.A. Briceño et al. [HadSpec]
Phys.Rev. **D97**, (2018) 054513

J.J. Dudek et al. [HadSpec]
Phys.Rev. **D93**, (2016) 094506

J.J. Dudek et al. [HadSpec]
Phys.Rev.Lett. **113**, (2014) 182001

Path to three-body physics from QCD

Follow same path in three-body sector — considerably more challenging

- 8 kinematic variables (vs. 2 in two-body)
- Link between lattice QCD observable to amplitudes involves integral equations

$$\{E_{\mathbf{n}}\} \longrightarrow \mathcal{K}_3 \longrightarrow \mathcal{T}_3 \longrightarrow m_{\text{res.}}, \Gamma_{\text{res.}}$$

Quantization Condition | Integral Equations | Analytic Continuation

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Quantization Condition

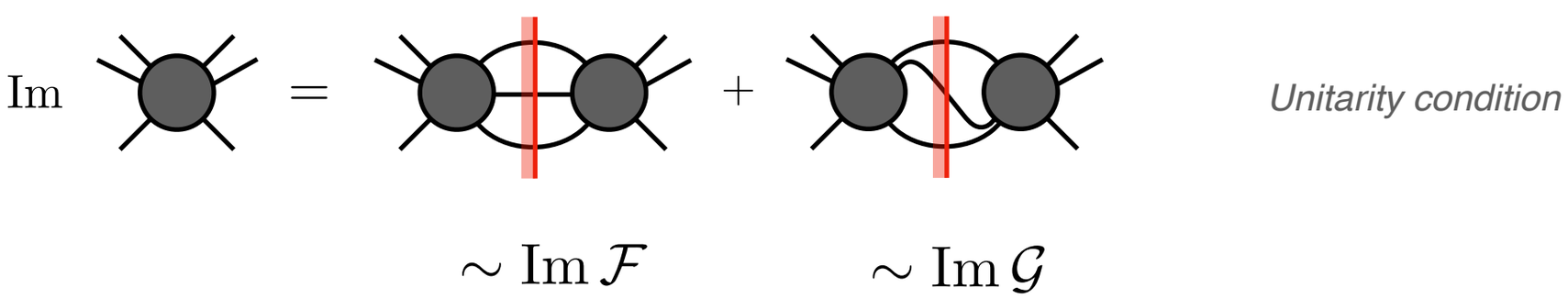
Integral Equations

Analytic Continuation

Scattering theory for three-body systems

Using principles of S matrix to constrain amplitude

- ***Lorentz Invariance***
- ***Unitarity***
- ***Analyticity***
- ***Crossing***



Unitarity condition

$$\mathcal{T}_3 = \mathcal{K}_3 - \int \mathcal{K}_3\,\mathcal{F}\,\mathcal{T}_3 - \iint \mathcal{K}_3\,\mathcal{G}\,\mathcal{T}_3$$

On-shell scattering equation

$\mathcal{K}_3$ *Unknown!*
Obtained from Lattice QCD

cf. two-body case: $\mathcal{T}_2 = \mathcal{K}_2 - \mathcal{K}_2\,\mathcal{F}\,\mathcal{T}_2$

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

AJ et al. [JPAC]
Phys. Rev. D **100**, 034508 (2019)

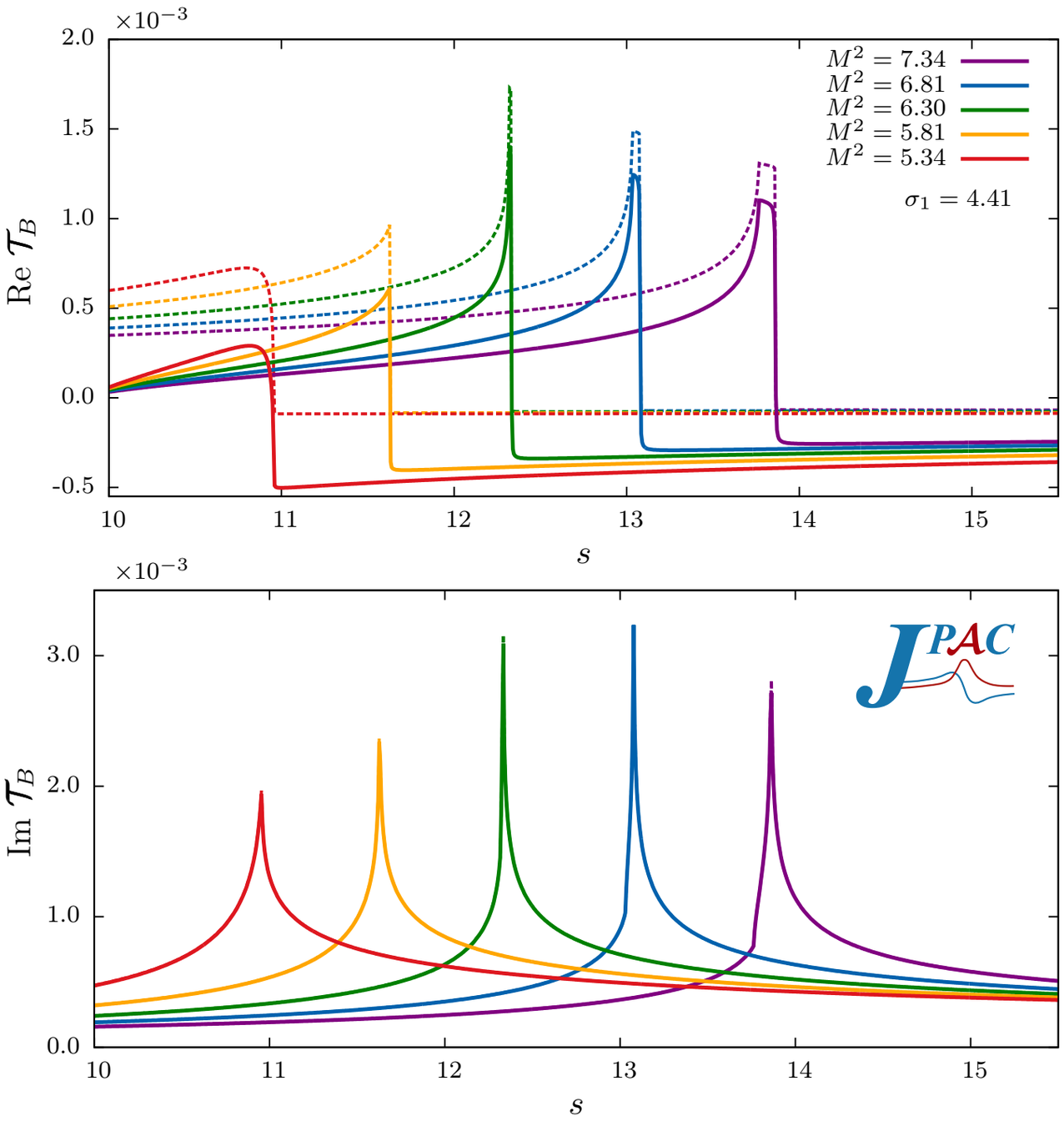
AJ, *in preparation*

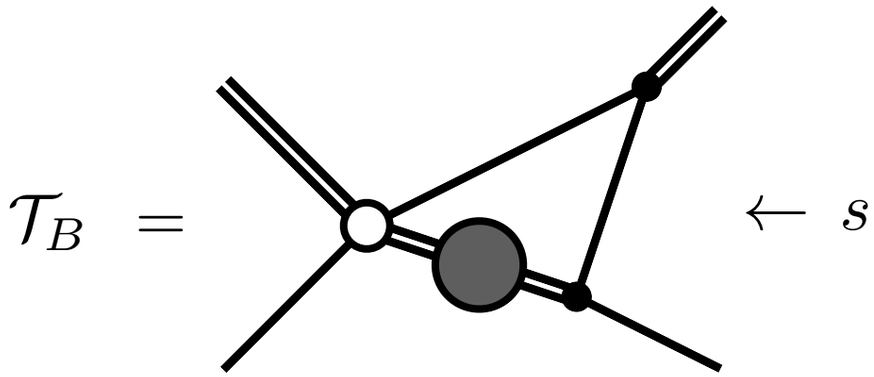
Scattering theory for three-body systems

Using principles of S matrix to constrain amplitude

- Informs the analytic structure — needed to understand resonance phenomena

e.g. effects of triangle singularities





$$\mathcal{T}_B \subset \mathcal{T}_3$$

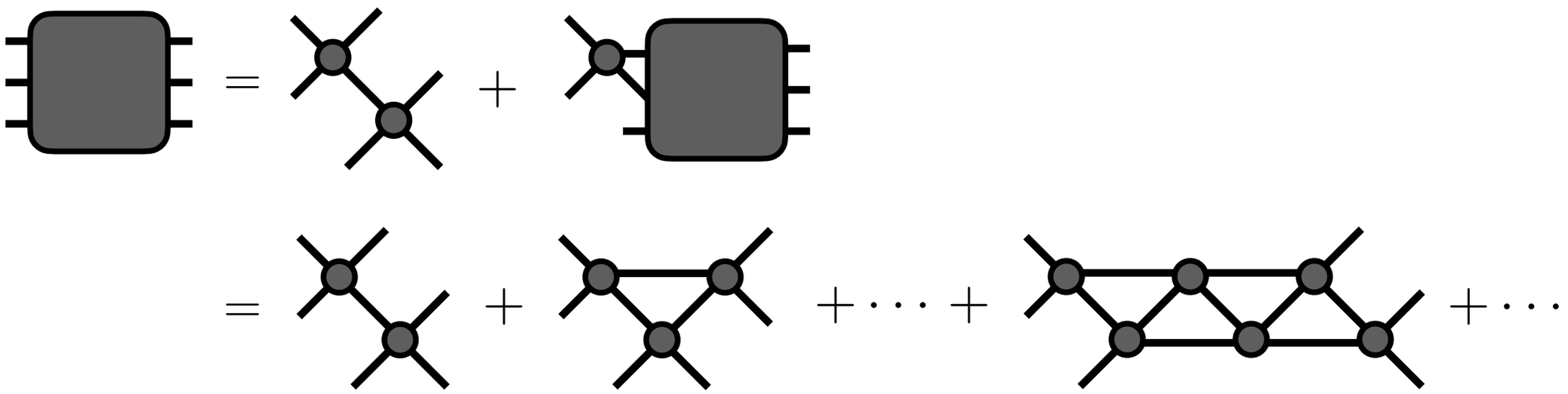
AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

Solving the three-body equations

Examine toy-model — $3\varphi \to 3\varphi$

- Assume exchange dominance — ***No short-range three-body forces***
- Scalar system — $J = 0$
- Two-hadron pair forms bound state — $2\varphi \to b$

Toy model version of $3N \to 3N$ *with* $2N \to d$



AJ, R. Briceño, S. Dawid, M.H. Islam, and C. McCarty
arXiv:2010.09820 (2020)

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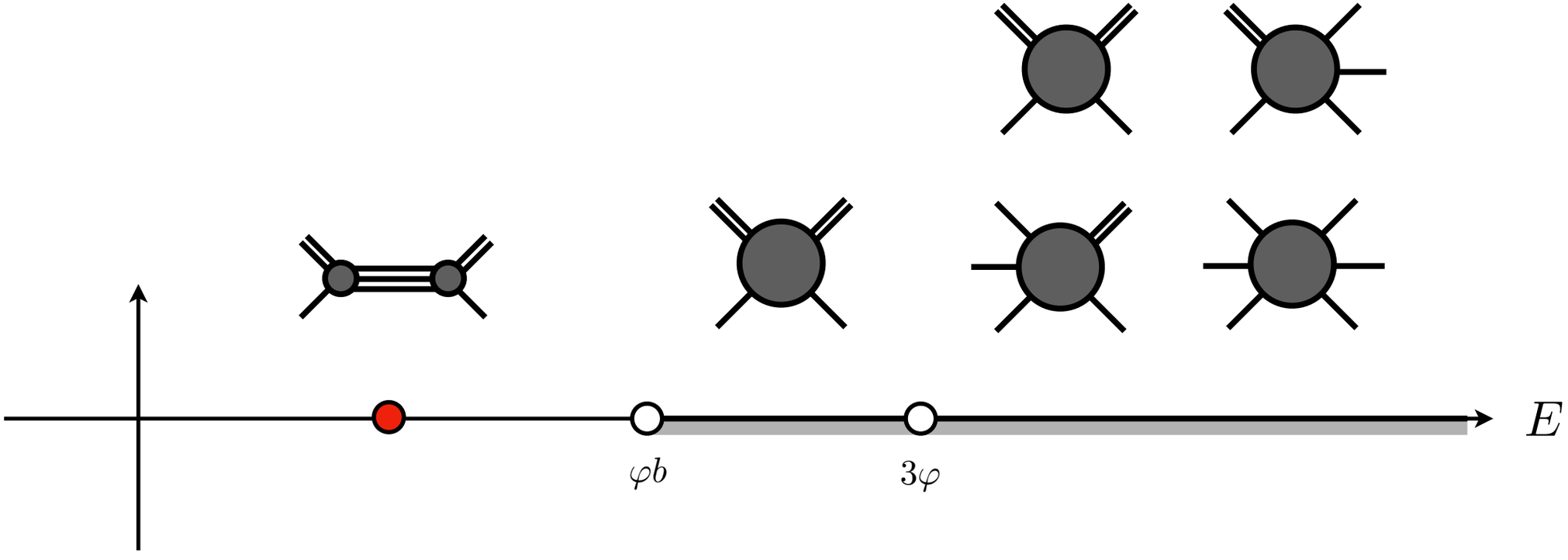

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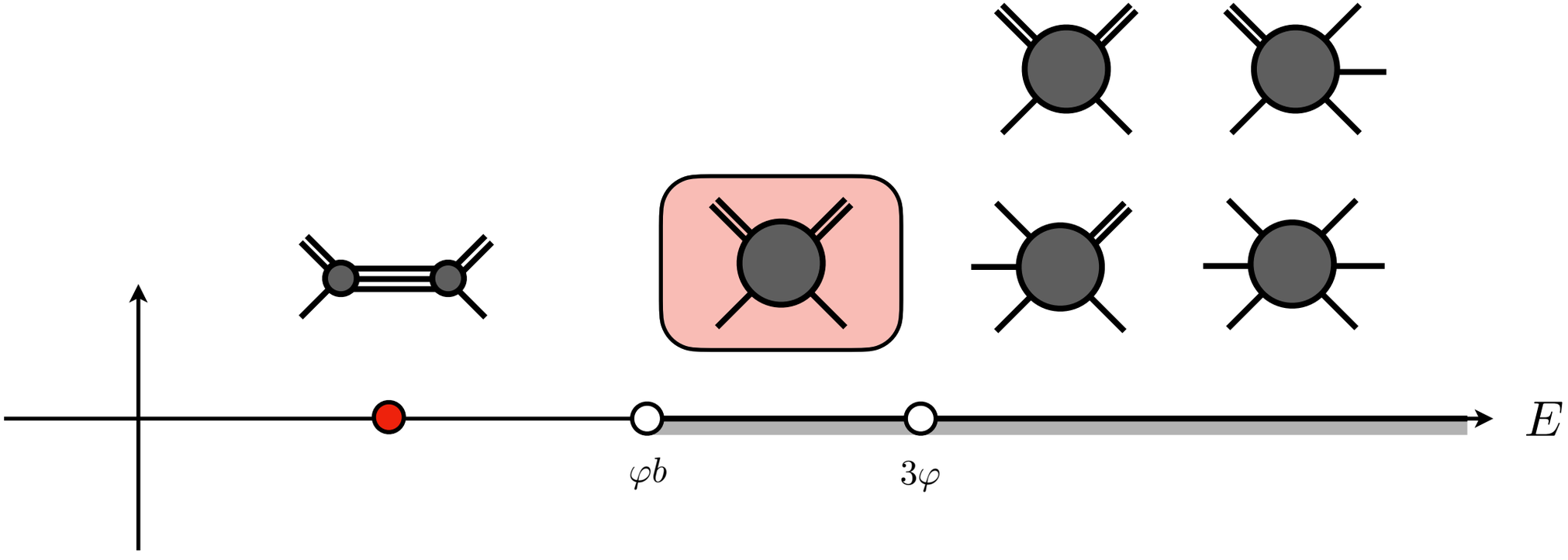

