Quirks of QCD: Twist-2 Operators on the Lattice

Tangereen Velveteen Bailey Claringbold   Chris Monahan

William & Mary

Abstract

When using deep inelastic scattering (DIS) to probe hadronic structure, we can use the operator product expansion to approximate the product of currents, using the operators' twist to suppress higher order effects. On the lattice, these operators experience power divergent mixing, which we aim to control by introducing the gradient flow. We study an example in perturbation theory.

Quantum Chromodynamics

Quantum Chromodynamics or QCD the the quantum field theory of the strong force. Some key feature of QCD are

- describes the interactions of quarks and gluons
- $SU(3)$ color symmetry group
- confinement: at low energy the coupling is large, quarks are constrained to hardons, and perturbation theory is invalid
- asymptotic freedom: at high energy the coupling is small, quarks can be treated as free particles, and perturbation theory can be used

Twist-2 Operators

- Deep Inelastic Scattering or DIS is used to probe the structure of hadrons.
- A high energy lepton scatters off of a quark within the hadron.
- Calculations require finding a matrix element of the product of quark currents.
- This is done through the Operator Product Expansion or OPE, is controlled by the twist of the operators.
- The twist t of an operator of mass dimension d and spin s has a twist $t = d - s$.
- This project deals with twist-2 operators.

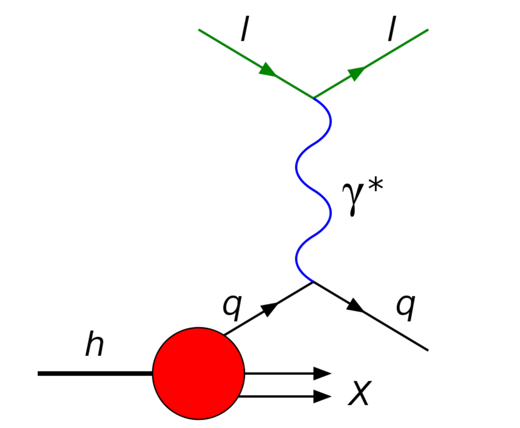


Figure 1. A diagram showing the process of deep inelastic scattering where a lepton l scatters off of a quark q inside of a hardon h. [1]

The Lattice Regulator

- QCD is non-perturbative at low energies and many quantities that we might which to theoretically calculate cannot be carried out analytically
- Do calculations numerically using a lattice regulator.
- Transform the theory onto a Euclidean lattice, with the quarks sitting on the lattice cites and the gluons on the links between them.
- The lattice also serves as a regulator for the divergences of the theory, parameterized by the lattice spacing a.
- The lattice breaks rotational symmetry,leading to the twist-2 operators to have power-divergent mixing under renormalization.

The Gradient Flow

In order to use the twist-2 operators with the lattice regulator, we can restore an approximate rotational symmetry by smearing the quark and gluon fields in physical space, rather than lattice space, by the flow time τ. From [3], we replace the quark fields $\psi, \bar{\psi}$ and the gluon field A_μ to with the smeared fields $\chi, \bar{\chi}$ and B_μ, such that the smeared fields match the unsmeared fields at flow time $\tau = 0$ and satisfy

$$\partial_\tau \chi = D_\mu D_\mu \chi \qquad \partial_\tau \bar{\chi} = \bar{\chi} \overleftarrow{D}_\mu \overleftarrow{D}_\mu \tag{1}$$

$$\partial_\tau B_\mu = D_\nu(\partial_\nu B_\mu - \partial_\mu B_\nu + [B_\nu, B_\mu]) \tag{2}$$

Solving these differential equations leads to different Feynman rules from the usual QCD ones. In the Feynman gauge, the quark and gluon fields pick up Gaussian factors in their momentum, parameterized by the flow time. The QCD interactions are only changed by replacing the original fields with the flowed fields and all occur at flow time $\tau = 0$. In addition, there are two kernels, one for the quarks and one for the gluon, and four flowed interaction vertices that couple propagators and kernels.

