Guesstimation:
Back of the envelope solutions to the world’s problems
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Why estimate?

We are continually bombarded by numbers, good, bad and bogus.
- 1000 tons of microplastics fall on US protected areas annually
- Every American produces 100 cubic-feet of garbage a year

Do they make sense?
How should we respond?

Answers/estimates lead to actions.

Estimates fall in one of three ‘Goldilocks’ categories:
1) Too big
2) Too small
3) Just right (in the middle)

‘Too big’ or ‘Too small’ lead to obvious actions.
‘In the middle’ requires more thought (and perhaps even actual calculation).
How to estimate (almost) anything

1) Write down the answer (to within a factor of 10).
   1) Establish lower and upper bounds,
   2) Then take the geometric mean for your answer.

How many clowns can fit in a Cooper Mini?

\[ 1 < n_{\text{clowns}} < 100 \]
\[ \rightarrow n_{\text{clowns}} \approx \sqrt{1 \times 100} = 10 \]

This ensures that your estimate will be within a factor of 10.

2) If you can’t write down the answer, break the question into smaller pieces and go to step 1

Dare to be imprecise!
Making Arithmetic Easy

Million = 1,000,000 = 10^6
Billion = 1,000,000,000 = 10^9
Trillion = 1,000,000,000,000 = 10^{12}

Counting zeroes gets tiresome,
→ use scientific notation

\[(3 \times 10^4) \times (2 \times 10^5) = (2 \times 3) \times 10^{4+5} = 6 \times 10^9\]

Arithmetic is now easy!

• Worry mostly about the exponents
• Use only ONE significant figure

3 × 3 = 10
9 / 4 = 2
\(\sqrt{10} = \pi\)
Millions, billions, trillions, gazillions
What do these big numbers really mean?

- The Jefferson Lab budget is about $100 Million
- The market capitalization of Uber about $93 Billion
- The federal budget deficit this year is a few $Trillion

Consider seconds:
One day = 24 hours × 60 min/hr × 60 s/min ~ 10^5 s
Thus 1 million sec = 10^6 s × (1 day / 10^5 s) = 10 days
1 billion sec = 10^9 s × (1 day / 10^5 s) = 10^4 days
~ 30 years
1 trillion sec ~ 3 × 10^4 years
Dear news organizations: Stop giving large numbers without context or proper comparison. The difference between a million and a billion is the difference between a sip of wine and 30 seconds with your daughter and a bottle of gin and all night with her.
How many people in the world are listening to a speaker now?

picking their nose now?
How many people in the world are listening to a speaker picking their nose now?

The fraction of time you spend doing something typical equals the fraction of people doing that something right now.

Time spent nose picking:

\[1 < t < 100 \text{ s/day}\]

\[\rightarrow t \approx \sqrt{1 \times 100} = 10 \text{ s/day} = \frac{10 \text{ s}}{10^5 \text{s}} = 10^{-4}\]

\[(7 \times 10^9) \times (10^{-4}) = 7 \times 10^5 = \text{almost a million people!}\]
What is the kinetic energy of a 1 km asteroid hitting the Earth (the dinosaur-killer)?

We need to estimate the mass and the velocity.

Assume an iron asteroid (divide by 3 for a rocky one):

\[ M = \rho V = (10^4 \text{kg/m}^3) \times (10^3 \text{m})^3 = 10^{13} \text{kg} \]

The asteroid’s velocity comes from the Sun’s gravity. So does the Earth’s velocity:

\[ V_\text{asteroid} \approx V_\text{Earth} = \frac{2\pi R_{\text{Earth-Sun}}}{1 \text{ year}} = \frac{2\pi \times 1.5 \times 10^{11} \text{m}}{\pi \times 10^7 \text{s}} = 3 \times 10^4 \text{ m/s} \]

\[ KE = \frac{1}{2} M v^2 = 0.5 \times (10^{13} \text{kg})(3 \times 10^4 \text{ m/s})^2 = 5 \times 10^{21} \text{J} \]

This is \(10^6\) megatons of TNT!
Why do we drive gasoline-powered cars?