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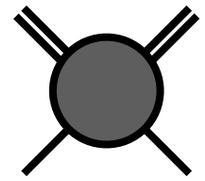

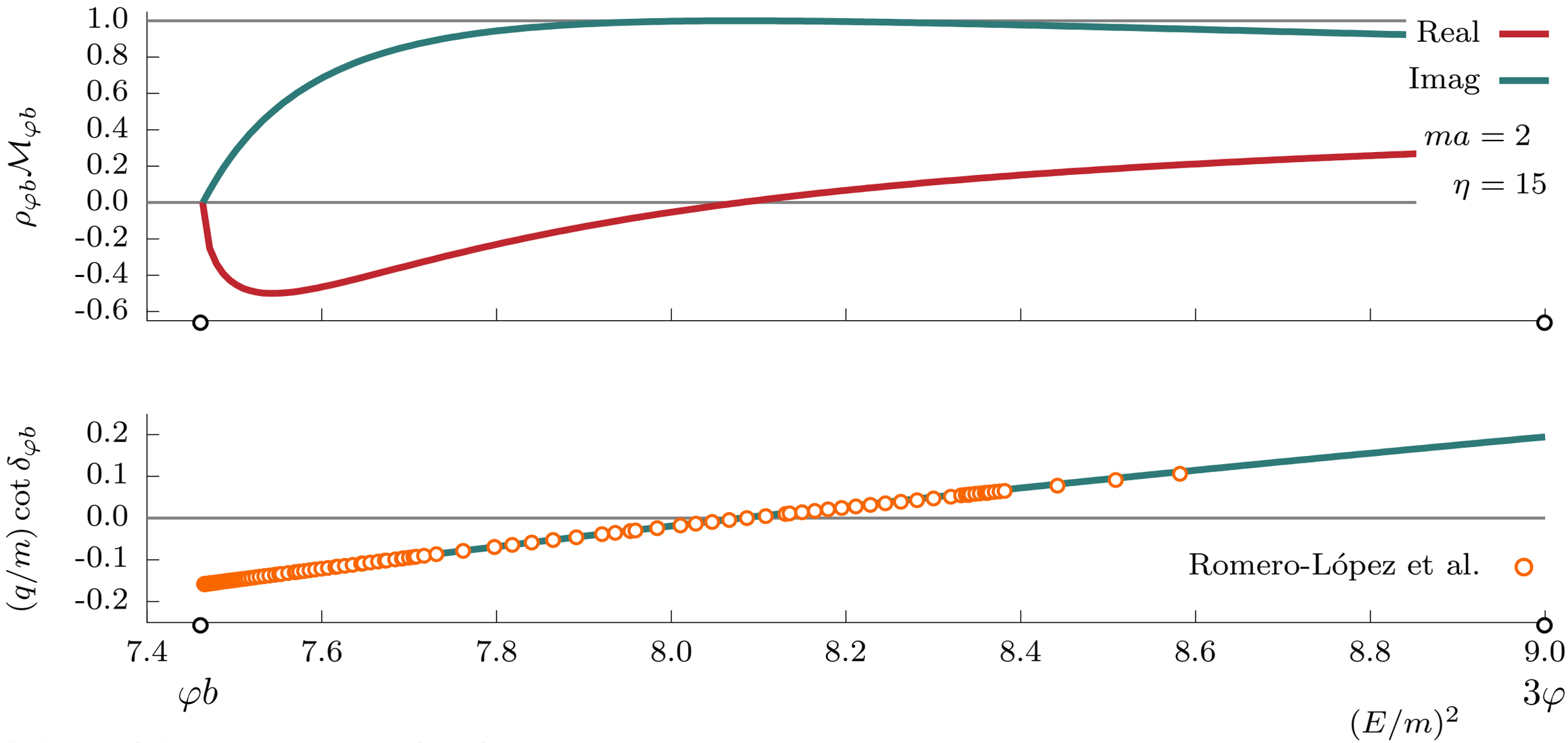

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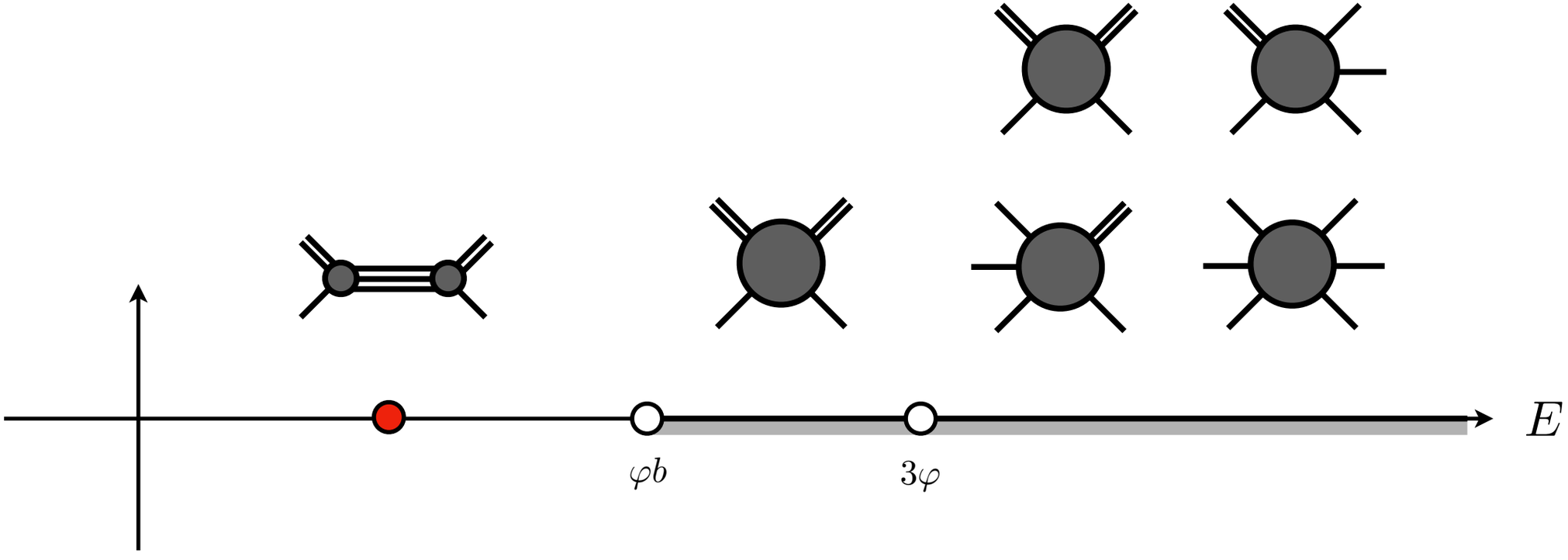
Above the three-body threshold

Methodology not limited to below 3-body threshold

- Allows for calculation of breakup / recombination amplitude



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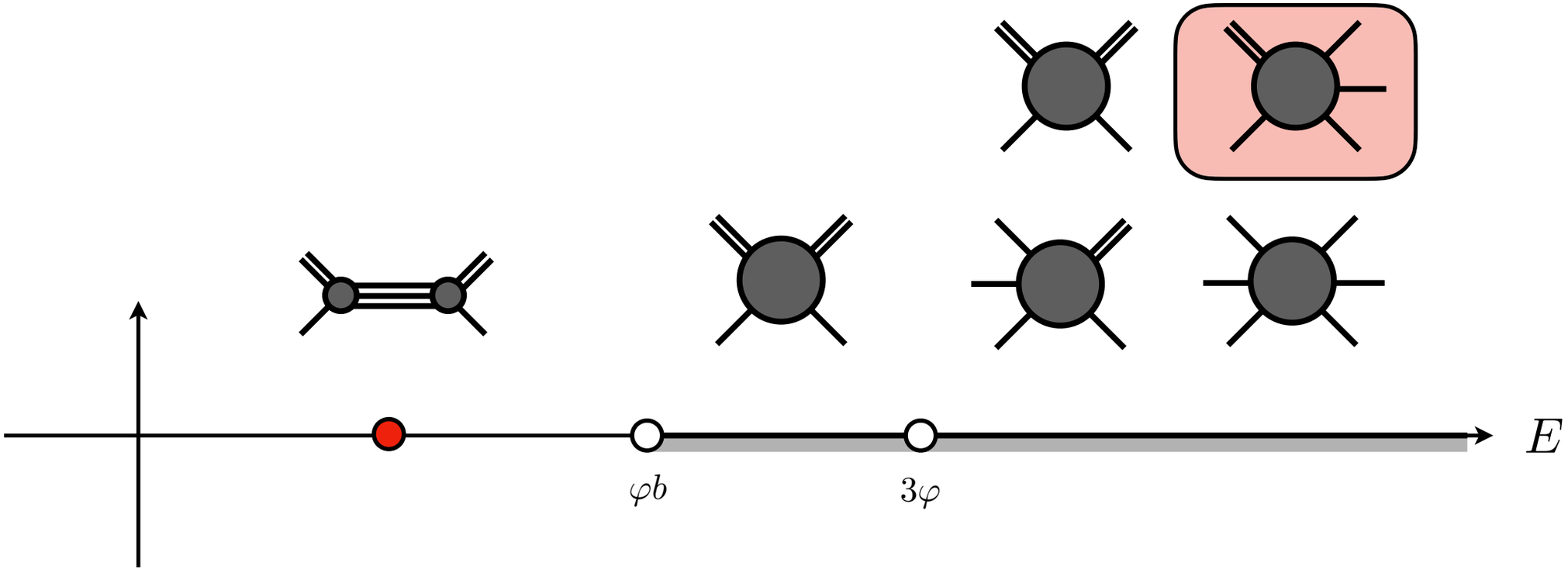

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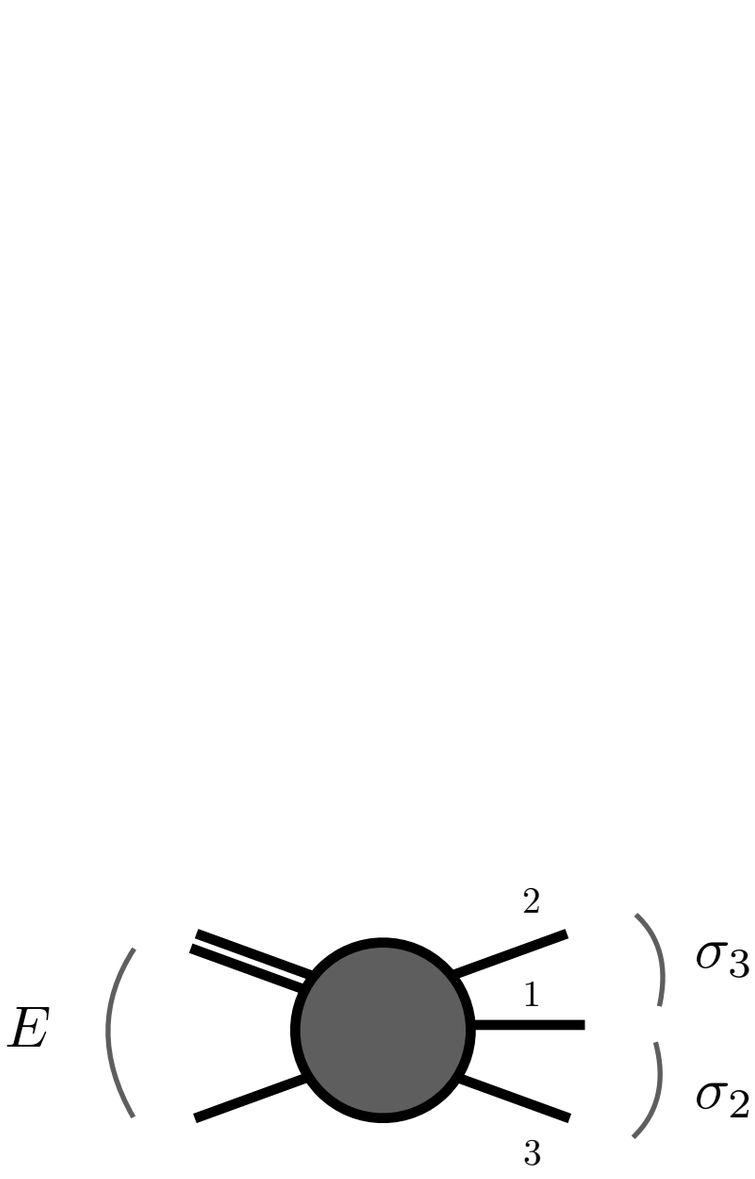