The Hyperoctahedral Group

For this project, we need two twist-2 operators that exhibit power-divergent mixing so that we can test if applying the gradient flow controls the mixing. To find two such operators, we examine the symmetry group of the Euclidean lattice, the hyperoctahedral group of order 4 or $H(4)$. There are 20 irreducible representations of $H(4)$, as described in [2]. Operators of definite twist can be constructed to transform under one of these irreducible representations. In this way, operators in different irreducible representations *do not* exhibit power-divergent mixing under renormalization; however, we wish to use two operators that *do* exhibit power-divergent mixing. We take an example of two operators in the same irreducible representation of $H(4)$ and check that they have the desired mixing before applying our prescription. We chose the simplest two such twist-2 operators which fall in the $\tau_1^{(4)}$ representation, following the convention of [2], with charge conjugation $\mathcal{C} = -1$: the rank-1 operator

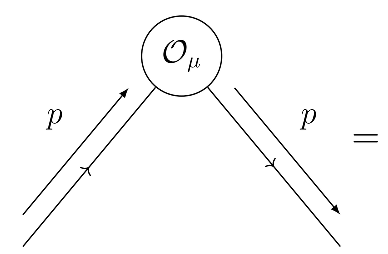
$$\mathcal{O}_\mu = \bar{\psi}\gamma_\mu\psi \tag{3}$$

and the rank-3 operator

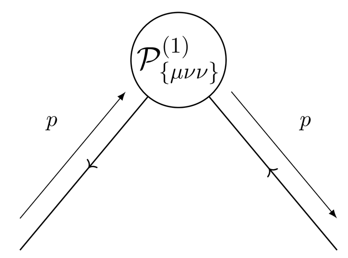
$$\mathcal{P}_{\{\mu\nu\nu\}} = \frac{1}{\sqrt{2}}\sum_{\nu=1}^{4} \bar{\psi}\gamma_{\{\mu}\overleftrightarrow{D}_\nu\overleftrightarrow{D}_{\nu\}}\psi = \frac{1}{\sqrt{2}}\sum_{\nu=1}^{4}\left(\bar{\psi}\gamma_\mu\overleftrightarrow{D}_\nu\overleftrightarrow{D}_\nu\psi + \bar{\psi}\gamma_\nu\overleftrightarrow{D}_\mu\overleftrightarrow{D}_\nu\psi + \bar{\psi}\gamma_\nu\overleftrightarrow{D}_\nu\overleftrightarrow{D}_\mu\psi\right). \tag{4}$$

Continuum without the Gradient Flow

For our example operators, we begin by checking their behavior in the continuum without the gradient flow. If there will be power-divergent mixing, there must be a term in one of the 1-loop diagrams that is proportional to the tree-level of the other diagram. In this case, we find that the 1-loop diagram of $P_{\{\mu\nu\nu\}}$ is proportional to the tree level diagram of O_μ. The tree-level diagrams for each operator are

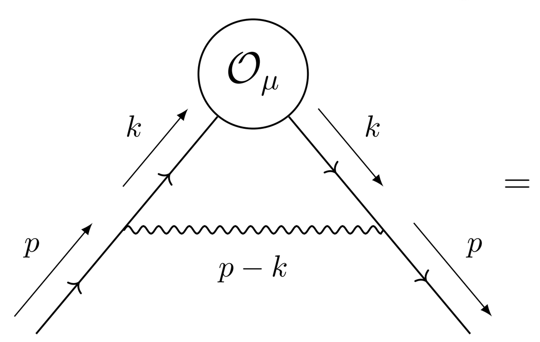


$$= \Gamma_{\mathcal{O}}^{(\text{tree})} = \gamma_\mu \quad \text{and}$$

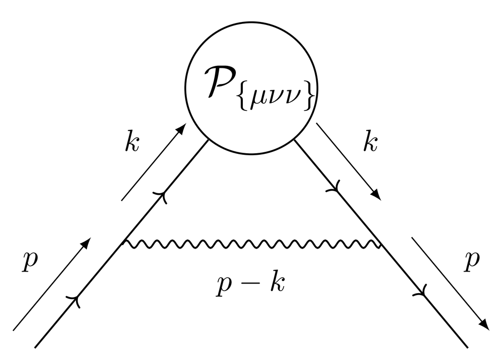


$$= \Gamma_{\mathcal{P}}^{(\text{tree})} = -2\sqrt{2}(\gamma_\mu p^2 + 2\not{p}p_\mu). \tag{5}$$

At 1-loop, O_μ will have one diagram and $P_{\{\mu\nu\nu\}}$ will have three, a vertex diagram and a sail diagram and its mirror (plus a tadpole diagram which vanishes). Evaluating these, we find