Gasoline energy density:

C (coal) < gasoline < CH₄ (methane) \(\Rightarrow\) CH₂

1.5 eV per chemical reaction: \(\text{CH}_2 + \text{O} \Rightarrow \text{CO}_2 + \text{H}_2\text{O}\)

At 14 g/mole, 1 kg contains \(10^2\) moles

Energy per kg:

\[
e = N_A e_{\text{reac}}
= \left(10^2 \frac{\text{mole}}{\text{kg}}\right) \left(6 \times 10^{23} \frac{\#}{\text{mole}}\right) (3 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV})
= 3 \times 10^7 \text{ J/kg}
\]

Reality: \(e = 4.5 \times 10^7 \text{ J/kg} = 3 \times 10^7 \text{ J/L}\)

Battery energy density: 100--200 Wh/kg \(\approx 4 - 7 \times 10^5 \text{ J/kg}\)

100 times less!
Why do we drive gasoline-powered cars (2)?

Gasoline energy transfer rate:

At the gas pump:
Fuel transferred: \( \sim 12 \text{ gal} = 50 \text{ L} \)
Time: \( 10 \text{ s} < t < 10^3 \text{ s} \)

\[
P = \frac{E}{t} = \frac{em}{t} = \frac{(3 \times 10^7 \text{ J/L}) \times (50 \text{ L})}{10^2 \text{ s}} = 10^7 \text{ W}
\]

That is 10 MegaWatts!

Home power cords only carry \( 10^3 \text{ W} \)
We’ll need a very thick power cord!
Can we grow our gasoline?

How much cropland would we need to grow corn for ethanol?

How much energy do we each consume?
How much energy do our cars consume?

People: 2500 Calories / day.

\[ 2.5 \times 10^3 \text{Cal} = 2.5 \times 10^6 \text{cal} = 10^7 \text{J} \]

Cars: 1 to 2 gallons (4 to 8 L) of gas / day

\[ (6 \text{ L}) \times (3 \times 10^7 \text{J/L}) = 2 \times 10^8 \text{J} \]

We would need **20 times** more cropland to grow all our ethanol.
Can we grow our gasoline?

How much cropland would we need to grow corn for ethanol?

How much energy do we eat?

How much energy do our cars consume?

People: 2500 Calories/day. $2.5 \times 10^3 \text{ Cal} = 2.5 \times 10^6 \text{ cal} = 10^7 \text{ J}$

Cars: 1 to 2 gallons (4 to 8 L) of gas/day ($6 \text{ L} \times (3 \times 10^7 \text{ J/L}) = 2 \times 10^8 \text{ J}$)

We would need 20 times more cropland to grow all our ethanol.

Why are we burning food in our cars?

Excuse me. I’m going to need this to run my car.

Why are we burning food in our cars?
How many foul balls land in beer cups at ball games?
How many foul balls land in beer cups at ball games?

- Foul balls per game: $1 < n < 10^3 \Rightarrow n = 30$
- Not caught by fan: $0.1 < c < 1 \Rightarrow c = 0.3$
- Seats w/ beer: $0.01 < b < 1 \Rightarrow b = 0.1$
- Cup: seat area: $a = (0.1 \text{ m})^2 / (1 \text{ m})^2 = 10^{-2}$

$$N = ncba = (30 / \text{ game})(0.3)(0.1)(10^{-2})$$

$$= 10^{-2} / \text{ game}$$

30 teams, 80 games/team-season

$\Rightarrow 2000 \text{ games/season}$

$\Rightarrow 20 \text{ beer-landings/season}$
What do windmills cost?

Dominion Virginia Power (2019) is building two windmills that can power 3000 homes for $300 million

Is this a lot or a little?

Windmill lifespan: $10 \text{ yrs} < T < 100 \text{ yrs}$

$\Rightarrow T \approx 30 \text{ years}$

Monthly electric bill: $10 < \text{bill} < 10^3$

$\Rightarrow$ $100$/month $\Rightarrow$ $10^3$/year

Value of electricity provided:

$V = (3 \times 10^3 \text{ homes})(30 \text{ yrs})(10^3 /\text{home yr}) = 10^8$

$10^8 < 3 \times 10^8$

OOPS!
How much trash does the US generate yearly?
How much trash does the US generate yearly?