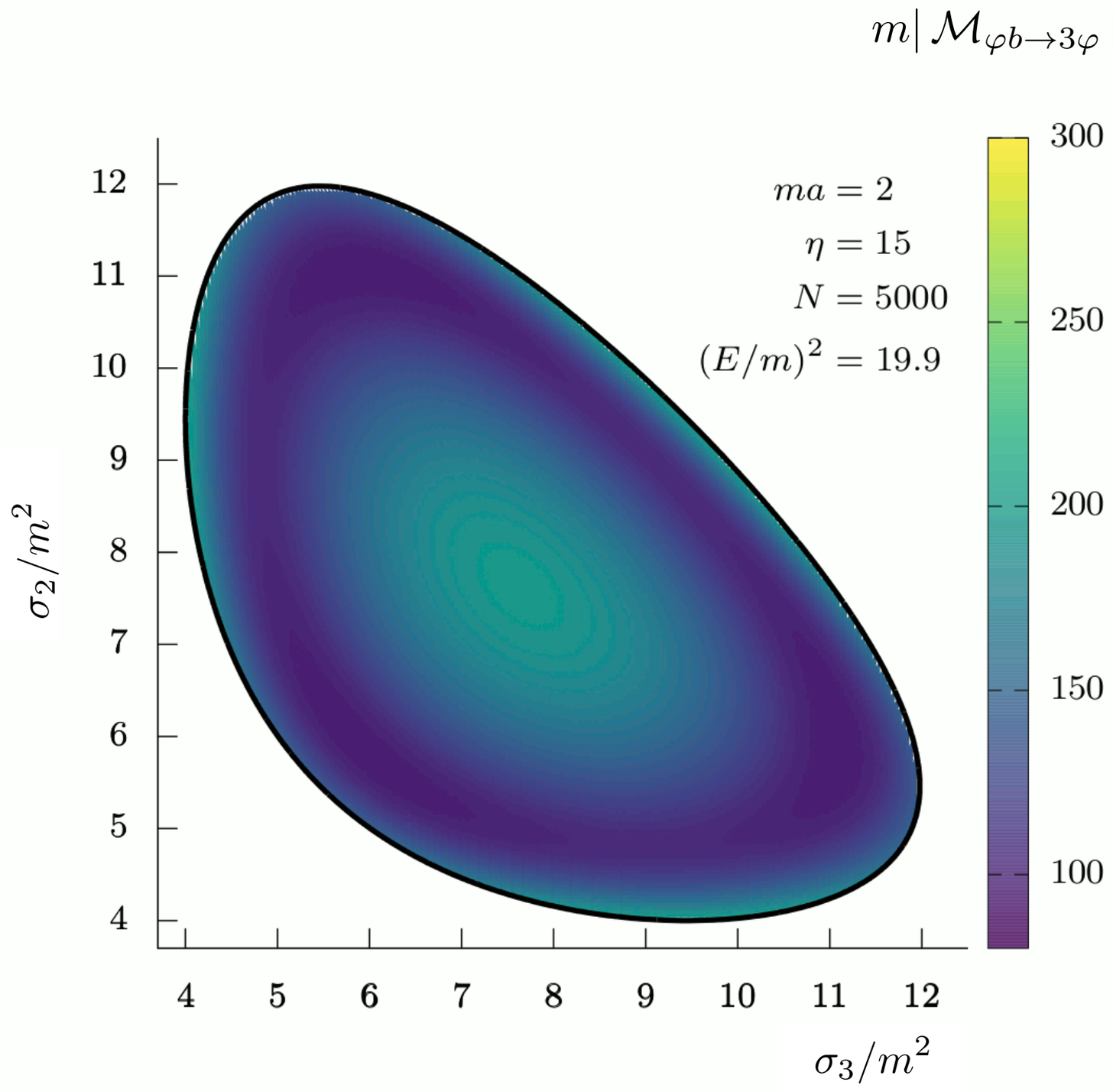

Unfortunately… pretty boring…

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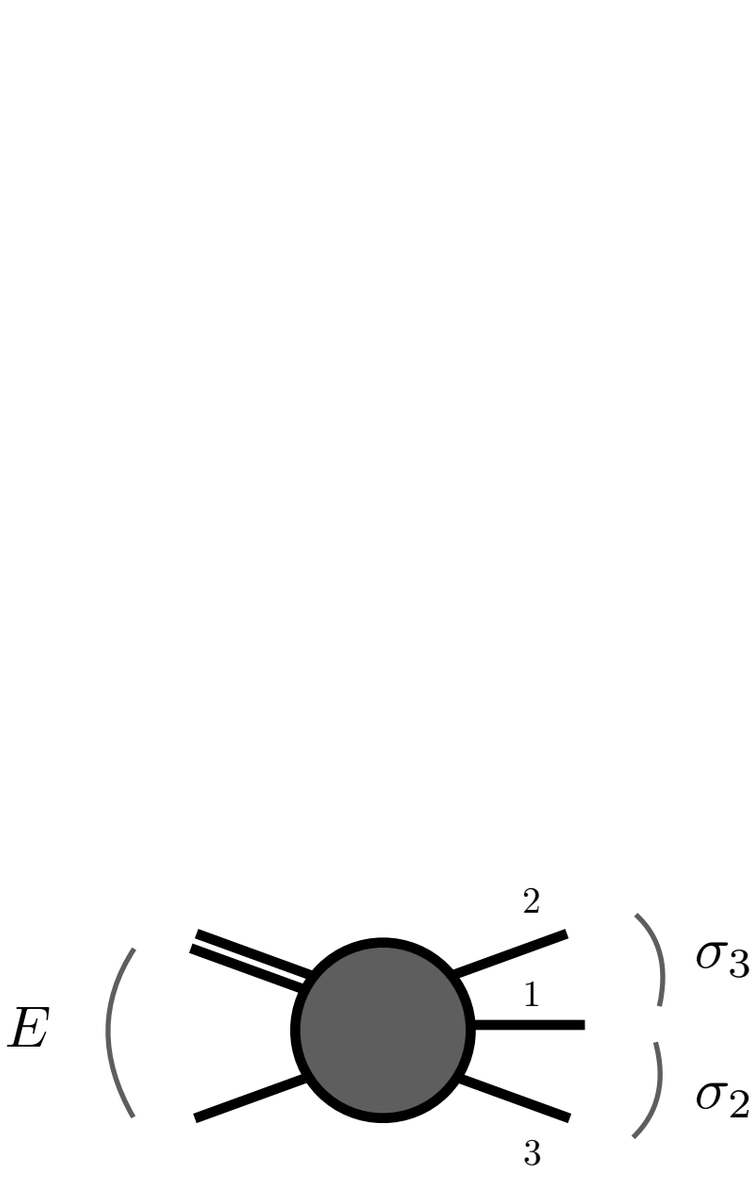

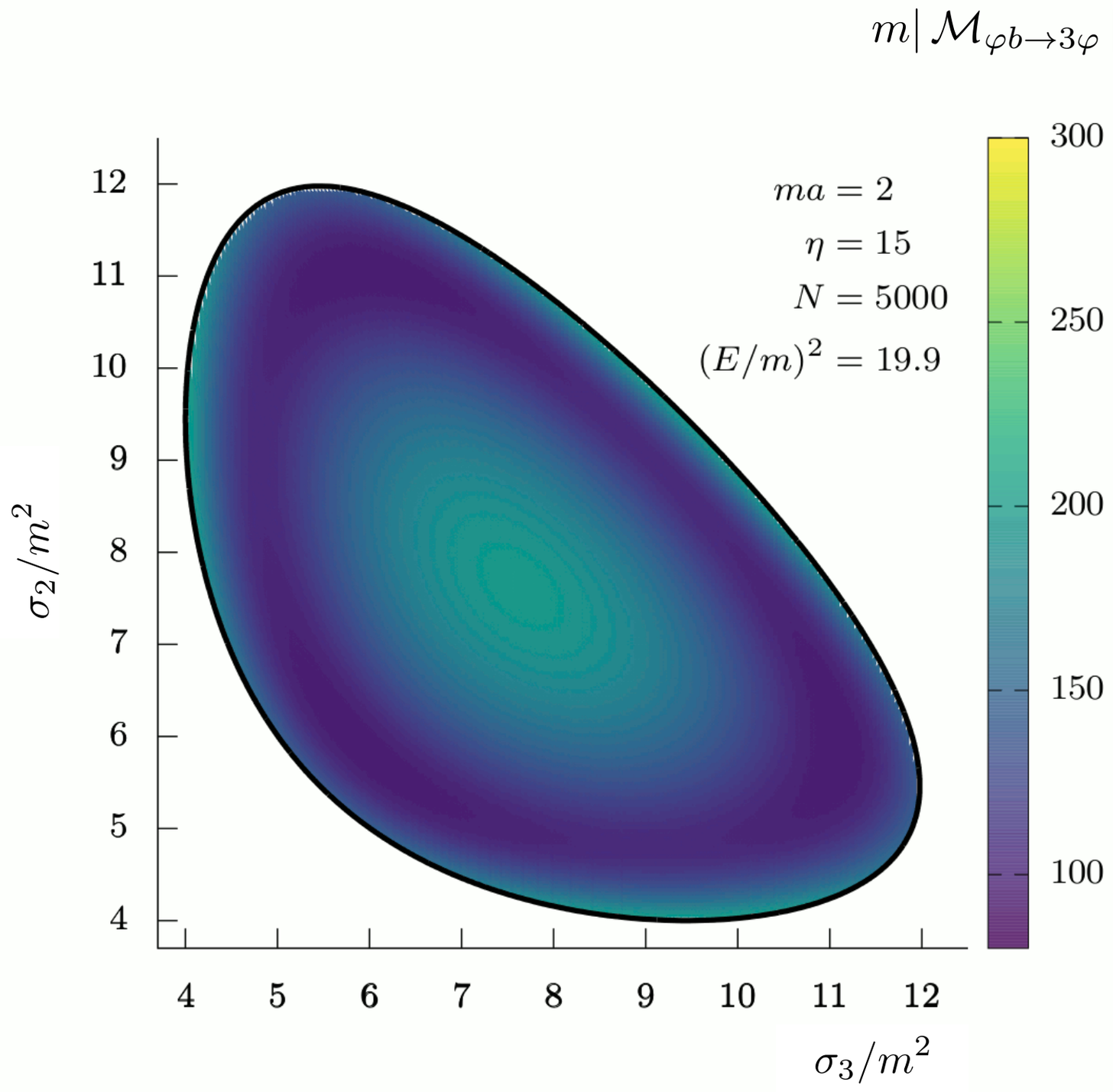

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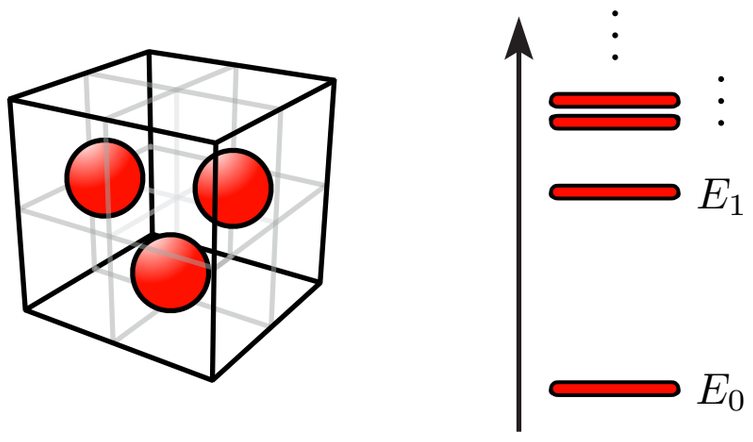
Connecting Lattice QCD to three-body amplitudes

For a given K matrix, can solve integral equations

- K matrices given from lattice QCD



$$\det\Big(1 + \mathcal{K}_3 \, (\mathcal{F}_L + \mathcal{G}_L) \Big)_{E=E_\mathbf{n}} = 0$$

Quantization condition

$$\mathcal{T}_3 = \mathcal{K}_3 - \int \mathcal{K}_3 \, \mathcal{F} \, \mathcal{T}_3 - \iint \mathcal{K}_3 \, \mathcal{G} \, \mathcal{T}_3$$

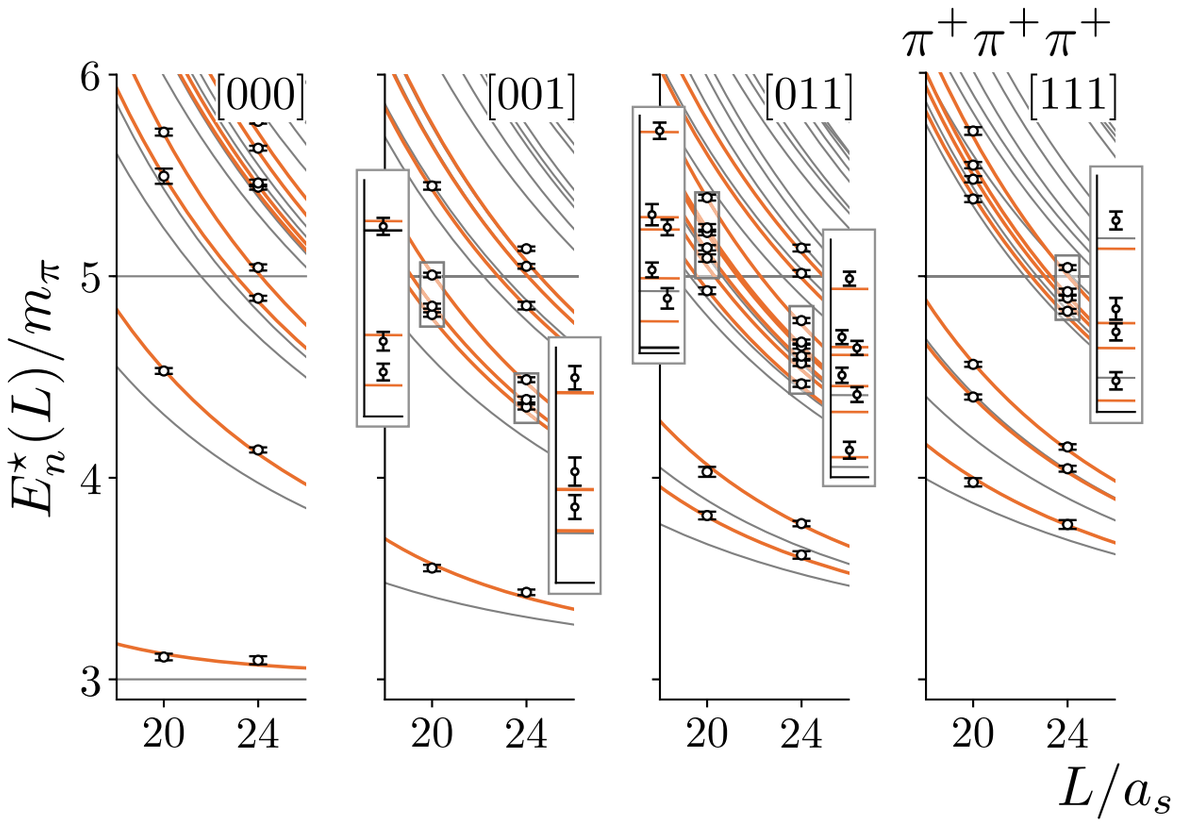
On-shell scattering equation

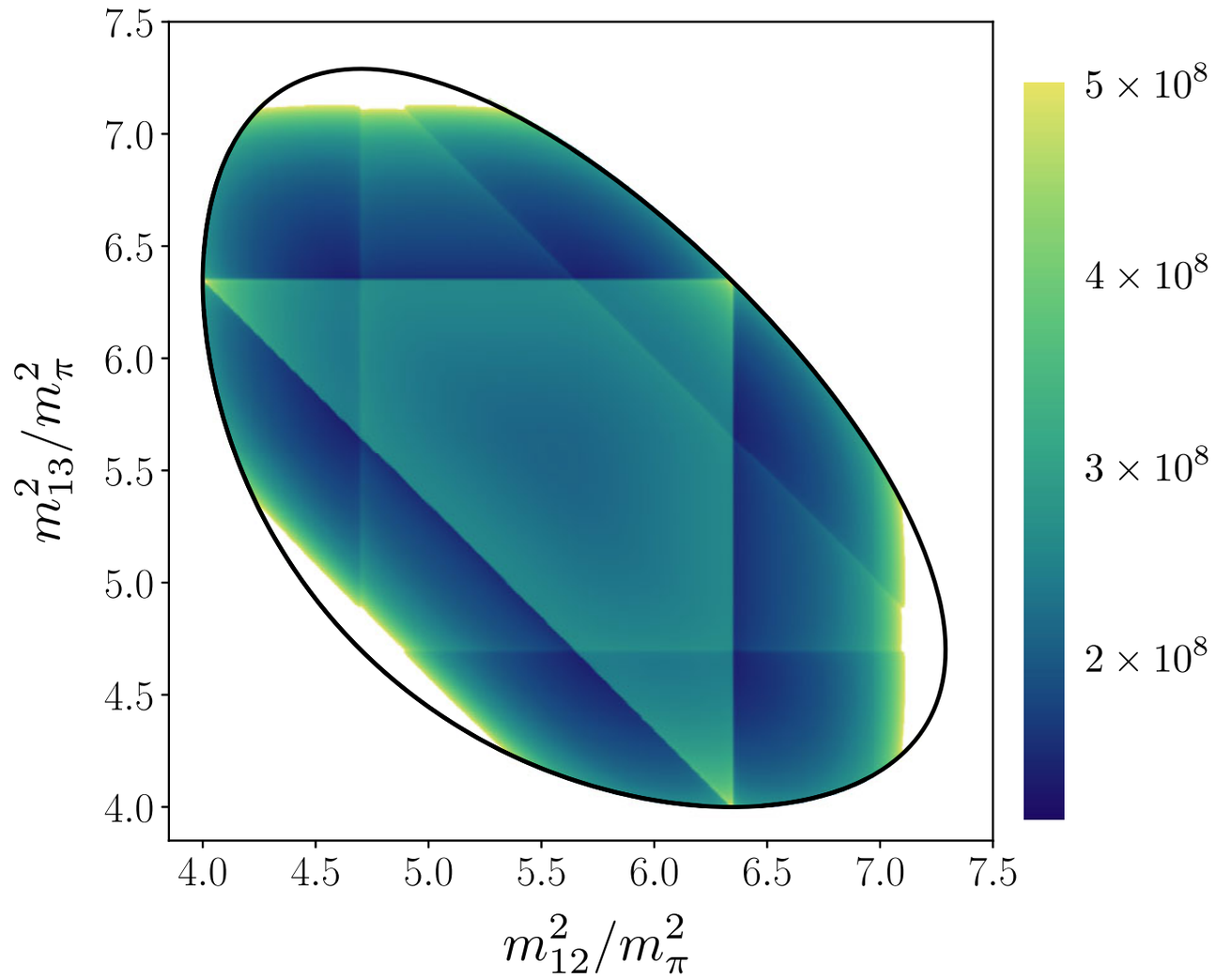
M. Hansen and S. Sharpe
Phys. Rev. D **90**, 116003 (2014), Phys. Rev. D **95**, 034501 (2017)

M. Mai and M. Döring
Eur. Phys. J. A **53**, 240 (2017), Phys. Rev. Lett. **122**, 062503 (2019)

AJ, *in preparation*

Applications to $3\pi^+$





M. Hansen et al. [HadSpec]
Phys. Rev. Lett. **126**, (2021) 012001

Future extensions

Many avenues of interest - Formal, phenomenology, computational (including lattice QCD)

Higher partial waves

e.g. $(3\pi)_{J=1}$

w/
R. Briceño
Effy John (REYES)