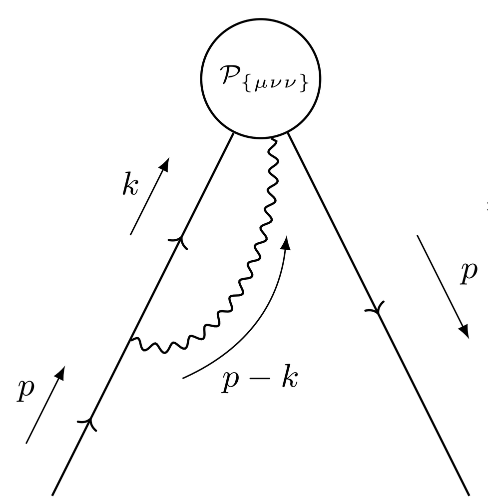


$$= \Gamma^\mu_{\mathcal{O},1} = \frac{ig^2C_F}{(4\pi)^2}\gamma^\mu\left[\frac{1}{\epsilon} - \gamma_E - 1 + \log(4\pi) - \log\left(\frac{p^2}{\mu^2}\right)\right], \tag{6}$$



$$= \Gamma^\mu_{\mathcal{P},1,1} = \frac{-2i\sqrt{2}g^2C_F}{(4\pi)^2}(\gamma^\mu p^2 + 2\not{p}p^\mu), \tag{7}$$

and



$$= \Gamma^\mu_{\mathcal{P},1,3} = \sqrt{2}g^2C_F\left[\tfrac{(2D+4)\gamma^\mu}{D}\int_0^1 \mathrm{d}x\ \left(J_2^{(2)} + x^2J_2^{(0)}\right) + \left((4-2D)\not{p}p^\mu + 4p^2\gamma^\mu\right)\iint_0^1 \mathrm{d}x\ xJ_2^{(0)}\right]. \tag{8}$$

From these diagrams, we can see that both $\Gamma^\mu_{\mathcal{P},1,1}$ and $\Gamma^\mu_{\mathcal{P},1,3}$ contain terms proportional to $\Gamma^{(\text{tree})}_{\mathcal{O}} = \gamma^\mu$. So $\mathcal{O}_\mu$ and $P_{\{\mu\nu\nu\}}$ will power-divergently mix in the continuum without the gradient flow, which is what we were hoping to find for our example.

Continuum with the Gradient Flow

Next, we repeat the calculation of the fermion bilinears of our example operators in the continuum with the gradient flow. The tree level diagrams pick up Gaussian factors from the flowed quark propagators so

$$\Gamma_{\mathcal{O}}^{\text{tree}} = \gamma_\mu e^{-2\tau p^2} \text{ and } \Gamma_{\mathcal{P}}^{(\text{tree})} = -2\sqrt{2}(\gamma_\mu p^2 + 2\not{p}p_\mu)e^{-2\tau p^2}, \tag{9}$$

where τ is the flow time of the operator's vertex.

At 1-loop order with the gradient flow, there are now three diagrams for the $\mathcal{O}_\mu$ operator and five for the $\mathcal{P}_{\{\mu\nu\nu\}}$ operator (plus a tadpole diagram which vanishes).