1 < trash < 100 pounds per person per week
⇒ 500 lbs / person - year
Double it to include businesses ⇒ 1000 lbs / person - year
(1/2 ton / person-year) × (3 × 10^8 people) = 2 × 10^8 tons/year

Landfill space needed for a century’s trash?

\[ V = \frac{(2 \times 10^8 \text{ ton/yr}) \times (10^2 \text{ yr})}{0.2 \text{ ton/m}^3} = 10^{11} \text{ m}^3 \]

10 m < h < 10^3 m \quad \rightarrow \quad A = V / h = 10^9 \text{ m}^2 = 10^3 \text{ km}^2

That is Virginia Beach or Los Angeles or 10^{-4} of the total area of the US

Landfill space is not a problem.
How much trash does the US generate yearly?

Landfill space is not a problem.
Should you recycle that plastic drink bottle?

What is its value?
Energy density: Plastic ~ gasoline
Mass:
\[ 1 \text{ g (paperclip)} < m < 100 \text{ g (4 oz)} \]
\[ m = 10 \text{ g} \]
\[ m \approx 1\% \text{ of a liter of gasoline} \]

That is worth < $0.01

...after collecting, cleaning, sorting and processing.
Landfill space is not a problem.

Reduce! Reuse! Recycle?
Plastic Rain Is the New Acid Rain

“Researchers find that over 1,000 metric tons of microplastic fall on 11 protected areas in the US annually, equivalent to over 120 million plastic water bottles.” <insert environmental doom and gloom here>

Is this a little or a lot?

It falls on 6% of the US land area.

\[
\rho = \frac{10^3 \text{tons}}{(6 \times 10^{-2})(10^7 \text{km}^2)} \approx 10^{-3} \text{tons/km}^2 = 1 \text{ gram/km}^2
\]
Lightening the Load

How much fuel will the airline save if you urinate immediately before flying across country?

Airplane glide ratio: $1 < r < 10^2 \Rightarrow r \approx 10$

Pee: $0.1 < V < 1\text{ L} \rightarrow V \approx 0.3\text{ L} \rightarrow M_{\text{pee}} \approx 0.3\text{ kg}$

Distance traveled: $d = 5 \times 10^3\text{ km} = 5 \times 10^6\text{ m}$

Distance descended: $h = d/r = 5 \times 10^5\text{ m}$

Energy: $E = M_{\text{pee}}gh = (0.3\text{ kg}) \left(10 \frac{\text{m}}{\text{s}^2}\right)(5 \times 10^5\text{ m}) = 2 \times 10^6\text{ J}$

Efficiency: $0.1 < \varepsilon < 1 \rightarrow \varepsilon \approx 0.3$

Fuel needed: $V_{\text{fuel}} = \frac{E}{\varepsilon e} = \frac{2 \times 10^6\text{ J}}{0.3(3 \times 10^7\text{ J/L})} = 0.2\text{ L}$

That’s about 20¢ of fuel.
Atomic bombs and confetti

Enrico Fermi was about 10 miles from the Trinity A-bomb test:

About 40 seconds after the explosion the air blast reached me. I tried to estimate its strength by dropping from about six feet small pieces of paper before, during, and after the passage of the blast wave. Since, at the time, there was no wind I could observe very distinctly and actually measure the displacement of the pieces of paper that were in the process of falling while the blast was passing. The shift was about 2 1/2 meters, which, at the time, I estimated to correspond to the blast that would be produced by ten thousand tons of T.N.T.

How did he do that?
Estimating the A-Bomb

Work done by the shock wave

\[ W = P \Delta V \]

\( \Delta V \) (hemispherical shell)

\[ \Delta V = (2\pi r^2)d = 6(1.6 \times 10^4 \text{ m})^2(2.5\text{ m}) = 4 \times 10^9 \text{ m}^3 \]

Human surface area \( \sim 1 \text{ m}^2 \)

- \( P < 10^4 \text{ N/m}^2 \) (knock you down)
- \( P > 10^2 \text{ N/m}^2 \) (negligible)

\( \Rightarrow P = 10^3 \text{ N/m}^2 \)

\[ W = P \Delta V = (10^3 \text{ N/m}^2)(4 \times 10^9 \text{ m}^3) = 4 \times 10^{12} \text{ J} = 1 \text{ kT} \]

Now multiply by a few for other energy channels
We get \( E = 4 \text{ kT} \)
Fermi got \( E = 10 \text{ kT} \)
The actual blast was 20 kT

Not too bad!
How fast would we need to flap our arms to fly?