Resonant sub-channels

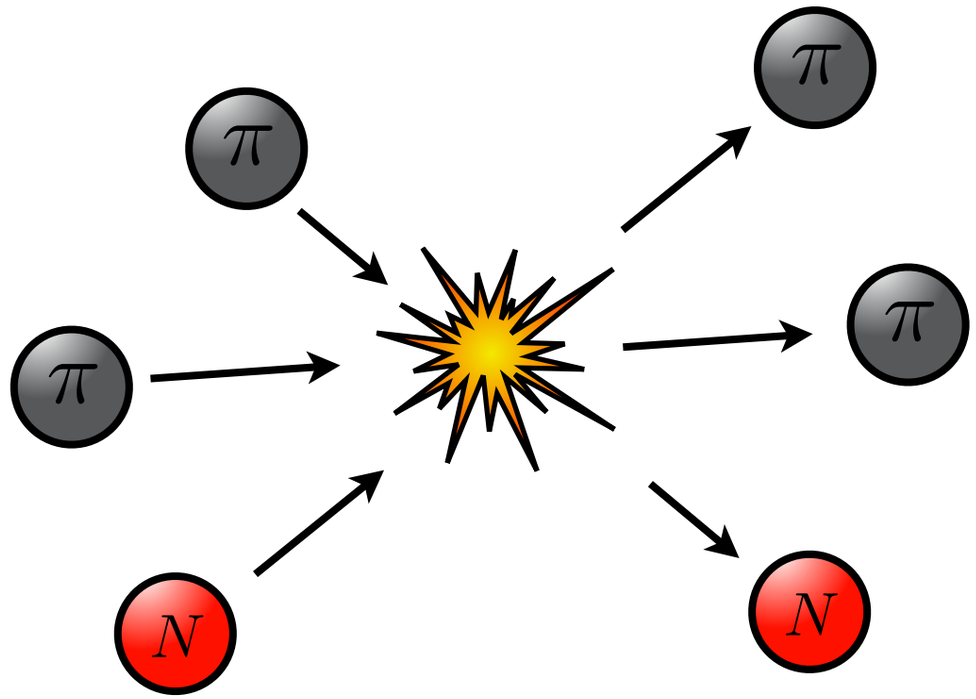
e.g. $\rho\pi \to 3\pi$

w/
R. Briceño
Taylor Powell (ODU)

Coupled $\mathbf{2 \to 3}$

e.g. $\pi\eta \to 3\pi$

w/
R. Briceño
S. Dawid (IU)
Taylor Powell (ODU)



Analytic continuation

$E \to \mathrm{Re}\, E + i\, \mathrm{Im}\, E$

w/
R. Briceño
Md. Islam (ODU)
S. Dawid (IU)

Systems with Spin

e.g. $\pi\pi N \to \pi\pi N$

Topic for another talk…

Matrix elements

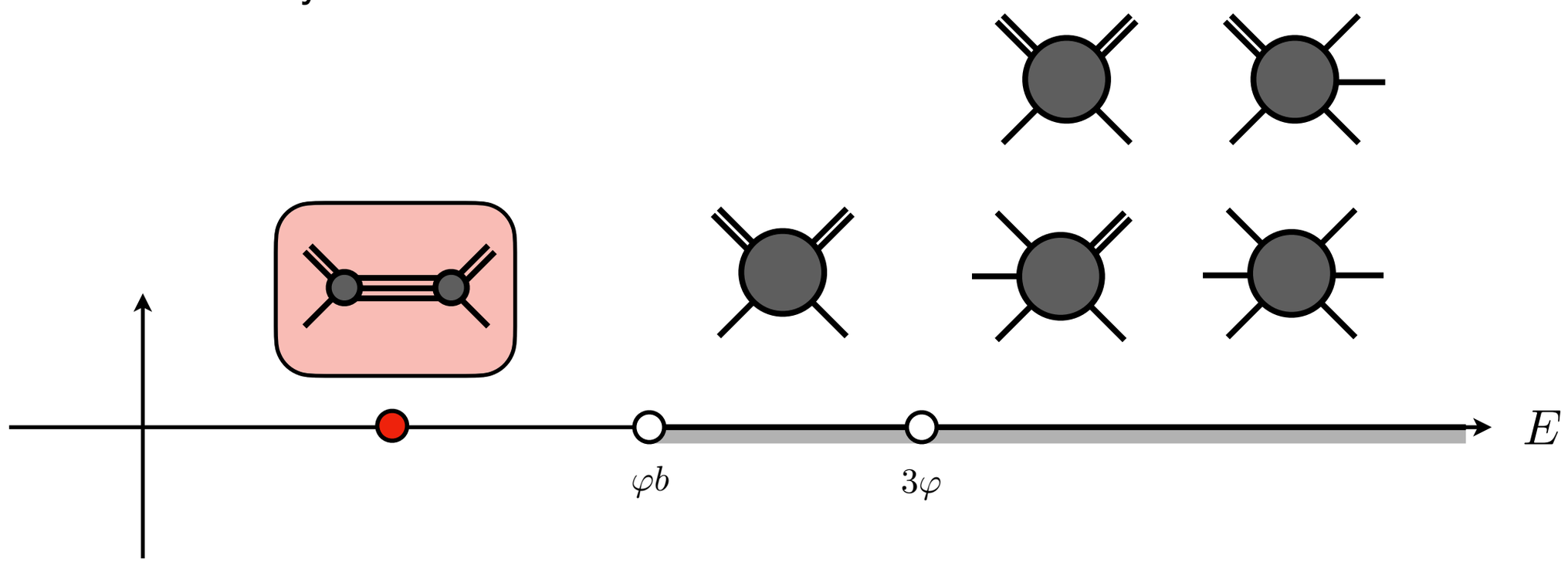
e.g. $2\pi + \gamma^\star \to 2\pi$

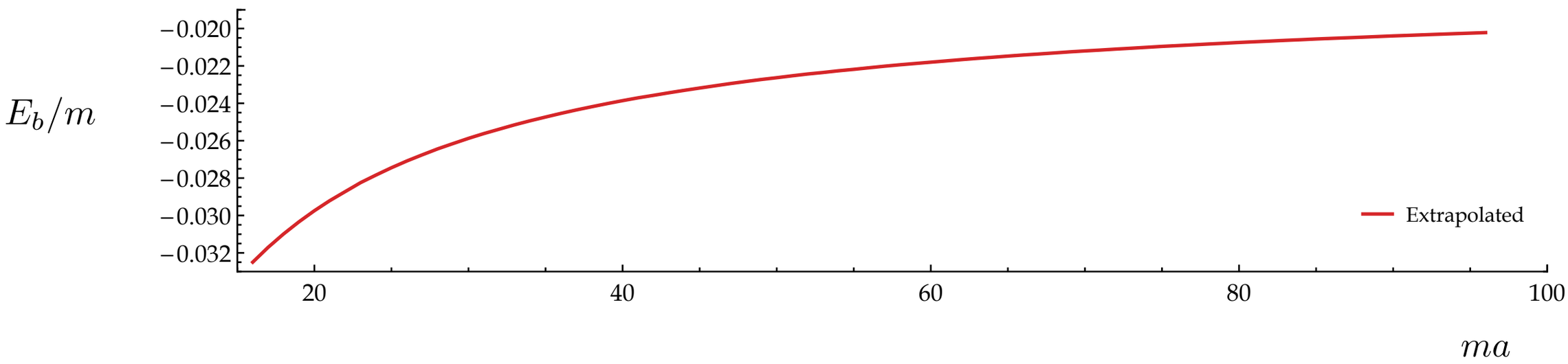
w/
R. Briceño
M. Hansen
K. Sherman (ODU)
F. Ortega-Gama (WM)

Future extensions

Analytic continuation to three-body bound state

e.g. toy model triton



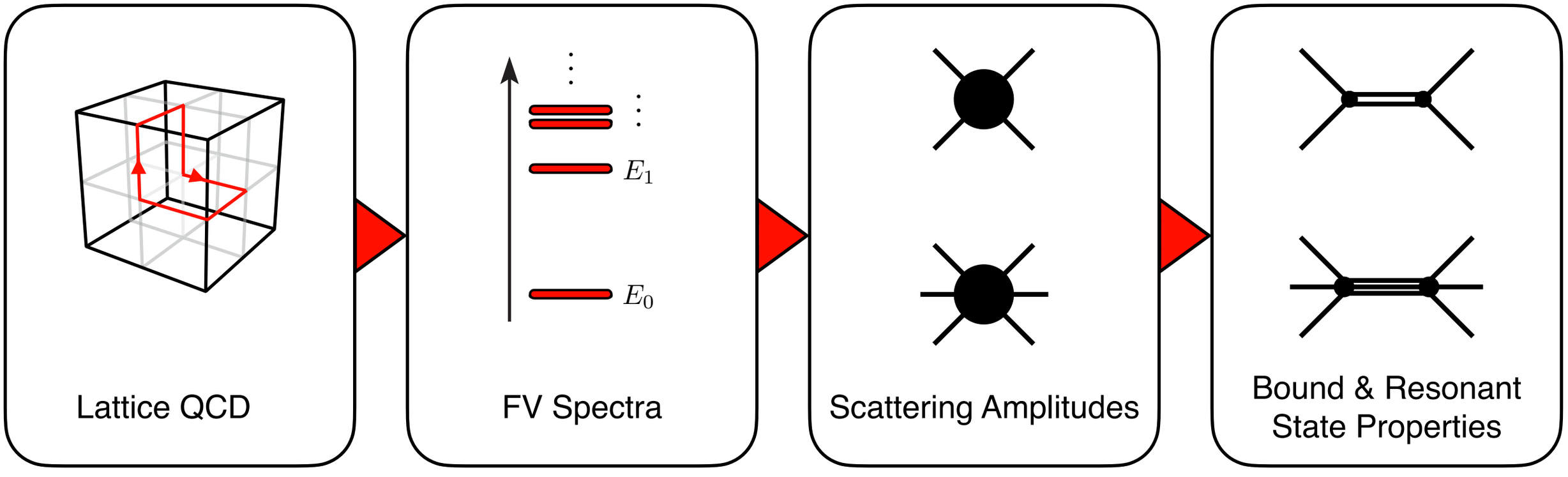


AJ, R. Briceño, S. Dawid, M.H. Islam
In preparation

Path to three-body physics from QCD

Three-body resonances from QCD are within reach

- Clear systematic path via Lattice QCD
- Scattering theory constraints allow for model-independent analyses



Summary

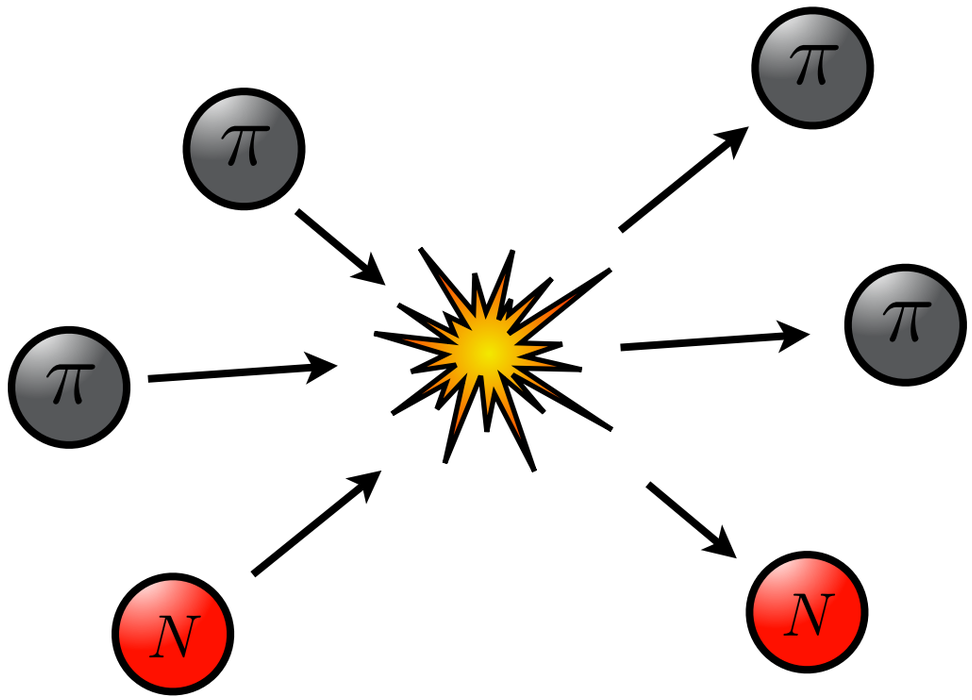
Three-body interactions play a key role in many outstanding problems in hadron physics

- Most excited states strongly couple to three (or more) particle channels

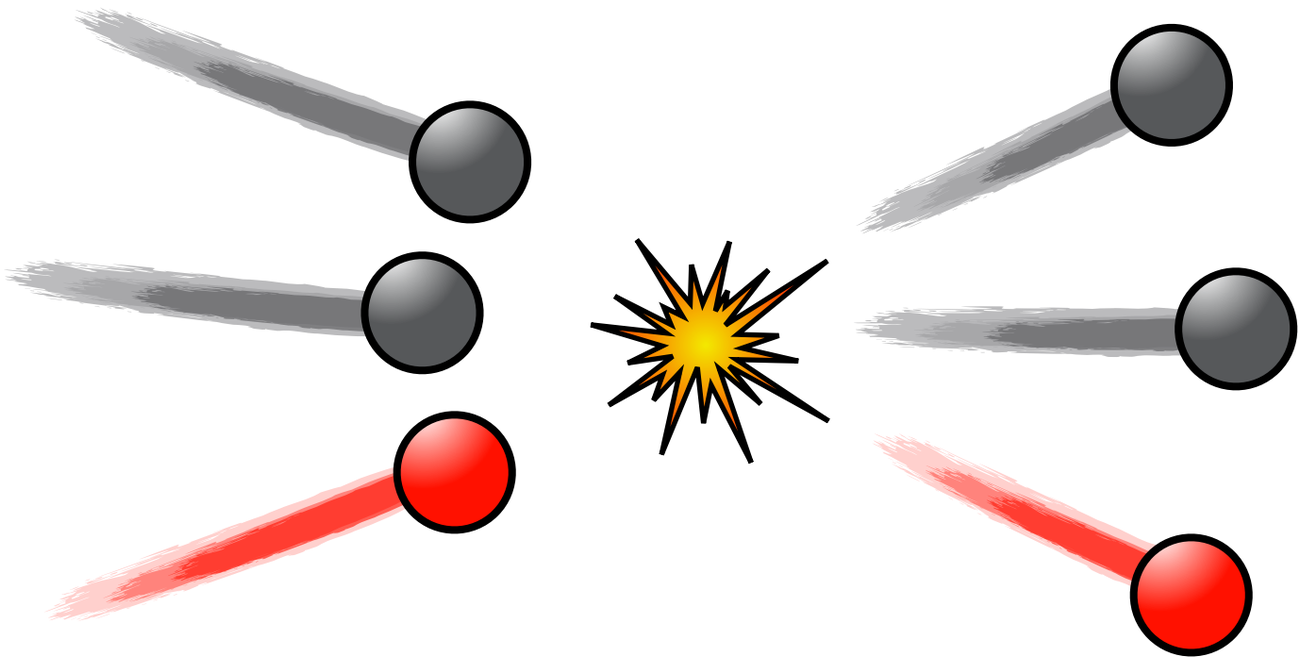
Framework in place to determine three-body hadron physics from lattice QCD

- Rapid development in formalisms relating lattice QCD observables to amplitudes
- Scattering phenomenology is advancing in tandem
- First applications of these ideas appearing in the literature

Much more to come!