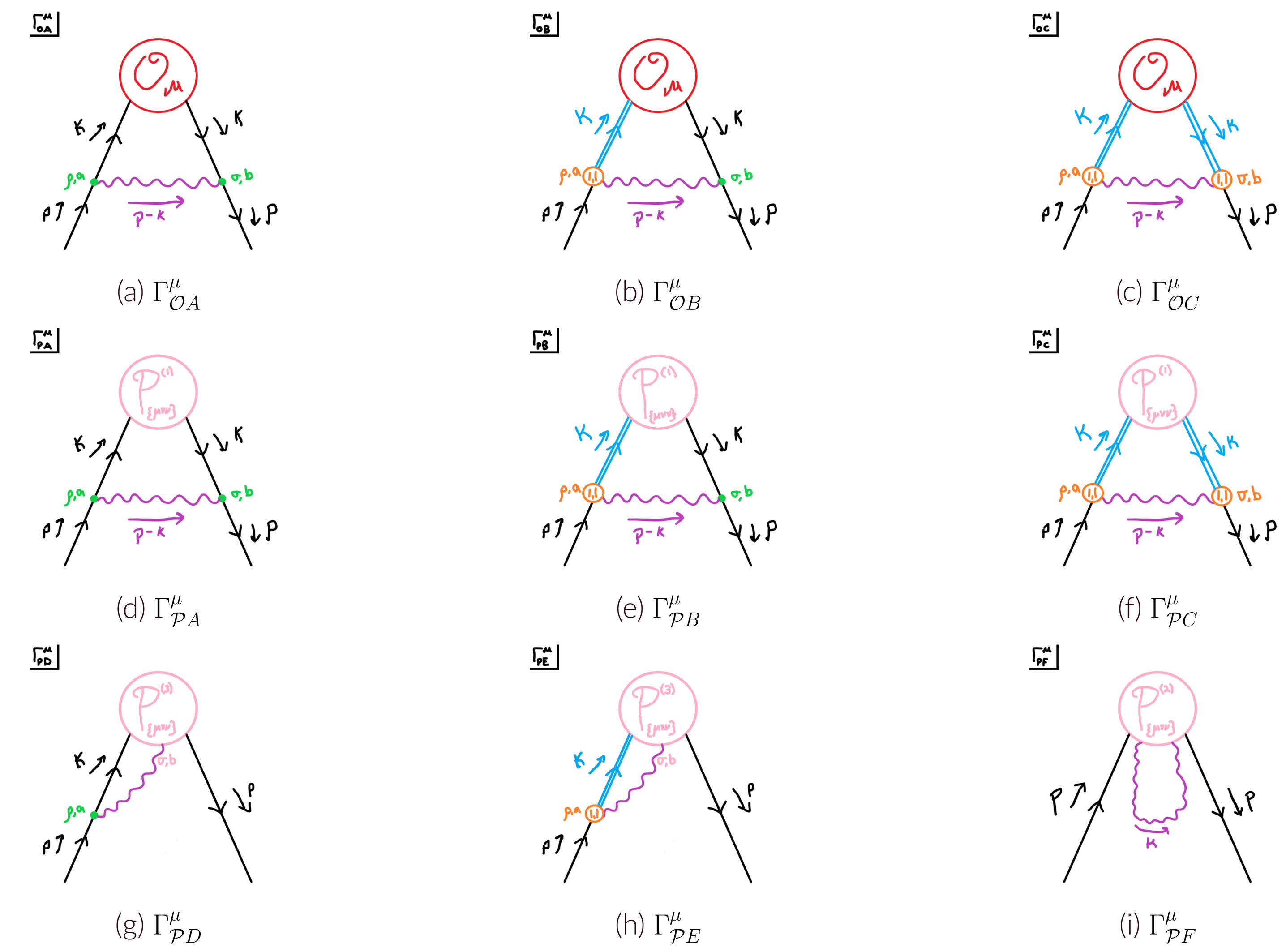


Figure 2. The leading order corrections to $\overline{\mathcal{O}}^\mu$ and $\overline{\mathcal{P}}^\mu$.

Importantly, we can see that the one of the $\mathcal{P}_{\{\mu\nu\nu\}}$ diagrams contains a term proportional to the tree level diagram of O_μ:

$$\Gamma^\mu_{\mathcal{P}A} = 2\sqrt{2}g^2C_F[a(p^2\gamma^\mu + 2p^\mu\not{p}) + b\not{p}\gamma^\mu] \tag{10}$$

where

$$a = m^2p^2K_{2,4}(2t) + \left(\frac{(D-2)(D+1)}{D}p^2 + m^2\right), K_{2,2}(2t) - \frac{(D-2)^2}{D}K_{2,0}(2t) \tag{11}$$

$$b = \frac{2imp^2(4-D)}{D}K_{2,2}(2t), \text{ and } K_{n,l}(x) = \int_k \frac{e^{-xk^2}}{(k^2+m^2)^n k^l}. \tag{12}$$

Conclusion

So far, we have demonstrated that our example operators will exhibit power-divergent mixing under renormalization in both the continuum with and without the gradient flow. This is promising because we want to find that they will also have this kind of mixing on the lattice without the gradient flow. Demonstrating this is the next step in this project. After that, all that remains is to repeat the calculation once more on the lattice but with the gradient flow. If our hypothesis is correct, we will see that the operators no longer exhibit power-divergent mixing in this case, due to the pseudo-restoration of the rotational symmetry to the lattice from the smearing of the fields by the gradient flow.

References

[1] Em2. Deep inelastic scattering. *Wikipedia Commons*, 2007.

[2] M. Göckeler, R. Horsley, E. M. Ilgenfritz, H. Perlt, P. Rakow, G. Schierholz, and A. Schiller. Lattice operators for moments of the structure functions and their transformation under the hypercubic group. *Physical Review D*, 54(9):5705–5714, Nov 1996.

[3] Martin Luscher. Chiral symmetry and the Yang–Mills gradient flow. *JHEP*, 04:123, 2013.