\[ F = \frac{\Delta p}{\Delta t} = W_{you} \]
\[ \Delta p_{air} = m\Delta v = (\rho_{air}A_{arm}d_{flap})(d_{flap}/\Delta t) \]
\[ f = \frac{1}{\Delta t} = \sqrt{\frac{W_{you}}{\rho_{air}A_{arm}d_{flap}^2}} \]

Estimate:
\[ A_{arm} = (1 \text{ m})(0.1 \text{ m}) = 0.1 \text{ m}^2 \]
\[ 1 \text{ cm} < d_{flap} < 1 \text{ m} \Rightarrow d_{flap} = 0.1 \text{ m} \]
\[ \rho_{air} = 1 \text{ kg/m}^3 \]
\[ W_{you} = mg = (10^2 \text{ kg}) \left(10 \frac{\text{N}}{\text{kg}}\right) = 10^3 \text{N} \]

Solve: \[ f = \sqrt{\frac{W_{you}}{\rho_{air}A_{arm}d_{flap}^2}} = \left(\frac{10^3 \text{N}}{(1 \text{ kg/m}^3)(0.1 \text{ m}^2)(0.1 \text{ m})^2}\right)^{\frac{1}{2}} = 10^3 \text{Hz} \]

Faster than a mosquito!
How much coal does a coal-plant need?

A 1-GW$_{e}$ coal-to-electricity power plant needs

$$E = 3(10^9 \text{ W})(\pi \times 10^7 \text{ s/yr}) = 10^{17} \text{ J/yr}$$

At an energy density for coal of $2 \times 10^7 \text{ J/kg}$ (assuming 50% of gasoline), we will need

$$M = \frac{10^{17} \text{ J/yr}}{2 \times 10^7 \text{ J/kg}} = 5 \times 10^9 \text{ kg/yr}$$

This is
- $5 \times 10^6$ tons per year or
- $5 \times 10^4$ 100-ton railroad cars per year or
- $5 \times 10^2$ 100-car coal trains per year or
- 1.5 100-car coal trains per day!
How much Uranium does a nuclear-plant need?

The power plant still needs $10^{17}$ J/yr.
Fissioning one Uranium-235: 200 MeV or $\sim 10^{-3} m$

$$m = \frac{E}{c^2} = \frac{10^{17} \text{ J/yr}}{(10^{-3})(3 \times 10^8 \text{ m/s})^2} = 10^3 \text{ kg/yr} = 1 \text{ ton/yr}$$

5% enriched fuel $\rightarrow$ 20 tons/year

This has a volume of 1 $\text{m}^3$.

It will fit under your dinner table.
If the Sun were made of gerbils, the Earth would be incinerated. Compare the power density of mammals and the sun.

Humans: \[ P = \frac{(2.5 \times 10^3 \text{kcal/day})(4 \times 10^3 \text{J/kcal})}{10^5 \text{s/day}} = 100 \text{W} \]

\[ M = 100 \text{ kg} \]
\[ \frac{P}{M} = \frac{100 \text{ W}}{100 \text{ kg}} = 1 \text{ W/kg} \]

Sun: \[ P = (10^3 \text{W/m}^2)4\pi(1.5 \times 10^{11} \text{m})^2 = 3 \times 10^{26} \text{ W} \]

\[ M = 2 \times 10^{30} \text{kg} \text{ (from the orbital period of the Earth)} \]
\[ \frac{P}{M} = \frac{3 \times 10^{26} \text{ W}}{2 \times 10^{30} \text{ kg}} = 10^{-4} \text{ W/kg} \]

Humans have $10^4$ times the power density of the Sun!

(and gerbils are even worse!)
How much fuel do we need to travel to the stars?

Distance:
\[ d = 4 \text{ light-years} \]

Time:
\[ t < (40 \text{ yr}) \]

Velocity:
\[ v = \frac{d}{t} = \frac{(4 \text{ l-y})}{(40 \text{ y})} = 0.1c \approx 3 \times 10^7 \text{ m/s} \]

Ship: Santa Maria (10^2 tons) \(< m \) < aircraft carrier (10^5 tons)
\[ m = 10^4 \text{ tons} = 10^7 \text{ kg} \]

Energy: \( KE = 0.5mv^2 = 0.5(10^7 \text{ kg})(3 \times 10^7 \text{ m/s})^2 \]
\[ = 5 \times 10^{21} \text{ J} \]

Fuel needed: \( m_{\text{fuel}} = \frac{5 \times 10^{21} \text{ J}}{4 \times 10^9 \text{ J/ton}} = 10^{12} \text{ tons} \)

We need \(10^8\) times more (chemical) fuel than payload!
Flying to the center of the galaxy

You are here
Flying to the center of the galaxy

How dangerous are the protons of the interstellar “vacuum”?