Extra Slides



On-shell scattering equations

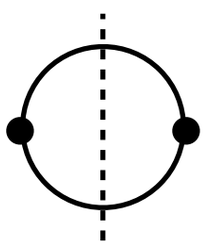
S-matrix unitarity fixes on-shell structure of scattering amplitudes

$2 \to 2$ scattering

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

On-shell two-particle rescattering

$$\mathrm{Im}\,\mathcal{I} = -\rho \sim$$



K-matrices

- *Unknown real function characterizing short-distance physics*
- *Parameterize with analytic function and determine from lattice QCD*
- *Scheme dependent (unphysical)*

On-shell scattering equations

S-matrix unitarity fixes on-shell structure of scattering amplitudes

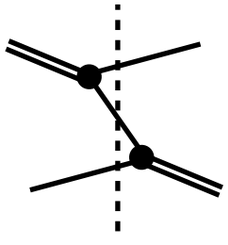
$2 \to 2$ scattering

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

$3 \to 3$ scattering

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3$$
$$- \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \int\int \mathcal{K}_3 G \mathcal{M}_3$$

On-shell exchange

$\operatorname{Im} G = -\Delta \sim$ 

K-matrices

- *Unknown real function characterizing short-distance physics*
- *Parameterize with analytic function and determine from lattice QCD*
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Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3\mathcal{I}\mathcal{M}_2 - \mathcal{K}_2\mathcal{I}\mathcal{M}_3 - \int \mathcal{K}_3\mathcal{I}\mathcal{M}_3$$
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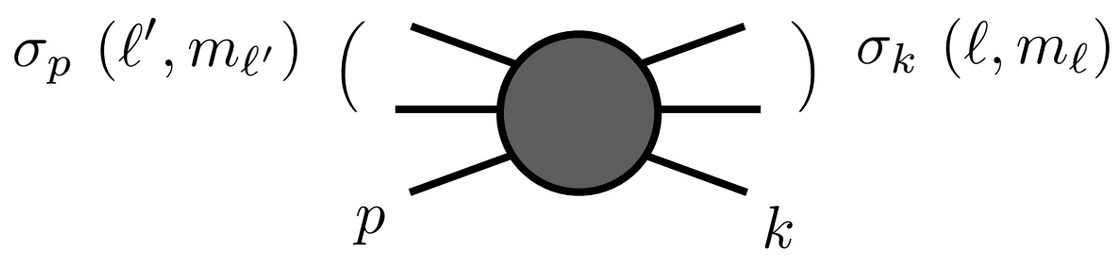

- *Many equivalent forms (Hansen-Sharpe, Blanton-Sharpe, Döring-Mai, JPAC-AJ,Mikhasenko)*
- *Matrix equation in pair angular momenta*
- *Integral equation in spectator momenta*
- *Singular kernels*
- *Scheme dependent K-matrices (unphysical)*

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Consider toy model: $3\varphi \to 3\varphi$ such that $2\varphi \to b$

AJ, R. Briceño, S. Dawid, M.H. Islam, and C. McCarty
arXiv:2010.09820

Solving three-body scattering equations

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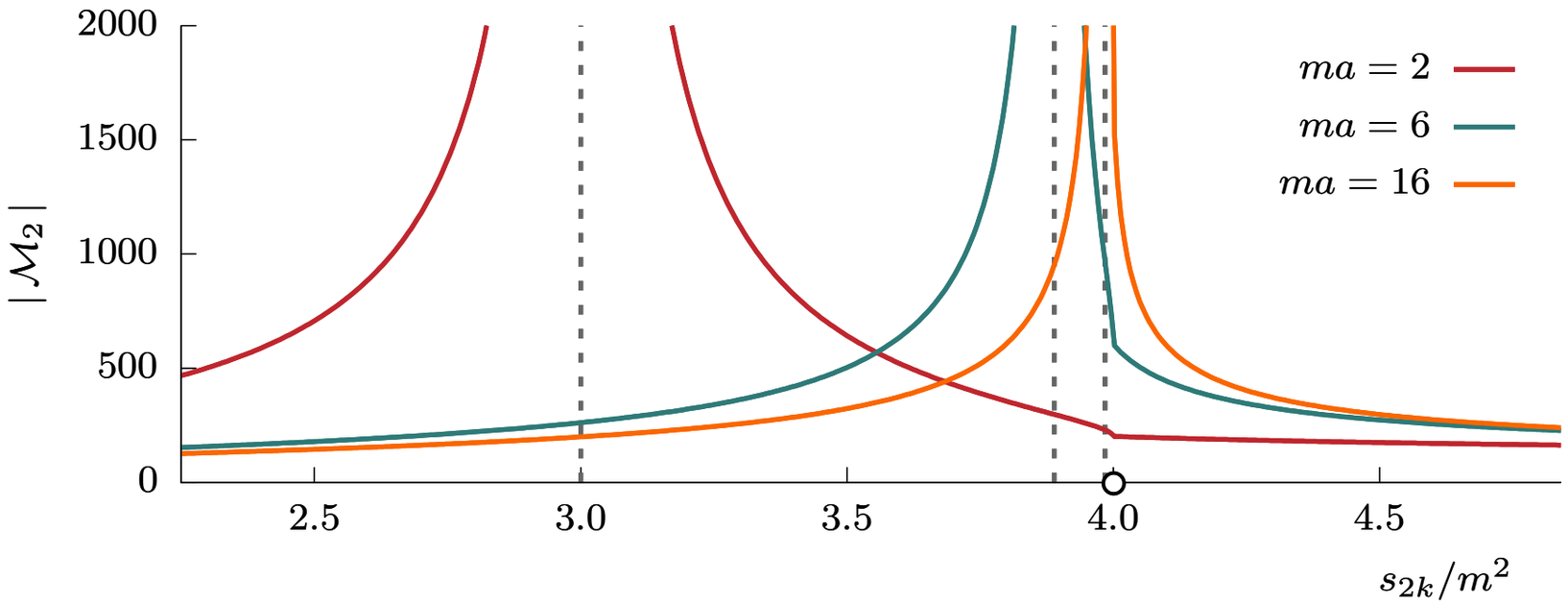
Consider toy model: $3\varphi \to 3\varphi$ such that $2\varphi \to b$

$$\mathcal{K}_2^{-1} \sim -\frac{1}{a}$$



Solving three-body scattering equations

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Assume $\mathcal{K}_3 = 0$

$$\mathcal{M}_3\big|_{\mathcal{K}_3=0} \equiv \mathcal{D} \qquad \Longrightarrow \qquad \mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int G \mathcal{D}$$

Solving three-body scattering equations

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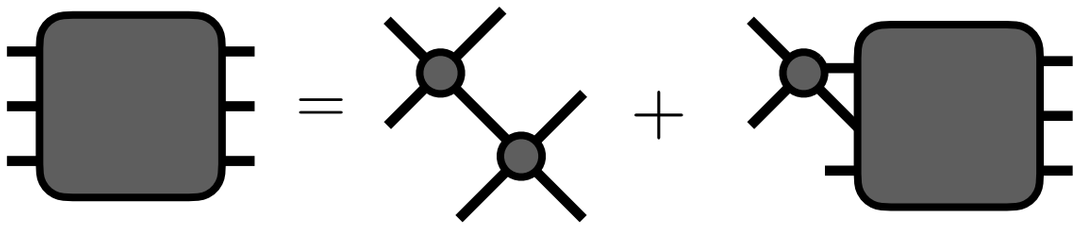
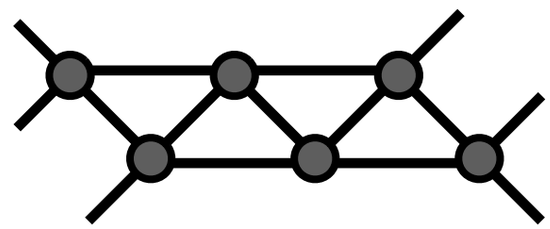
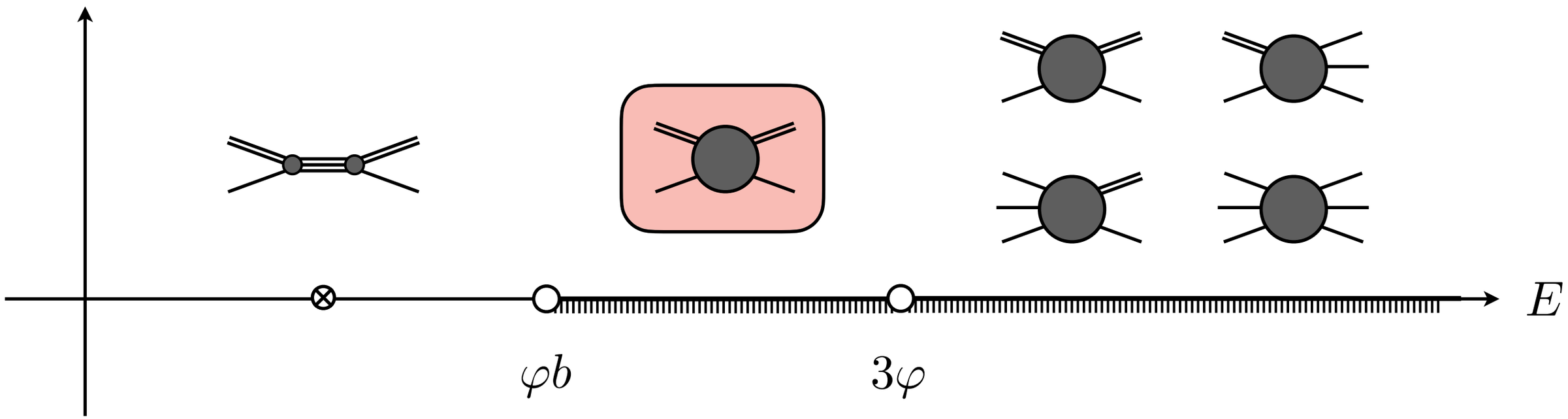
$i\mathcal{D} =$  $+ \cdots +$  $+ \cdots$

Solving three-body scattering equations

Focus on case where 2-body systems forms bound state

- Consider energies below three particle threshold

$$\lim_{\sigma_p,\sigma_k\to\sigma_b} i\mathcal{M}_3 = ig_b \frac{i}{\sigma_p-\sigma_b}\, i\mathcal{M}_{\varphi b}\, \frac{i}{\sigma_k-\sigma_b} ig_b$$



Solving three-body scattering equations

Convert integral equation to linear equation

- Introduce regulators N (matrix size) and ϵ (pole shift)
- Recover amplitude in $N \to \infty,\ \epsilon \to 0^+$ limit

$$\mathcal{M}_{\varphi b} = \lim_{N\to\infty} \lim_{\epsilon\to 0^+} \mathcal{M}_{\varphi b}(N, \epsilon)$$

Several methods

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys.* ***A305*** *(1982) 217*

Solving three-body scattering equations

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S-matrix unitarity provides a way to check quality of solutions

- Deviation from unitarity guides quality of solution

$$\mathrm{Im}\, \mathcal{M}^{-1}_{\varphi b}(E) = -\rho_{\varphi b}(E)$$

$$\Delta\rho_{\varphi b}(E; N) \equiv \left| \frac{\mathrm{Im}\left[\mathcal{M}^{-1}_{\varphi b}(E; N)\right] + \rho_{\varphi b}(E)}{\rho_{\varphi b}(E)} \right| \times 100$$

Several methods

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys.* ***A305*** *(1982) 217*

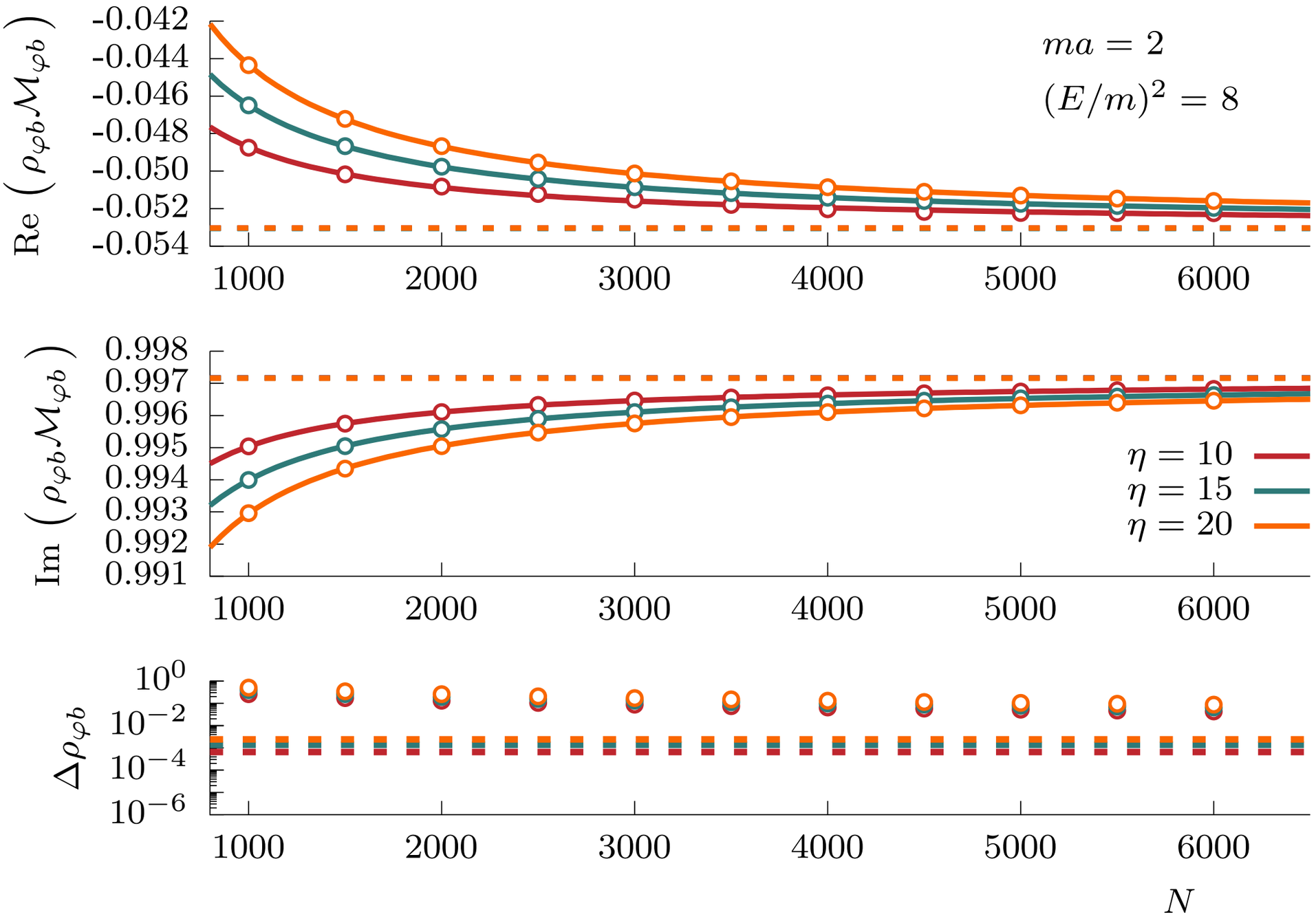
$N \to \infty$ Extrapolations

Compute multiple N solutions — extrapolate to $N \to \infty$ limit

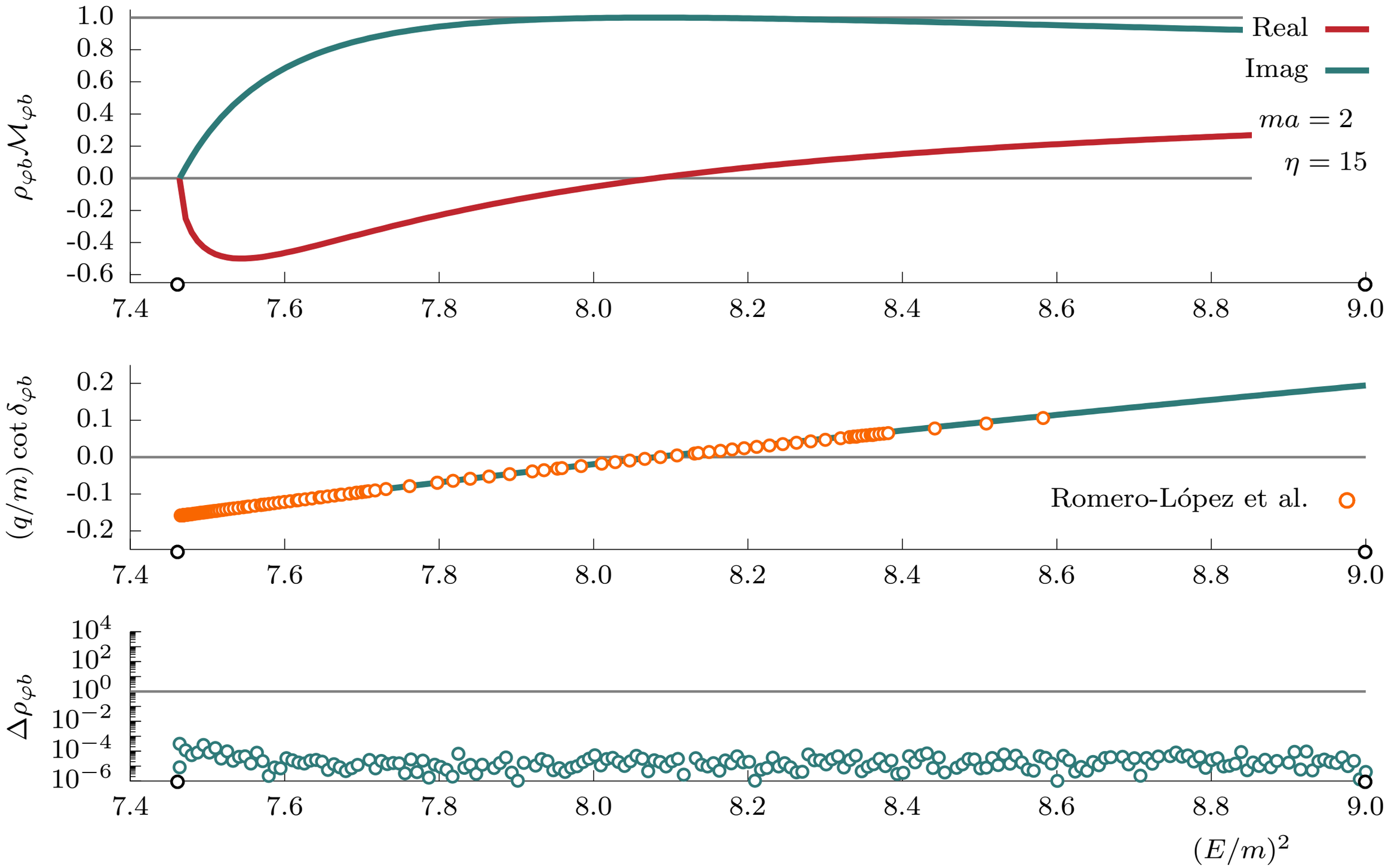
$$\mathcal{M}_{\varphi b}(E;N) \approx \mathcal{M}_{\varphi b}(E) + \frac{\alpha}{N}$$

- Unitarity deviation greatly improved

$$\epsilon \propto \frac{\eta}{N}$$



Example of solution



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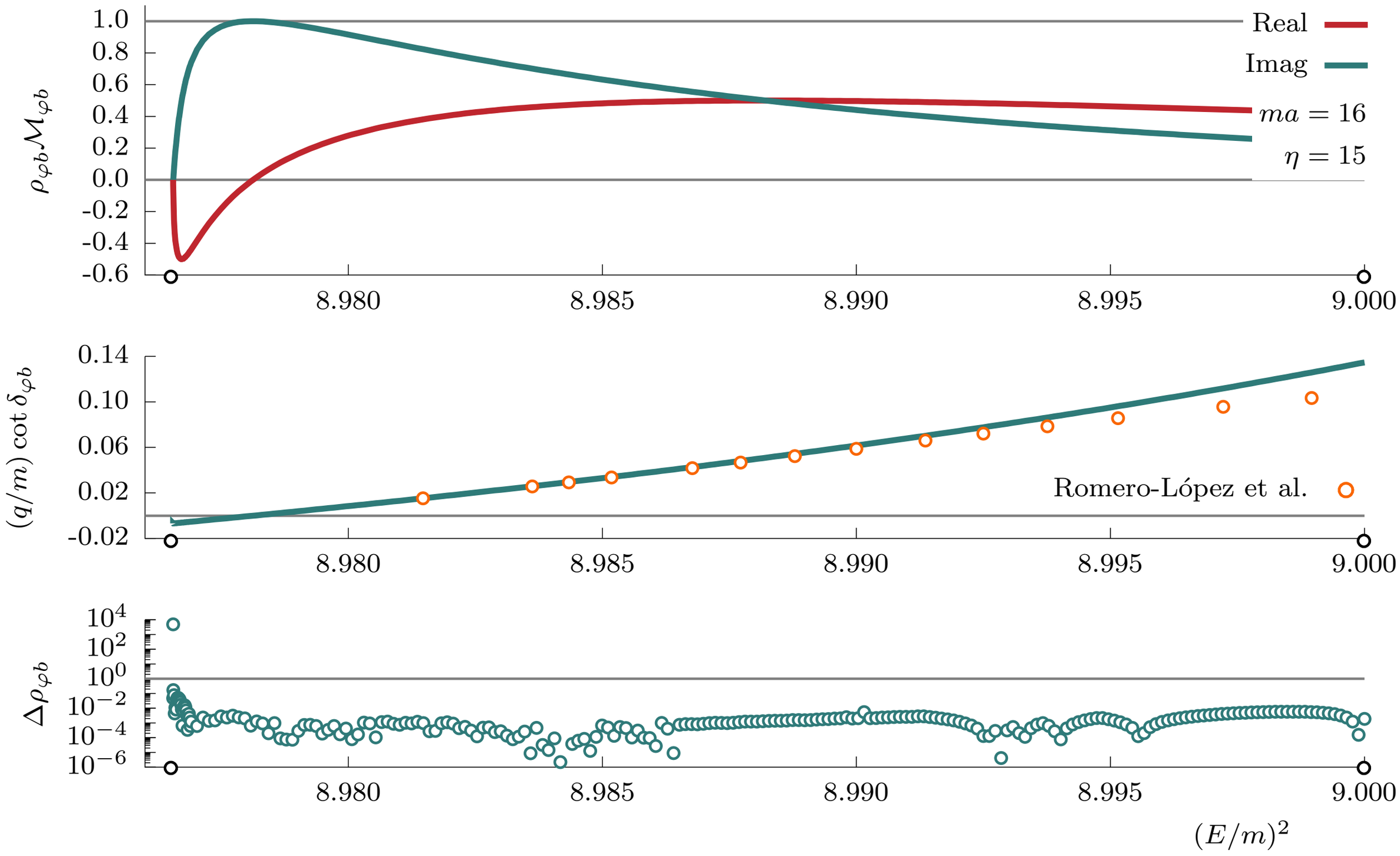
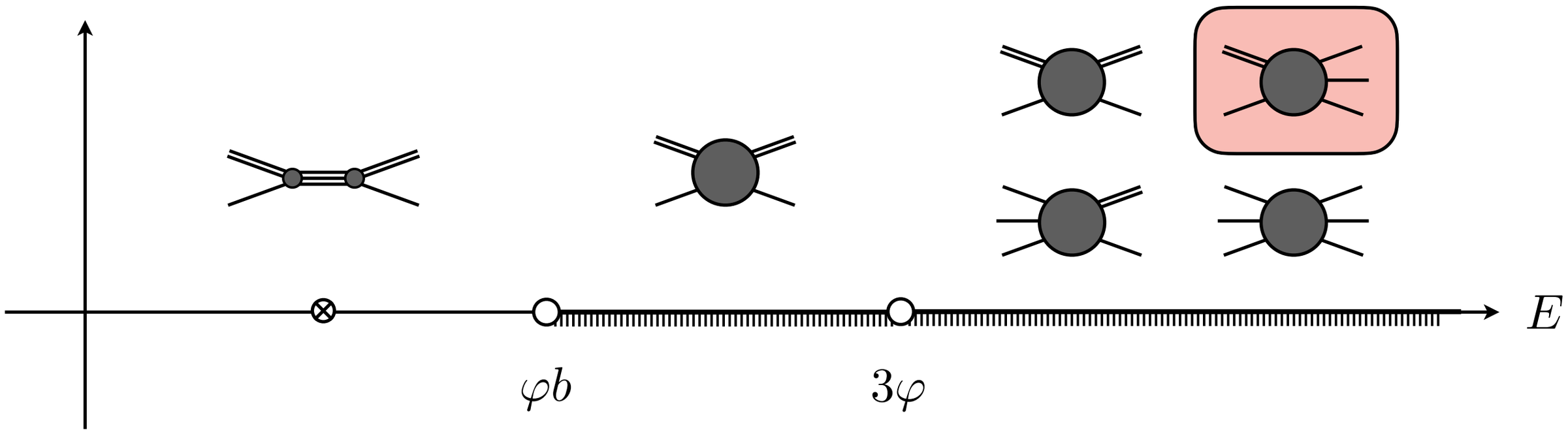


Above the three-body threshold

Methodology not limited to below 3-body threshold

- Allows for calculation of breakup / recombination amplitude

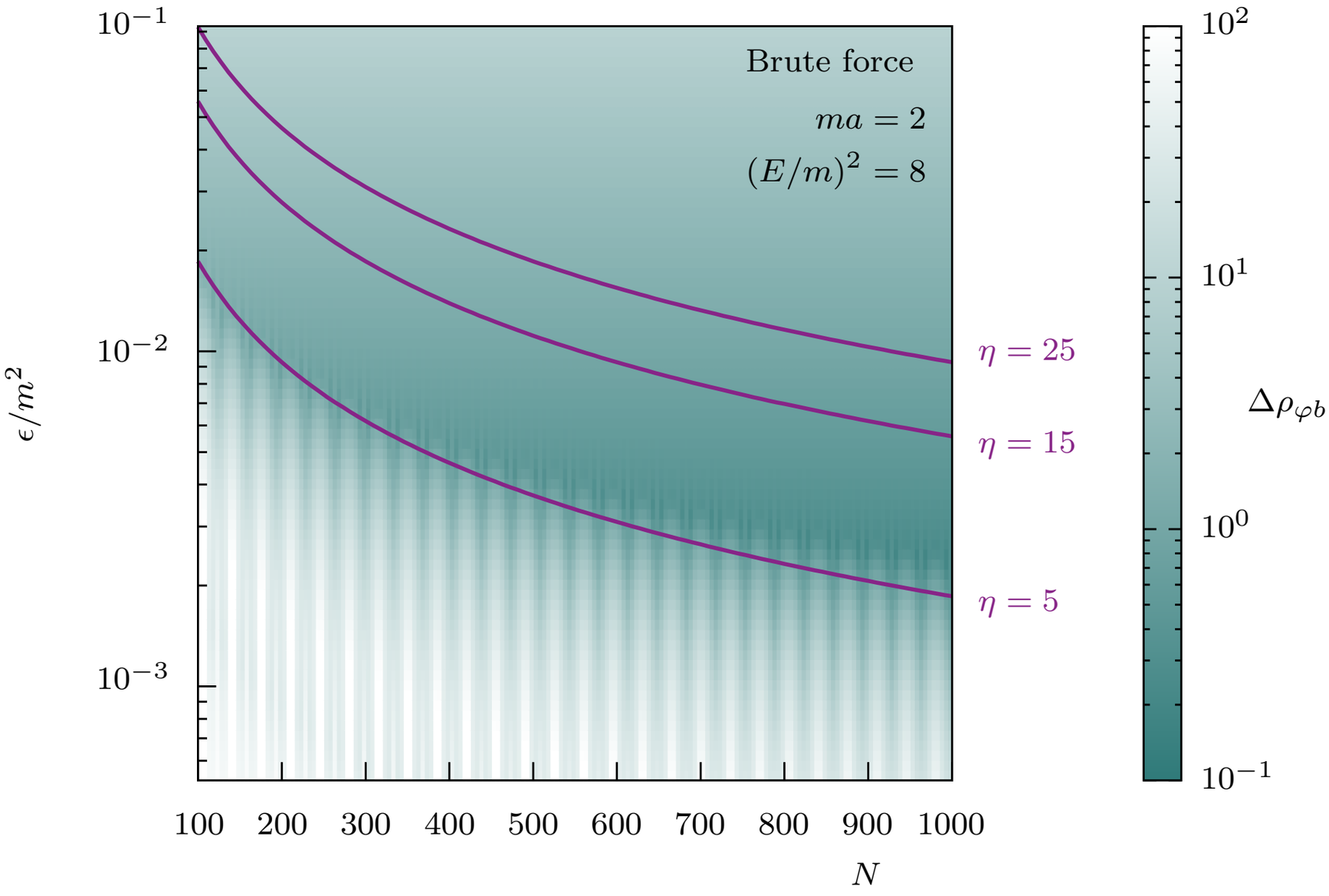
$$\mathcal{M}_{\varphi\to 3\varphi}(\sigma_p) = -\lim_{\sigma_k\to\sigma_b} \frac{\sigma_k-\sigma_b}{g}\,\mathcal{M}_3(\sigma_p,\sigma_k)$$



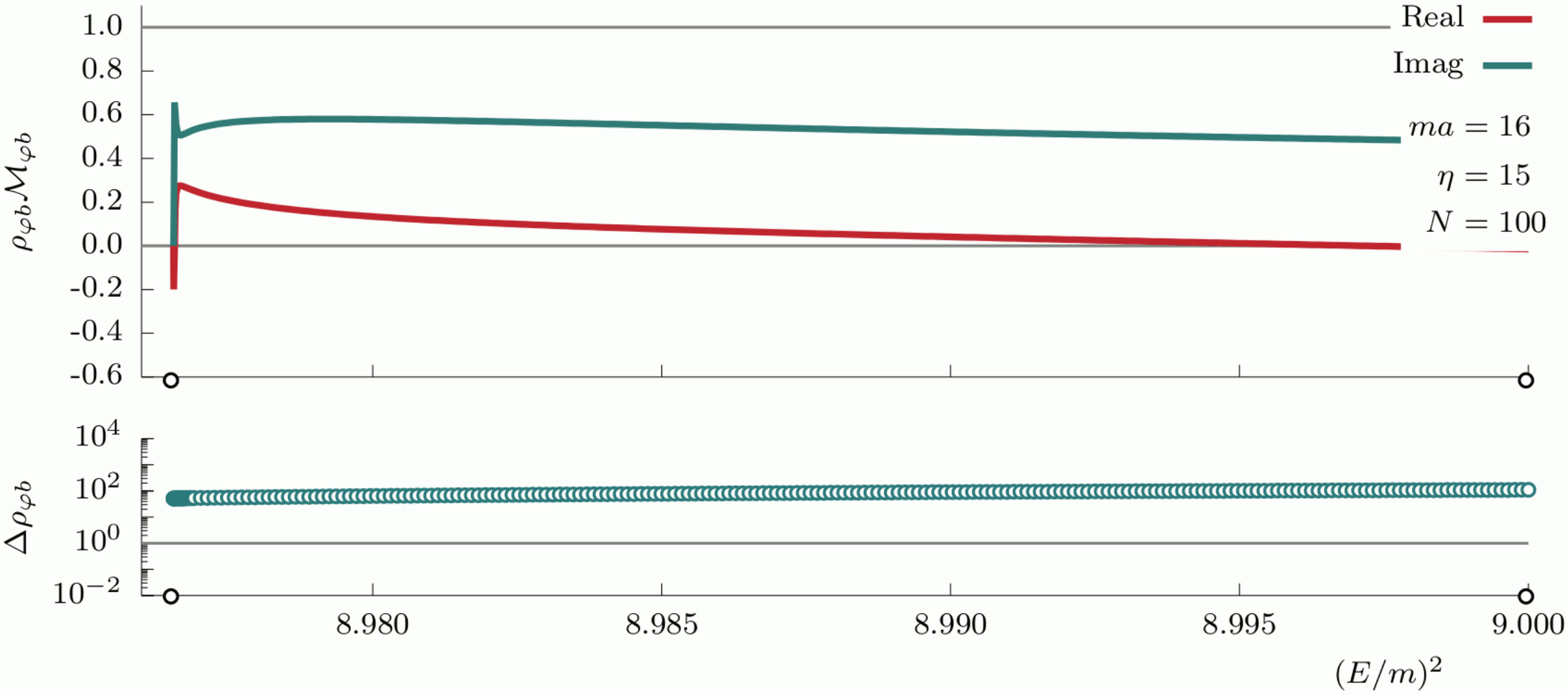
$\epsilon \to 0$ limit

Ensure $\epsilon \to 0$ through $N \to \infty$ limit

$$\left[\sum_x \Delta x - \int \mathrm{d}x\right] \frac{1}{x^2 - x_0^2 + i\epsilon} \sim e^{-2\pi\epsilon/\Delta x} \qquad \Longrightarrow \qquad \epsilon \propto \frac{\eta}{N}$$