Distance: \( d \sim 3 \times 10^4 \) ly
Time = 30 yrs
\( \Rightarrow \) Relativistic contraction \( \gamma \approx 10^3 \)

\( \rho \approx 1 \) proton/cm\(^3\)

\[ E = \gamma mc^2 = (10^3)(1 \text{ GeV}) = 1 \text{ TeV} \]

\[ c = 3 \times 10^{10} \text{ cm/s} \Rightarrow 3 \times 10^{13} \text{ prot/cm}^2\text{-sec pass through you} \]
Each proton deposits \( E = 2 \) MeV/cm\(^3\)

1 cm\(^2\) of area:
\( \Rightarrow P = (3 \times 10^{13} \text{ p/s})(2 \text{ MeV/(p} \cdot \text{cm}^3))(10^{-13} \text{ J/MeV}) = 6 \text{ W/cm}^3 \)

Lethal dose = \( 10^3 \text{ rad} = 0.01 \text{ J/g} = 0.01 \text{ J/cm}^3 \)
Tissue boils: \( 10^3 \text{ J/g} = 10^3 \text{ J/cm}^3 \Rightarrow 200 \text{ s} = 3 \) minutes

**Speed Kills!**
Estimating Uncertainties

Assume your upper and lower bounds are $\pm 2\sigma$ uncertainties. Add powers of ten in quadrature.

Example: how many planets are there in the universe?

<table>
<thead>
<tr>
<th>Quantity</th>
<th>Bounds</th>
<th>Estimate</th>
<th>$\log_{10}$ Error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Low</td>
<td>Hi</td>
<td>Range</td>
</tr>
<tr>
<td>#planets per star</td>
<td>$10^{-1}$</td>
<td>100</td>
<td>$10^3$</td>
</tr>
<tr>
<td>#stars per galaxy</td>
<td>$10^9$</td>
<td>$10^{13}$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>#galaxies</td>
<td>$10^9$</td>
<td>$10^{13}$</td>
<td>$10^4$</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$log_{10}(\text{Error}) = \frac{\log_{10}(\text{Range})}{4}$

Answer: $3 \times 10^{22}$ to within a factor of $10^{1.5} = 30$
Summary

To estimate:
• Dare to be imprecise!
  • Estimate to within a factor of 10
  • Take the geometric mean of lower and upper bounds
• Break up the question into smaller pieces if necessary.
• Remember a few key numbers

Using estimation and without relying on experts we found:
• Landfill space is not a problem
• Batteries suck
• Burning ethanol in cars is a really really bad idea
• You won’t save the world by recycling plastic water bottles
• Speed kills!
Post-talk questions
Gold in cell phone

- $1 < t < 100 \text{ um} \Rightarrow 10 \text{ um} = 10^{-5} \text{ m} = 10^{-3} \text{ cm}$
- $1 \text{ mm}^2 < \text{ area} < 10^3 \text{ mm}^2 \Rightarrow 30 \text{ mm}^2 = 0.3 \text{ cm}^2$
- $V = At = (10^{-3} \text{ cm})(0.3 \text{ cm}^2) = 3 \times 10^{-4} \text{ cm}^3$
- $M = \rho V = (20 \text{ g/cm}^3)(3 \times 10^{-3} \text{ cm}^3)$
  \[ = 6 \times 10^{-3} \text{ g} = 6 \text{ mg} \text{ (reality = 34 mg)} \]
- $M_{\text{cell phone}} \sim 2 \text{ oz} \sim 60 \text{ g}$
- Purity of gold = $6 \text{ mg} / 60 \text{ g} \sim 10^{-4}$
Calories burned thinking hard

- Average energy used by brain: 1% < e < 100% → 10% = 10 W (reality 20%)
- Intense thinking: harder, but not 10 times harder 1 < f < 10 → f = 3
- 10 W * f = 30 W
- Reality: 6000 cal/day for chess champs (factor of 2-3)