Evolution of solutions



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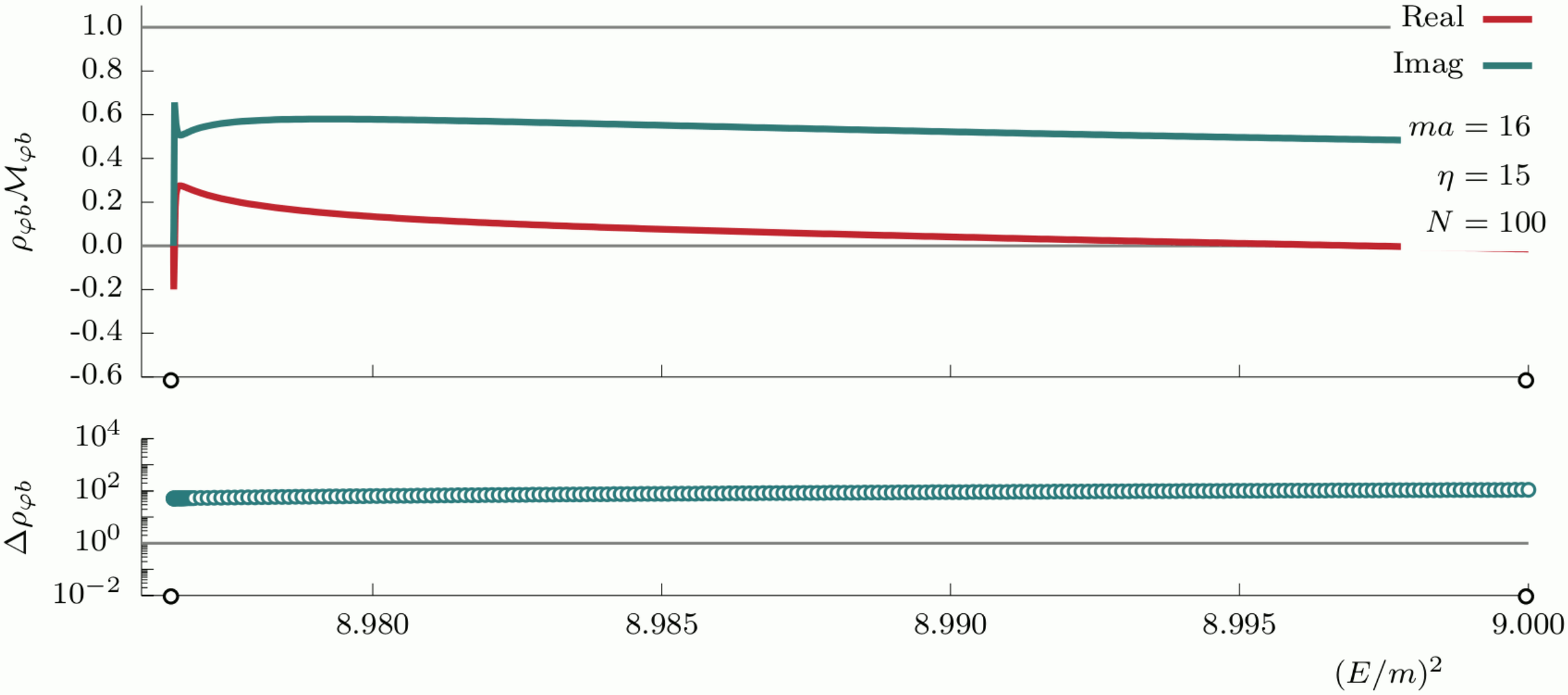
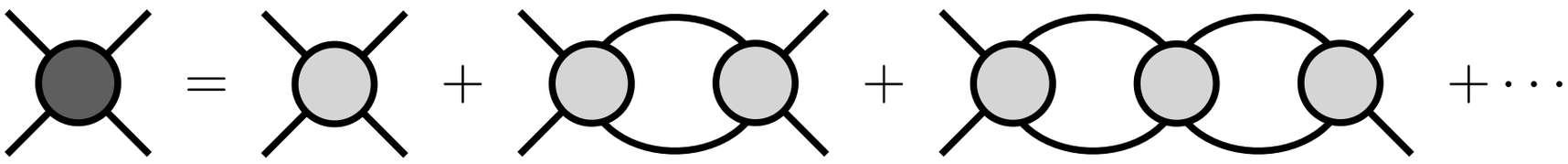


On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2 \to 2}$



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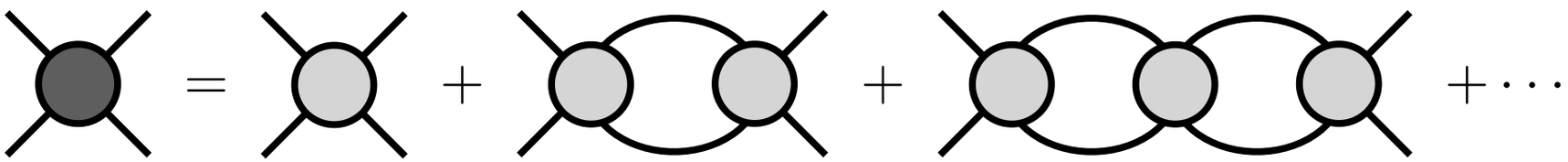

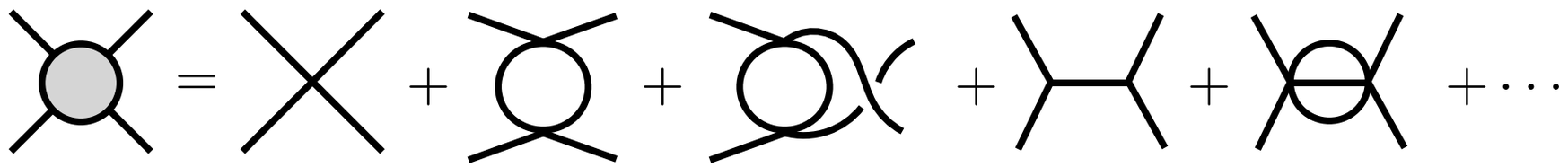

All 2PI diagrams - left hand cuts and higher multi-particle thresholds

On-shell scattering amplitudes from RFT

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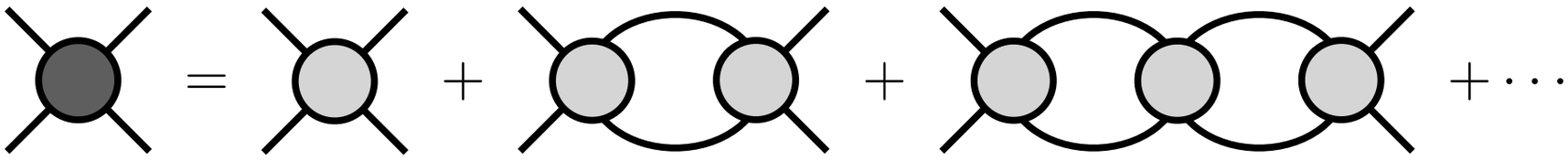

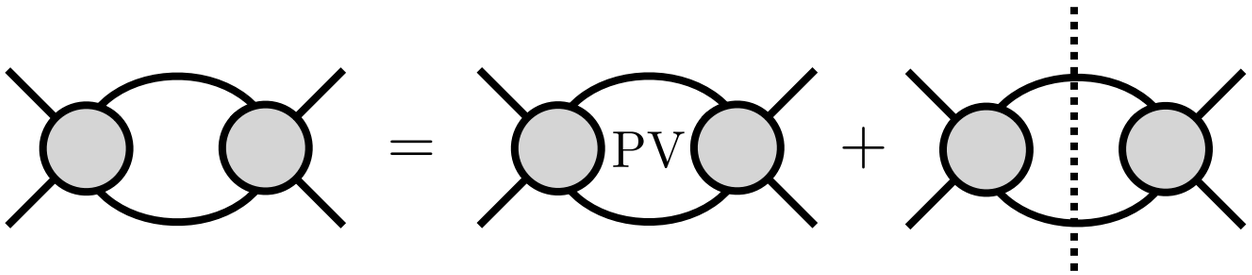

$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\rm th}}$$

On-shell scattering amplitudes from RFT

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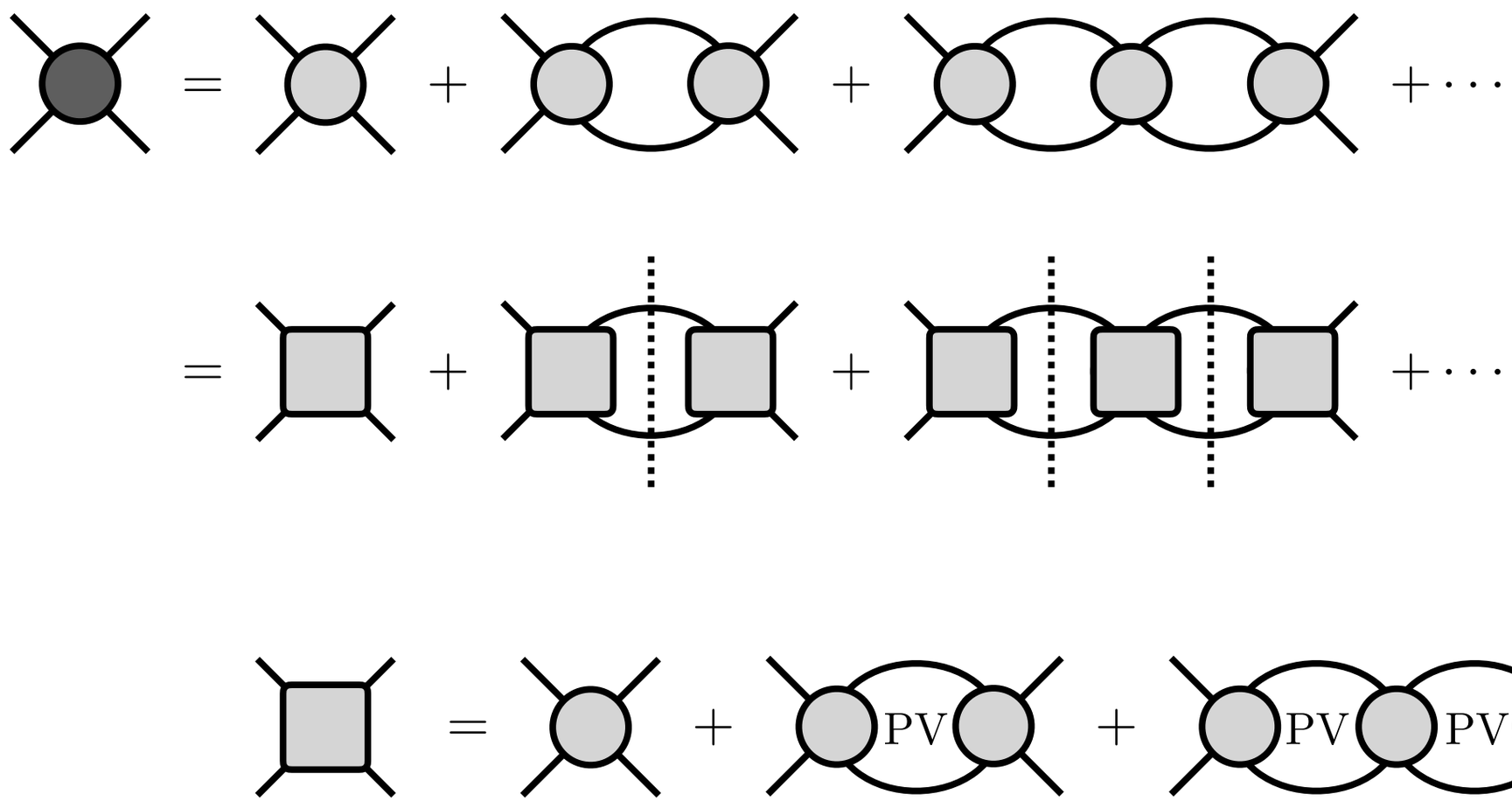

K matrix — unknown dynamical function unconstrained by unitarity

On-shell scattering amplitudes from RFT

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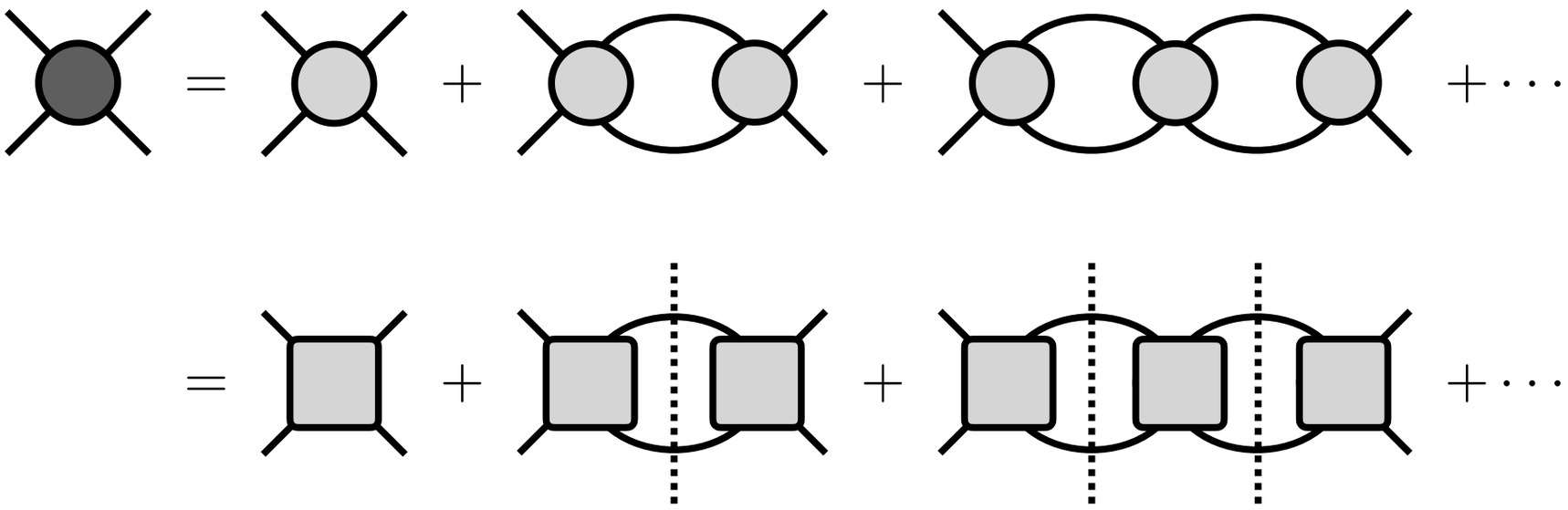


$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i\rho \mathcal{M}_2$$

For given K matrix, obtain on-shell solution for amplitude

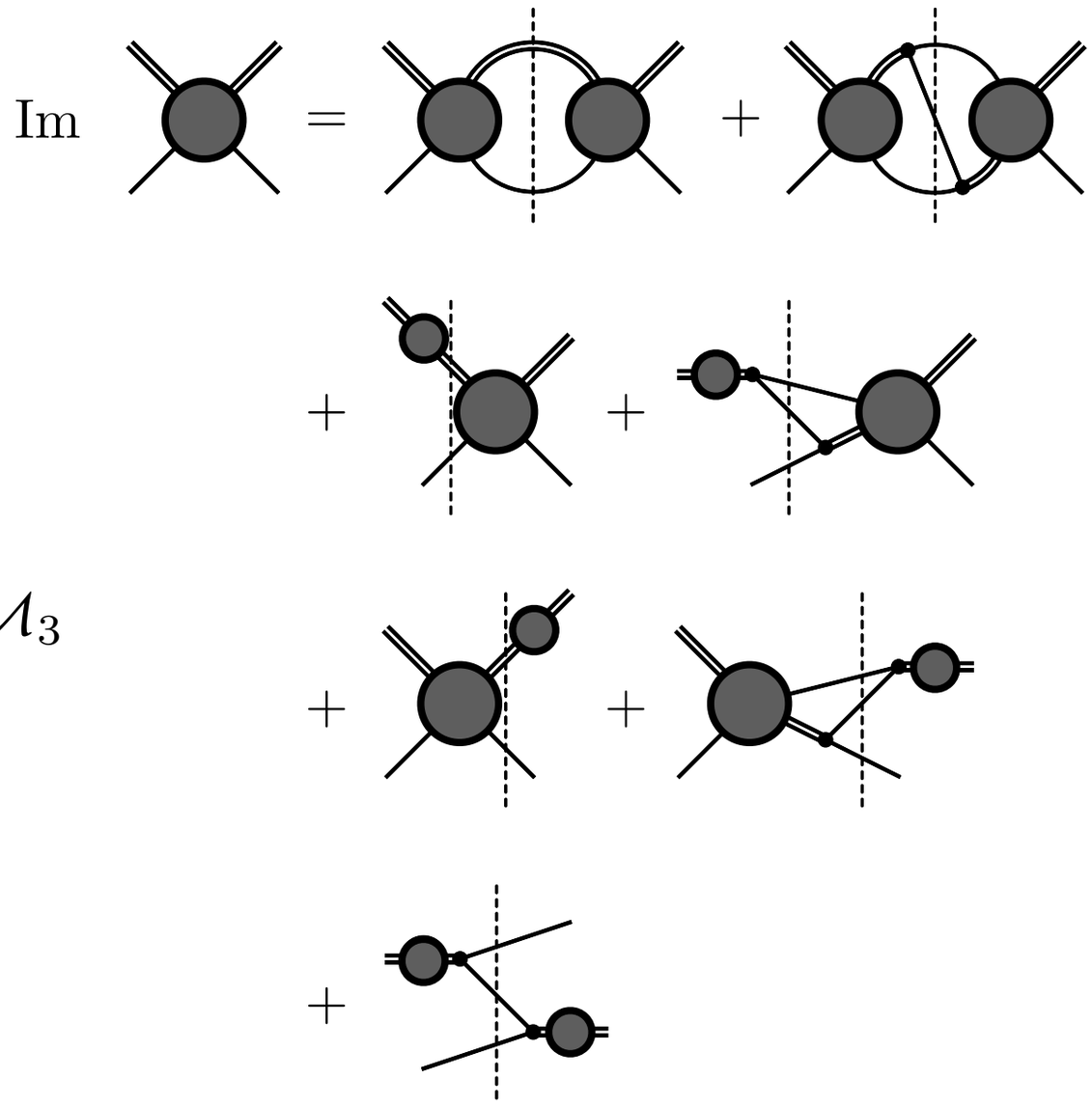
On-shell scattering amplitudes from unitarity

Start with *S* matrix unitarity

- Provides constrain for amplitude on real axis in physical region
- On-shell amplitude satisfies linear equation — check unitarity constraint

$$\mathcal{M}_3 = \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_2 + \int \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_3$$

R is a different short-distance function



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

M. Mikhasenko, AJ et al. [JPAC]
Phys. Rev. D **98**, 096021 (2018)

M. Mikhasenko, AJ et al. [JPAC]
JHEP **08**, 080 (2019)

Extension to FV

M. Mai and M. Döring
Eur. Phys. J. A **53**, 240 (2017)

Equivalence of relativistic methods

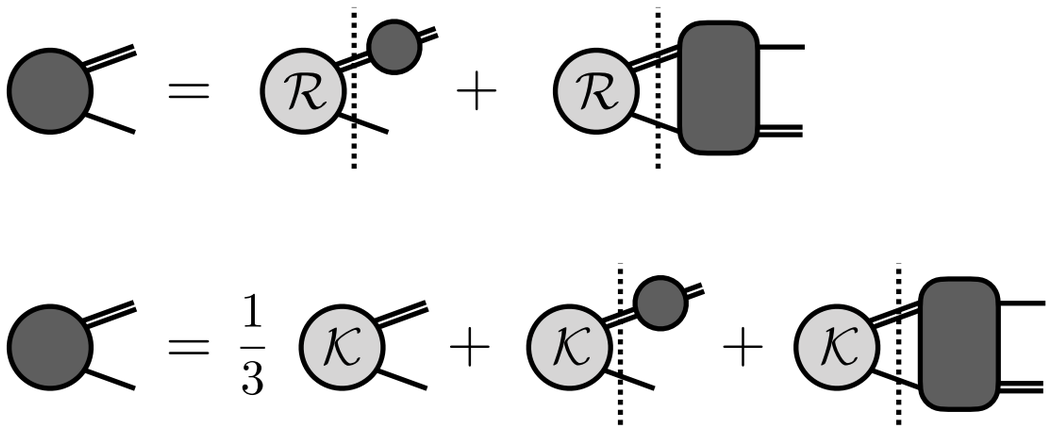
The RFT vs FVU methods can be summarized as "Bottom up" vs "Top down"

On-shell scattering equations are equivalent

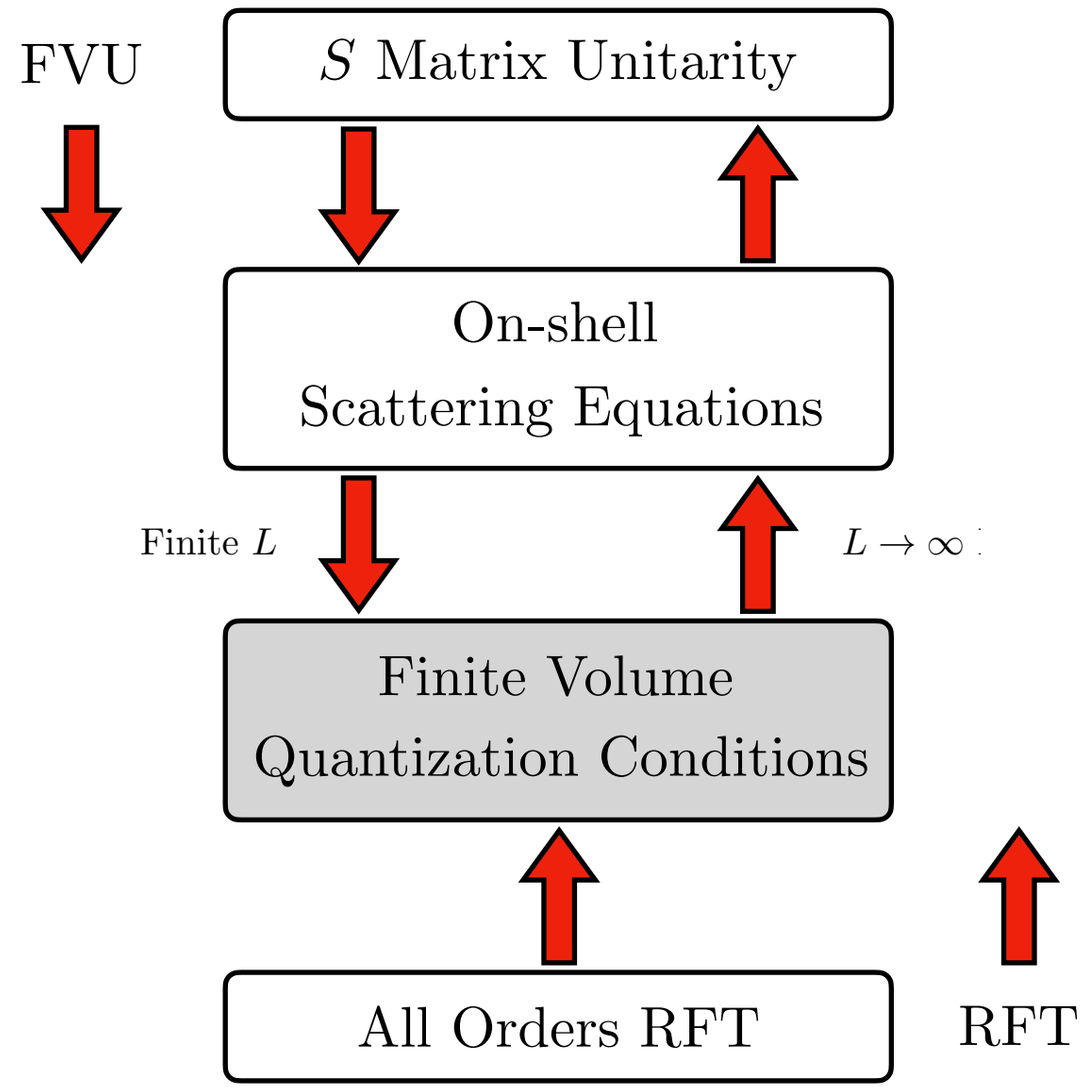
AJ et al.
Phys. Rev. D **100**, 034508 (2019)

Quantization conditions are equivalent

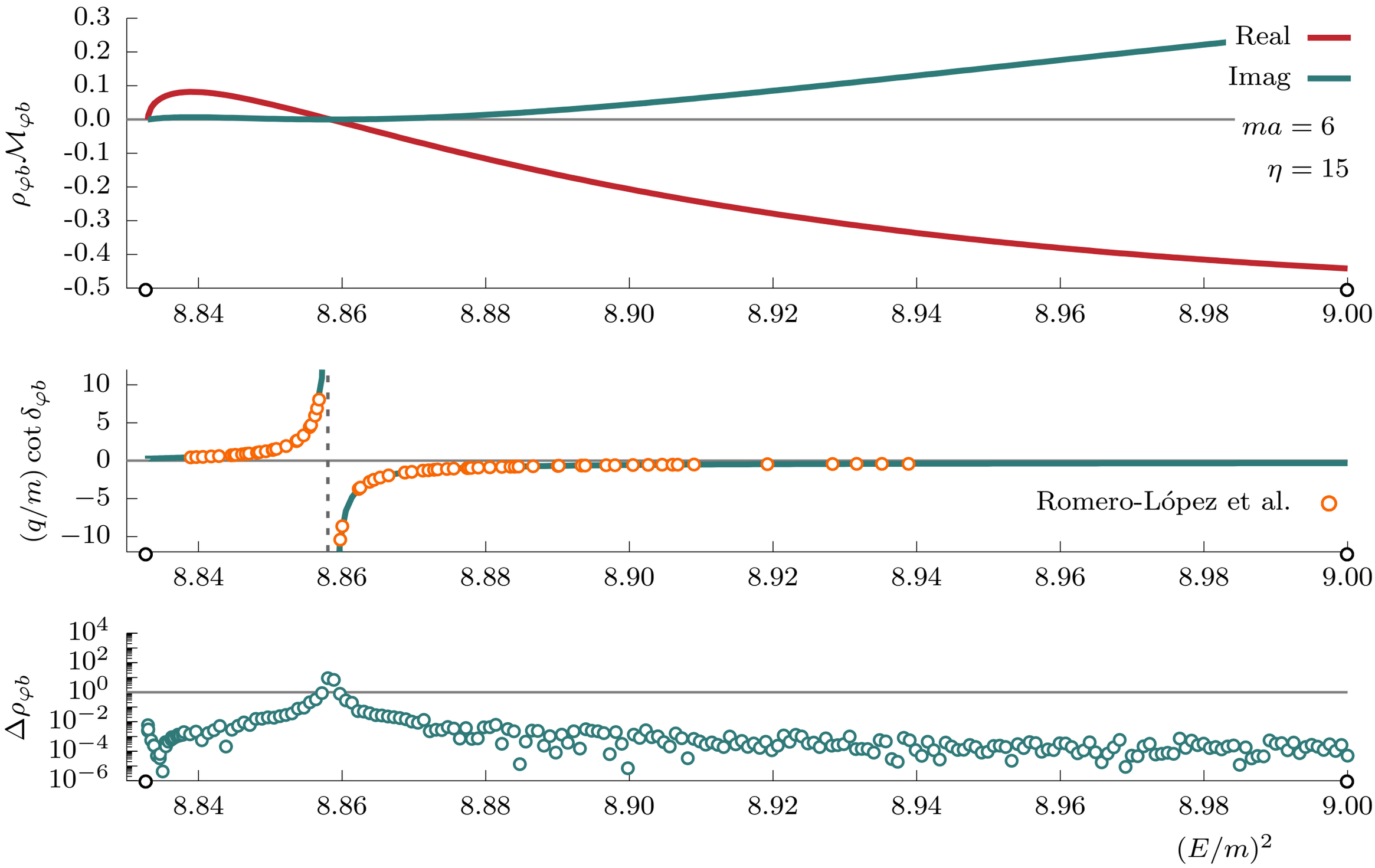
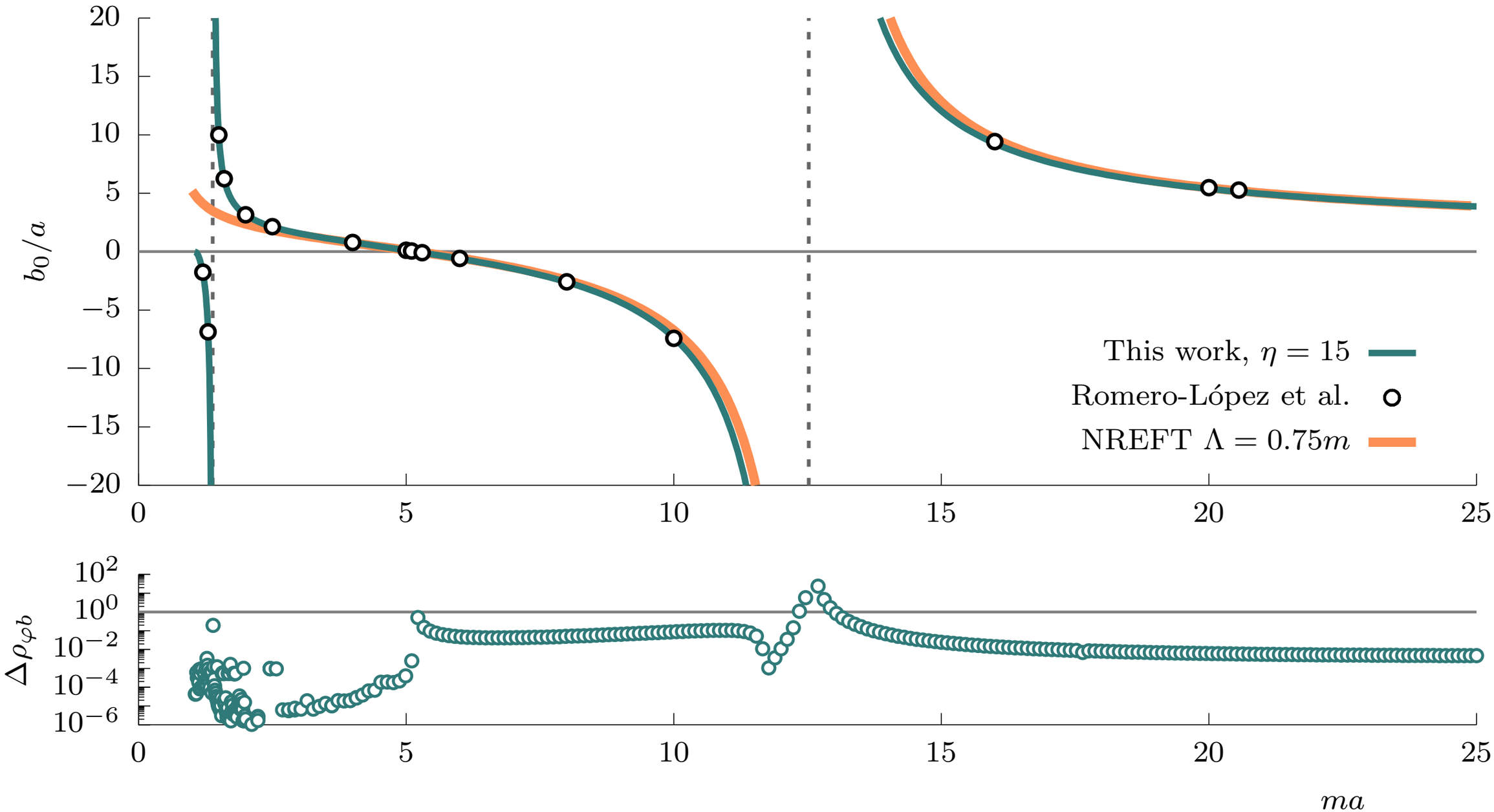
T. Blanton and S. Sharpe
arXiv:2007.16190 (2020)



Small details, e.g. aspects of symmetrization — see literature for information



Example of solution — Extrapolated result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